SYMPLECTIC HOMOLOGY OF DISC COTANGENT BUNDLES OF DOMAINS IN EUCLIDEAN SPACE

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Abstract. Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$, and D^*V denote its disc cotangent bundle. We compute symplectic homology of D^*V, in terms of relative homology of loop spaces on the closure of V. We use this result to show that the Floer-Hofer-Wysocki capacity of D^*V is between $2r(V)$ and $2(n+1)r(V)$, where $r(V)$ denotes the inradius of V. As an application, we study periodic billiard trajectories on V.

1. Introduction

1.1. **Main result.** Let us consider the symplectic vector space $T^*\mathbb{R}^n$, with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ and the standard symplectic form $\omega_n := dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$. For any bounded open set $U \subset T^*\mathbb{R}^n$ and real numbers $a < b$, one can define a $\mathbb{Z}_2$-module $\mathrm{SH}^{[a,b)}_*(U)$, which is called *symplectic homology*. This invariant was introduced in [7]. Our first goal is to compute $\mathrm{SH}^{[a,b)}_*(U)$, when U is a disk cotangent bundle of a domain in $\mathbb{R}^n$.

First let us fix notations. For any domain (i.e. connected open set) $V \subset \mathbb{R}^n$, its disc cotangent bundle $D^*V \subset T^*\mathbb{R}^n$ is defined as

$$D^*V := \{(q,p) \in T^*\mathbb{R}^n \mid q \in V, |p| < 1\}.$$

We use the following notations for loop spaces:

- $\Lambda(\mathbb{R}^n) := W^{1,2}(S^1, \mathbb{R}^n)$, where $S^1 := \mathbb{R}/\mathbb{Z}$.
- $\Lambda^{<a}(\mathbb{R}^n) := \{\gamma \in \Lambda(\mathbb{R}^n) \mid \text{length of } \gamma < a\}$.
- For any subset $S \subset \mathbb{R}^n$, we set

$$\Lambda(S) := \{\gamma \in \Lambda(\mathbb{R}^n) \mid \gamma(S^1) \subset S\}, \quad \Lambda^{<a}(S) := \Lambda(S) \cap \Lambda^{<a}(\mathbb{R}^n).$$

Then, the main result in this note is the following:

Theorem 1.1. *Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$, and $\bar{V}$ denote its closure in $\mathbb{R}^n$. For any $a < 0$ and $b > 0$, there exists a natural isomorphism*

$$\mathrm{SH}^{[a,b)}_*(D^*V) \cong H_*\big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big).$$

Moreover, for any $0 < b^- < b^+$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SH}^{[a,b^-)}_*(D^*V) & \xrightarrow{\cong} & H_*\big(\Lambda^{<b^-}(\bar{V}), \Lambda^{<b^-}(\bar{V}) \setminus \Lambda(V)\big) \\ \downarrow & & \downarrow \\ \mathrm{SH}^{[a,b^+)}_*(D^*V) & \xrightarrow[\cong]{} & H_*\big(\Lambda^{<b^+}(\bar{V}), \Lambda^{<b^+}(\bar{V}) \setminus \Lambda(V)\big). \end{array}$$

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The left vertical arrow is a natural map in symplectic homology, and the right vertical arrow is induced by inclusion.

1.2. **Floer-Hofer-Wysocki capacity and periodic billiard trajectories.** By using symplectic homology, one can define the *Floer-Hofer-Wysocki capacity*, which is denoted as c_{FHW}. The Floer-Hofer-Wysocki capacity was introduced in [8]. We recall its definition in Section 2.4. The Floer-Hofer-Wysocki capacity of a disk cotangent bundle D^*V is important in the study of *periodic billiard trajectories* on V (for precise definition, see Definition 6.3):

Proposition 1.2. *Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then, there exists a periodic billiard trajectory γ on V with at most $n+1$ bounce times such that length of $\gamma = c_{\mathrm{FHW}}(D^*V)$.*

Remark 1.3. The idea of using symplectic capacities to study periodic billiard trajectory is due to Viterbo [14]. See also [5], in which a result similar to Proposition 1.2 (Theorem 2.13 in [5]) is proved. Proposition 1.2 is essentially the same as Theorem 13 in [11]. However, our formulation of symplectic homology in this note is a bit different from that in [11], in which we used Viterbo's symplectic homology [13]. Hence we include a proof of Proposition 1.2 in Section 6, for the sake of completeness.

Given Proposition 1.2, it is natural to ask if one can compute $c_{\mathrm{FHW}}(D^*V)$ by using only elementary (i.e. singular) homology theory. The following corollary of our main result gives an answer to this question. For any $x \in \bar{V}$, c_x denotes the constant loop at x.

Corollary 1.4. *Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$, and $b > 0$. Let us define $\iota^b : (\bar{V}, \partial V) \to \big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big)$ by $\iota^b(x) := c_x$. Denote by $(\iota^b)_*$ the map on homology induced by ι^b. Then,*

$$c_{\mathrm{FHW}}(D^*V) = \inf\{b \mid (\iota^b)_n : H_n(\bar{V}, \partial V) \to H_n(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)) \text{ vanishes}\}.$$

To prove Corollary 1.4, we need to combine our main result Theorem 1.1 with results in [10]. Corollary 1.4 is proved in Section 6.

1.3. **Floer-Hofer-Wysocki capacity and inradius.** Using Corollary 1.4, one can obtain a quite good estimate of $c_{\mathrm{FHW}}(D^*V)$ by using the inradius of V. First let us define the notion of the inradius:

Definition 1.5. Let V be a domain in $\mathbb{R}^n$. The *inradius* of V, which is denoted as $r(V)$, is the supremum of radii of balls in V. In other words, $r(V) := \sup_{x \in V} \mathrm{dist}(x, \partial V)$.

Our estimate of the Floer-Hofer-Wysocki capacity is the following:

Theorem 1.6. *Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then, there holds $2r(V) \le c_{\mathrm{FHW}}(D^*V) \le 2(n+1)r(V)$.*

Combined with Proposition 1.2, Theorem 1.6 implies the following result:

Corollary 1.7. *Let V be a bounded domain with smooth boundary in $\mathbb{R}^n$. There exists a periodic billiard trajectory on V with at most $n+1$ bounce times and length between $2r(V)$ and $2(n+1)r(V)$.*

Remark 1.8. Let $\xi(V)$ denote the infimum of the lengths of periodic billiard trajectories on V. Corollary 1.7 shows that $\xi(V) \le 2(n+1)r(V)$. When V is *convex*, this result was already established as Theorem 1.3 in [5]. On the other hand, the main result in [11] is that $\xi(V) \le \text{const}_n r(V)$ for any domain V with smooth boundary in $\mathbb{R}^n$. A weaker result $\xi(V) \le \text{const}_n \text{vol}(V)^{1/n}$ was obtained in [14], [9].

Theorem 1.6 is proved in Section 7. Here we give a short comment on the proof. Actually, the lower bound is immediate from Corollary 1.4, and the issue is to prove the upper bound. By Corollary 1.4, it is enough to show that if $b > 2(n+1)r(V)$, then $(\iota^b)_*[(\bar{V}, \partial V)] = 0$. We will prove this by constructing a $(n+1)$-chain in $\big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big)$ which bounds $(\bar{V}, \partial V)$. Details are carried out in Section 7.

1.4. **Organization of the paper.** In Section 2, we recall the definition and main properties of symplectic homology, following [7]. In Section 3, we recall Morse theory for Lagrangian action functionals on loop spaces, following [1], [3]. The goal in these sections is to fix a setup for the arguments in Sections 4, 5, 6.

In Section 4, we prove our main result Theorem 1.1. The proof consists of two steps:

Step1: In Theorem 4.2, we prove an isomorphism between Floer homology of a quadratic Hamiltonian on $T^*\mathbb{R}^n$ and Morse homology of its fiberwise Legendre transform.

Step2: By taking a limit of Hamiltonians, we deduce Theorem 1.1 from Theorem 4.2.

Our proof of Theorem 4.2 is based on [2]: we construct an isomorphism by using so called *hybrid moduli spaces*. However, since we will work on $T^*\mathbb{R}^n$, proofs of various C^0-estimates for (hybrid) Floer trajectories are not automatic. Techniques in [2] (in which the authors are working on cotangent bundles of *compact* manifolds) do not seem to work directly in our setting. To prove C^0- estimates for Floer trajectories in our setting, we combine techniques in [2] and [7]. Proofs of C^0- estimates are carried out in Section 5.

In Section 6, we discuss the Floer-Hofer-Wysocki capacity and periodic billiard trajectories. The goal of this section is to prove Proposition 1.2 and Corollary 1.4.

In Section 7, we prove Theorem 1.6. This section can be read almost independently from the other parts of the paper.

2. Symplectic homology

We recall the definition and main properties of symplectic homology. We basically follow [7].

2.1. **Hamiltonian.** For $H \in C^\infty(T^*\mathbb{R}^n)$, its *Hamiltonian vector field* X_H is defined as $\omega_n(X_H, \cdot\,) = -dH(\,\cdot\,)$.

For $H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ and $t \in S^1$, $H_t \in C^\infty(T^*\mathbb{R}^n)$ is defined as $H_t(q,p) := H(t,q,p)$. $\mathscr{P}(H)$ denotes the set of periodic orbits of $(X_{H_t})_{t \in S^1}$, i.e.

$$\mathscr{P}(H) := \{x \in C^\infty(S^1, T^*\mathbb{R}^n) \mid \dot{x}(t) = X_{H_t}(x(t))\}.$$

$x \in \mathscr{P}(H)$ is called *nondegenerate* if 1 is not an eigenvalue of the Poincaré map associated with x. We introduce the following conditions on $H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$:

- (H0): Every element in $\mathscr{P}(H)$ is nondegenerate.
- (H1): There exists $a \in (0,\infty) \setminus \pi\mathbb{Z}$ such that $\sup_{t \in S^1} \|H_t - Q^a\|_{C^1(T^*\mathbb{R}^n)} < \infty$, where $Q^a(q,p) := a(|q|^2 + |p|^2)$.

Remark 2.1. The class of Hamiltonians considered in this note is a bit different from that in [7]. To put it more precisely, (H1) is more restrictive than conditions (6), (7) in [7]. On the other hand, we do not need condition (8) in [7]. It is easy to see that our definition of symplectic homology is equivalent to that in [7], see Remark 2.6.

Lemma 2.2. *For any $H \in C^\infty(S^1 \times T\mathbb{R}^n)$ which satisfies (H1), $\mathscr{P}(H)$ is C^0-bounded. In particular, if H also satisfies (H0), then $\mathscr{P}(H)$ is a finite set.*

Proof. Suppose that there exists $H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ which satisfies (H1) and $\mathscr{P}(H)$ is *not* C^0-bounded. Then there exists a sequence $(x_j)_{j=1,2,\ldots}$ in $\mathscr{P}(H)$ such that $R_j := \max_{t \in S^1} |x_j(t)|$ goes to ∞ as $j \to \infty$. Define $v_j : S^1 \to T^*\mathbb{R}^n$ and $h^j \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ by

$$v_j(t) := x_j(t)/R_j, \qquad h^j(t,q,p) := H(t, R_j q, R_j p)/R_j^2.$$

It is easy to show that $v_j \in \mathscr{P}(h^j)$. Moreover, since $\sup_{t \in S^1} \|dH_t - dQ^a\|_{C^0} < \infty$,

$$\lim_{j \to \infty} \sup_{t \in S^1} \|dh^j_t - dQ^a\|_{C^0} = 0. \tag{1}$$

By definition, $\max_{t \in S^1} |v_j(t)| = 1$. In particular, $(v_j)_j$ is C^0-bounded. Moreover, since $\partial_t v_j = X_{h^j_t}(v_j)$, (1) shows that $(v_j)_j$ is C^1-bounded. Hence, up to a subsequence, $(v_j)_j$ converges in $C^0(S^1, T^*\mathbb{R}^n)$. We denote the limit by v.

By the triangle inequality,

$$\int_0^1 |X_{h^j_t}(v_j(t)) - X_{Q^a}(v(t))|\, dt \le \int_0^1 |X_{h^j_t}(v_j(t)) - X_{Q^a}(v_j(t))|\, dt + \int_0^1 |X_{Q^a}(v_j(t)) - X_{Q^a}(v(t))|\, dt.$$

As $j \to \infty$, the first term on the RHS goes to 0 by (1), and the second term on the RHS goes to 0 since v_j converges to v in C^0. Therefore, for any $0 \le t_0 \le 1$,

$$\begin{aligned} v(t_0) - v(0) &= \lim_{j \to \infty} v_j(t_0) - v_j(0) \\ &= \lim_{j \to \infty} \int_0^{t_0} X_{h^j_t}(v_j(t))\, dt = \int_0^{t_0} X_{Q^a}(v(t))\, dt, \end{aligned}$$

hence $v \in \mathscr{P}(Q^a)$. On the other hand, it is clear that $\max_{t \in S^1} |v(t)| = 1$. This is a contradiction, since $a \notin \pi\mathbb{Z}$ implies that the only element in $\mathscr{P}(Q_a)$ is the constant loop at $(0, \ldots, 0)$. □

$H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ is called *admissible* if it satisfies (H0) and (H1).

2.2. **Truncated Floer homology.** Let $J = (J_t)_{t \in S^1}$ be a time dependent almost complex structure on $T^*\mathbb{R}^n$, such that:

(J1): For any $t \in S^1$, J_t is compatible with ω_n, i.e. $g_{J_t}(\xi, \eta) := \omega_n(\xi, J_t\eta)$ is a Riemannian metric on $T^*\mathbb{R}^n$.

Let $H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ be an admissible Hamiltonian. For any $x_-, x_+ \in \mathscr{P}(H)$, we introduce the Floer trajectory space in the usual manner:

$$\mathscr{M}_{H,J}(x_-, x_+) := \\ \{u : \mathbb{R} \times S^1 \to T^*\mathbb{R}^n \mid \partial_s u - J_t(\partial_t u - X_{H_t}(u)) = 0, \lim_{s \to \pm\infty} u(s) = x_\pm\}.$$

We set $\bar{\mathscr{M}}_{H,J}(x_-, x_+) := \mathscr{M}_{H,J}(x_-, x_+)/\mathbb{R}$, where $\mathbb{R}$ acts on $\mathscr{M}_{H,J}(x_-, x_+)$ by shift in the s-variable.

The *standard complex structure* J_{std} on $T^*\mathbb{R}^n$ is defined as

$$J_{\mathrm{std}}(\partial_{p_i}) := \partial_{q_i}, \qquad J_{\mathrm{std}}(\partial_{q_i}) := -\partial_{p_i}.$$

Now we state our first C^0-estimate. It is proved in Section 5.

Lemma 2.3. *There exists a constant $\varepsilon > 0$ which satisfies the following property:*

> *For any admissible Hamiltonian $H \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ and $J = (J_t)_{t \in S^1}$ which satisfies (J1) and $\sup_t \|J_t - J_{\mathrm{std}}\|_{C^0} < \varepsilon$, $\mathscr{M}_{H,J}(x_-, x_+)$ is C^0-bounded for any $x_-, x_+ \in \mathscr{P}(H)$.*

We recall the definition of Floer homology. For any $\gamma \in C^\infty(S^1, T^*\mathbb{R}^n)$, we set

$$\mathscr{A}_H(\gamma) := \int_{S^1} \gamma^*\left(\sum_i p_i dq_i\right) - H(t, \gamma(t))\, dt.$$

For real numbers $a < b$, the *Floer chain complex* $\mathrm{CF}_*^{[a,b)}(H)$ is the free $\mathbb{Z}_2$ module generated by $\{\gamma \in \mathscr{P}(H) \mid \mathscr{A}_H(\gamma) \in [a, b)\}$, indexed by the Conley-Zehnder index $\mathrm{ind}_{\mathrm{CZ}}$. For the definition of the Conley-Zehnder index, see Section 1.3 in [7].

Suppose that $J = (J_t)_{t \in S^1}$ satisfies (J1) and each J_t is sufficiently close to J_{std}. Lemma 2.3 shows that for generic J, $\bar{\mathscr{M}}_{H,J}(x_-, x_+)$ is a compact 0-dimensional manifold for any $x_-, x_+ \in \mathscr{P}(H)$ such that $\mathrm{ind}_{\mathrm{CZ}}(x_-) - \mathrm{ind}_{\mathrm{CZ}}(x_+) = 1$. We can thus define the Floer differential $\partial_{H,J}$ on $\mathrm{CF}_*^{[a,b)}(H)$ as

$$\partial_{H,J}([x_-]) := \sum_{\mathrm{ind}_{\mathrm{CZ}}(x_+) = \mathrm{ind}_{\mathrm{CZ}}(x_-) - 1} \sharp\bar{\mathscr{M}}_{H,J}(x_-, x_+) \cdot [x_+].$$

The usual gluing argument shows that $\partial_{H,J}^2 = 0$. $\mathrm{HF}_*^{[a,b)}(H, J) := H_*(\mathrm{CF}_*^{[a,b)}(H), \partial_{H,J})$ is called *truncated Floer homology.*

2.3. **Symplectic homology.** Suppose that we are given the following data:

- Admissible Hamiltonians $H^-, H^+ \in C^\infty(S^1 \times T^*\mathbb{R}^n)$
- $J^- = (J^-_t)_{t\in S^1}$, $J^+ = (J^+_t)_{t\in S^1}$, which satisfy (J1). Moreover, all J^-_t, J^+_t are sufficiently close to J_{std}.

We assume that $\mathrm{HF}^{[a,b)}_*(H^-, J^-)$, $\mathrm{HF}^{[a,b)}_*(H^+, J^+)$ are well-defined. If $H^- \le H^+$, i.e. $H^-(t,q,p) \le H^+(t,q,p)$ for any $t \in S^1$ and $(q,p) \in T^*\mathbb{R}^n$, one can define *monotonicity homomorphism*

$$\mathrm{HF}^{[a,b)}_*(H^-, J^-) \to \mathrm{HF}^{[a,b)}_*(H^+, J^+)$$

in the following way.

First we introduce the following conditions on $H \in C^\infty(\mathbb{R} \times S^1 \times T^*\mathbb{R}^n)$:

(HH1): There exists $s_0 > 0$ such that $H(s,t,q,p) = \begin{cases} H(s_0,t,q,p) & (s \ge s_0) \\ H(-s_0,t,q,p) & (s \le -s_0) \end{cases}.$

(HH2): $\partial_s H(s,t,q,p) \ge 0$ for any $(s,t,q,p) \in \mathbb{R} \times S^1 \times T^*\mathbb{R}^n$.

(HH3): There exists $a(s) \in C^\infty(\mathbb{R})$ such that:

- $a'(s) \ge 0$ for any s.
- $a(s) \in \pi\mathbb{Z} \implies a'(s) > 0$.
- Setting $\Delta(s,t,q,p) := H(s,t,q,p) - Q^{a(s)}(q,p)$, there holds

$$\sup_{(s,t)} \|\Delta_{s,t}\|_{C^1(T^*\mathbb{R}^n)} < \infty, \qquad \sup_{(s,t)} \|\partial_s \Delta_{s,t}\|_{C^0(T^*\mathbb{R}^n)} < \infty.$$

If H satisfies (HH1), (HH2), (HH3) and $H^\pm_t = H_{\pm s_0, t}$, H is called a homotopy from H^- to H^+. For any H^- and H^+ such that $H^- \le H^+$, there exists a homotopy from H^- to H^+. In fact, take $\rho \in C^\infty(\mathbb{R})$ such that

$$s \ge 1 \implies \rho(s) = 1, \quad s \le 0 \implies \rho(s) = 0,$$
$$0 < s < 1 \implies \rho(s) \in (0,1),\ \rho'(s) > 0.$$

Then $H(s,t,q,p) := \rho(s)H^+(t,q,p) + (1-\rho(s))H^-(t,q,p)$ is a homotopy from H^- to H^+.

Next we introduce conditions on $J = (J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$, a family of almost complex structures on $T^*\mathbb{R}^n$ parametrized by $\mathbb{R} \times S^1$:

(JJ1): There exists $s_1 > 0$ such that $J_{s,t}(q,p) = \begin{cases} J_{s_1,t}(q,p) & (s \ge s_1) \\ J_{-s_1,t}(q,p) & (s \le -s_1) \end{cases}.$

(JJ2): For any $(s,t) \in \mathbb{R} \times S^1$, $J_{s,t}$ is compatible with ω_n.

If J satisfies (JJ1), (JJ2) and $J^\pm_t = J_{\pm s_1, t}$, J is called a homotopy from J^- to J^+.

Let $H \in C^\infty(\mathbb{R} \times S^1 \times T^*\mathbb{R}^n)$ be a homotopy from H^- to H^+, and $J = (J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ be a homotopy from J^- to J^+. For any $x_- \in \mathscr{P}(H^-)$ and $x_+ \in \mathscr{P}(H^+)$, we define

$$\mathscr{M}_{H,J}(x_-, x_+) := \{u : \mathbb{R} \times S^1 \to T^*\mathbb{R}^n \mid \partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0,\ \lim_{s\to\pm\infty} u(s) = x_\pm\}.$$

Now we state our second C^0-estimate. It is proved in Section 5.

Lemma 2.4. *There exists a constant $\varepsilon > 0$ which satisfies the following property:*

> *If $J = (J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ satisfies $\sup_{s,t} \|J_{s,t} - J_{\mathrm{std}}\|_{C^0} < \varepsilon$, $\mathscr{M}_{H,J}(x_-, x_+)$ is C^0-bounded for any $x_- \in \mathscr{P}(H^-)$, $x_+ \in \mathscr{P}(H^+)$.*

Lemma 2.4 shows that, if J is generic and all $J_{s,t}$ are sufficiently close to J_{std}, $\mathscr{M}_{H,J}(x_-, x_+)$ is a compact 0-dimensional manifold for any $x_- \in \mathscr{P}(H^-)$, $x_+ \in \mathscr{P}(H^+)$ such that $\mathrm{ind}_{\mathrm{CZ}}(x_-) = \mathrm{ind}_{\mathrm{CZ}}(x_+)$. We define $\Phi : \mathrm{CF}_*^{[a,b)}(H^-, J^-) \to \mathrm{CF}_*^{[a,b)}(H^+, J^+)$ by

$$\Phi\big([x_-]\big) := \sum_{\mathrm{ind}_{\mathrm{CZ}}(x_+)=\mathrm{ind}_{\mathrm{CZ}}(x_-)} \sharp\mathscr{M}_{H,J}(x_-, x_+) \cdot [x_+].$$

The usual gluing argument shows that Φ is a chain map. The monotonicity homomorphism

$$\Phi_* : \mathrm{HF}_*^{[a,b)}(H^-, J^-) \to \mathrm{HF}_*^{[a,b)}(H^+, J^+)$$

is the homomorphism on homology induced by Φ. One can show that Φ_* does not depend on the choices of H and J, see Section 4.3 in [7].

Remark 2.5. Let H be an admissible Hamiltonian, and J^0, J^1 be S^1- dependent almost complex structures such that $\mathrm{HF}_*^{[a,b)}(H, J^0)$, $\mathrm{HF}_*^{[a,b)}(H, J^1)$ are well-defined. Then, one can show that the monotonicity homomorphism $\mathrm{HF}_*^{[a,b)}(H, J^0) \to \mathrm{HF}_*^{[a,b)}(H, J^1)$ is an i-somorphism. Hence $\mathrm{HF}_*^{[a,b)}(H, J)$ does not depend on J, and we denote it by $\mathrm{HF}_*^{[a,b)}(H)$. Moreover, for two admissible Hamiltonians H^-, H^+ satisfying $H^- \leq H^+$, the monotonicity homomorphism $\mathrm{HF}_*^{[a,b)}(H^-) \to \mathrm{HF}_*^{[a,b)}(H^+)$ is well-defined.

We define symplectic homology. Let U be a bounded open set in $T^*\mathbb{R}^n$. Let $\mathscr{H}_U$ denote the set consisting of admissible Hamiltonians H such that $H|_{S^1\times\bar{U}} < 0$. $\mathscr{H}_U$ is a directed set with relation

$$H^- \leq H^+ \iff H^-(t,q,p) \leq H^+(t,q,p) \qquad \big(\forall (t,q,p) \in S^1 \times T^*\mathbb{R}^n\big).$$

Then, for any $-\infty < a < b < \infty$, we define symplectic homology $\mathrm{SH}_*^{[a,b)}(U)$ by

$$\mathrm{SH}_*^{[a,b)}(U) := \varinjlim_{H\in\mathscr{H}_U} \mathrm{HF}_*^{[a,b)}(H).$$

If $U \subset V$, then obviously $\mathscr{H}_V \subset \mathscr{H}_U$. Hence there exists a natural homomorphism

$$\mathrm{SH}_*^{[a,b)}(V) \to \mathrm{SH}_*^{[a,b)}(U).$$

Moreover, for any $a^\pm, b^\pm \in \mathbb{R}$ such that $a^- \leq a^+$, $b^- \leq b^+$, $a^- < b^-$, $a^+ < b^+$, there exists a natural homomorphism $\mathrm{SH}_*^{[a^-,b^-)}(U) \to \mathrm{SH}_*^{[a^+,b^+)}(U)$.

Remark 2.6. As noted in Remark 2.1, the class of Hamiltonians considered here is different from that in [7]. However, our definition of symplectic homology given above is equivalent to the definition in [7] (see Section 1.6 in [7]). A key fact is that compact perturbations of quadratic Hamiltonians are admissible both in our sense and sense in [7].

2.4. **Floer-Hofer-Wysocki capacity.** Finally, we define the Floer-Hofer-Wysocki capacity , which is originally due to [8]. For any bounded open set U and $b > 0$, we define

$$\mathrm{SH}_*^{(0,b)}(U) := \varprojlim_{\varepsilon \to +0} \mathrm{SH}_*^{[\varepsilon,b)}(U).$$

When $U \subset V$, there exists a natural homomorphism $\mathrm{SH}_*^{(0,b)}(V) \to \mathrm{SH}_*^{(0,b)}(U)$. For any $p \in T^*\mathbb{R}^n$, we define

$$\Theta^b(p) := \varinjlim_{\varepsilon \to +0} \mathrm{SH}_{n+1}^{(0,b)}(B^{2n}(p:\varepsilon))$$

where $B^{2n}(p:\varepsilon)$ denotes the open ball in $T^*\mathbb{R}^n$ with center p and radius ε. It is known that $\Theta^b(p) \cong \mathbb{Z}_2$, see pp. 603–604 in [8].

Let U be a bounded domain (hence *connected*) in $T^*\mathbb{R}^n$. Taking $p \in U$ arbitrarily, we define the Floer-Hofer-Wysocki capacity of U as

$$c_{\mathrm{FHW}}(U) := \inf\{b \mid \mathrm{SH}_{n+1}^{(0,b)}(U) \to \Theta^b(p) \cong \mathbb{Z}_2 \text{ is onto}\}.$$

It's known that the above definition does not depend on the choice of p. See pp.604 in [8].

3. Loop space homology

In this section, we recall Morse theory on loop spaces for Lagrangian action functionals. We mainly follow [1], [3].

3.1. **Lagrangian action functional.** Recall that we used the notation $\Lambda(\mathbb{R}^n) := W^{1,2}(S^1, \mathbb{R}^n)$. Given $L \in C^\infty(S^1 \times T\mathbb{R}^n)$, we consider the action functional

$$\mathscr{S}_L : \Lambda(\mathbb{R}^n) \to \mathbb{R}; \quad \gamma \mapsto \int_{S^1} L(t, \gamma(t), \dot\gamma(t))\, dt.$$

We introduce the following conditions on L:

(L1): There exists $a \in (0,\infty) \setminus \pi\mathbb{Z}$ such that

$$\sup_{t \in S^1} \left\| L(t,q,v) - \left(\frac{|v|^2}{4a} - a|q|^2 \right) \right\|_{C^2(T\mathbb{R}^n)} < \infty.$$

(L2): There exists a constant $c > 0$ such that $\partial_v^2 L(t,q,v) \geq c$ for any $(t,q,v) \in S^1 \times T\mathbb{R}^n$.

Notice that (L1) implies the following estimates:

$$\begin{aligned}
\text{(L1)'}: \quad & |D^2 L(t,q,v)| \leq \text{const}, \\
& |\partial_q L(t,q,v)| \leq \text{const}(1+|q|), \quad |\partial_v L(t,q,v)| \leq \text{const}(1+|v|), \\
& |L(t,q,v)| \leq \text{const}(1+|q|^2+|v|^2).
\end{aligned}$$

Lemma 3.1. *If L satisfies (L1) and (L2), the following holds.*

(1) *$\mathscr{S}_L : \Lambda(\mathbb{R}^n) \to \mathbb{R}$ is a Fréchet C^1 function. Its differential $d\mathscr{S}_L$ is given by*

$$d\mathscr{S}_L(\gamma)(\xi) = \int_{S^1} \partial_q L(t,\gamma,\dot\gamma)\xi(t) + \partial_v L(t,\gamma,\dot\gamma)\dot\xi(t)\, dt.$$

Moreover, $d\mathscr{S}_L$ is Gâteaux differentiable. We denote the differential by $d^2\mathscr{S}_L$.

(2) *$\gamma \in \Lambda(\mathbb{R}^n)$ satisfies $d\mathscr{S}_L(\gamma) = 0$ if and only if $\gamma \in C^\infty(S^1, \mathbb{R}^n)$ and*

$$\partial_q L(t, \gamma, \dot{\gamma}) - \frac{d}{dt}\big(\partial_v L(t, \gamma, \dot{\gamma})\big) = 0.$$

Proof. Using (L1)' and (L2), the proof is the same as Proposition 3.1 in [3]. □

Let us set $\mathscr{P}(L) := \{\gamma \in \Lambda(\mathbb{R}^n) \mid d\mathscr{S}_L(\gamma) = 0\}$. $\gamma \in \mathscr{P}(L)$ is called *nondegenerate* if $d^2\mathscr{S}_L(\gamma)$ is nondegenerate as a symmetric bilinear form on $T_\gamma\Lambda(\mathbb{R}^n) = W^{1,2}(S^1, \mathbb{R}^n)$.

For each $\gamma \in \Lambda(\mathbb{R}^n)$, $D\mathscr{S}_L(\gamma) \in T_\gamma\Lambda(\mathbb{R}^n) = W^{1,2}(S^1, \mathbb{R}^n)$ is defined so that

$$\langle D\mathscr{S}_L(\gamma), \xi\rangle_{W^{1,2}} = d\mathscr{S}_L(\gamma)(\xi) \qquad \big(\forall \xi \in W^{1,2}(S^1, \mathbb{R}^n)\big).$$

We show that the pair $(\mathscr{S}_L, D\mathscr{S}_L)$ satisfies the Palais-Smale (PS) condition. First let us recall what the PS condition is:

Definition 3.2. Let M be a Hilbert manifold, $f : M \to \mathbb{R}$ be a C^1 function, and X be a continuous vector field on M. A sequence $(p_k)_k$ on M is called a *Palais-Smale (PS) sequence*, if $(f(p_k))_k$ is bounded, and $\lim_{k\to\infty} df(X(p_k)) = 0$. The pair (f, X) satisfies the *PS-condition*, if any PS sequence contains a convergent subsequence.

Lemma 3.3. *Suppose that $L \in C^\infty(S^1 \times T\mathbb{R}^n)$ satisfies (L1). Let $(\gamma_k)_k$ be a sequence on $\Lambda(\mathbb{R}^n)$ such that both $\mathscr{S}_L(\gamma_k)$ and $\|D\mathscr{S}_L(\gamma_k)\|_{W^{1,2}}$ are bounded. Then, $(\gamma_k)_k$ is C^0-bounded.*

Proof. Suppose that there exists a sequence $(\gamma_k)_k$ such that both $\mathscr{S}_L(\gamma_k)$, $\|D\mathscr{S}_L(\gamma_k)\|_{W^{1,2}}$ are bounded, and $m_k := \max_{t\in S^1} |\gamma_k(t)|$ goes to ∞ as $k \to \infty$. We define $\delta_k \in \Lambda(\mathbb{R}^n)$ and $l_k \in C^\infty(S^1 \times T\mathbb{R}^n)$ by

$$\delta_k(t) := \gamma_k(t)/m_k, \qquad l_k(t, q, p) := L(t, m_k q, m_k p)/{m_k}^2.$$

We show that $(\delta_k)_k$ is $W^{1,2}$-bounded. Since $(\delta_k)_k$ is obviously C^0-bounded, it is enough to show that $(\dot{\delta_k})_k$ is L^2-bounded. First notice that

$$\lim_{k\to\infty} \mathscr{S}_{l_k}(\delta_k) = \lim_{k\to\infty} \frac{\mathscr{S}_L(\gamma_k)}{{m_k}^2} = 0.$$

On the other hand, since L satisfies (L1),

$$\lim_{k\to\infty} \mathscr{S}_{l_k}(\delta_k) - \left(\int_{S^1} \frac{|\dot{\delta_k}|^2}{4a} - a|\delta_k|^2 \, dt\right) = 0.$$

Thus $(\dot{\delta_k})_k$ is L^2-bounded.

By taking a subsequence of $(\delta_k)_k$, we may assume that there exists $\delta \in \Lambda(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|\delta_k - \delta\|_{C^0} = 0$, and $\dot{\delta_k} \to \dot{\delta}\ (k \to \infty)$ weakly in L^2.

We prove that $d\mathscr{S}_l(\delta) = 0$, where $l(t, q, v) := |v|^2/4a - a|q|^2$. This means that $\delta \in C^\infty(S^1, \mathbb{R}^n)$ and $\ddot{\delta}(t) + 4a^2\delta(t) \equiv 0$. Since $a \notin \pi\mathbb{Z}$, this means that $\delta(t) \equiv 0$. However, since $\max_{t\in S^1} |\delta(t)| = \lim_{k\to\infty} \max_{t\in S^1} |\delta_k(t)| = 1$, this is a contradiction.

To prove $d\mathscr{S}_l(\delta) = 0$, first notice that

$$\lim_{k\to\infty} \|D\mathscr{S}_{l_k}(\delta_k)\|_{W^{1,2}} = \lim_{k\to\infty} \frac{\|D\mathscr{S}_L(\gamma_k)\|_{W^{1,2}}}{m_k} = 0.$$

Hence it is enough to show that for any $\xi \in C^\infty(S^1, \mathbb{R}^n)$ there holds

$$\lim_{k\to\infty} \big(d\mathscr{S}_l(\delta) - d\mathscr{S}_l(\delta_k)\big)(\xi) = 0, \qquad \lim_{k\to\infty} \big(d\mathscr{S}_l(\delta_k) - d\mathscr{S}_{l_k}(\delta_k)\big)(\xi) = 0.$$

To check the first claim, notice the following equation:

$$\big(d\mathscr{S}_l(\delta) - d\mathscr{S}_l(\delta_k)\big)(\xi) = \int_{S^1} \frac{\dot\delta(t) - \dot\delta_k(t)}{2a} \cdot \dot\xi(t) - 2a\big(\delta(t) - \delta_k(t)\big) \cdot \xi(t)\, dt.$$

Then, since $\dot\delta_k$ converges to $\dot\delta$ weakly in L^2, the RHS goes to 0 as $k \to \infty$. The second claim follows from $\lim_{k\to\infty} \|l - l_k\|_{C^1} = 0$. □

Corollary 3.4. *Suppose that $L \in C^\infty(S^1 \times T\mathbb{R}^n)$ satisfies (L1) and (L2). Then, the pair $(\mathscr{S}_L, D\mathscr{S}_L)$ satisfies the PS-condition on $\Lambda(\mathbb{R}^n)$.*

Proof. Suppose that $(\gamma_k)_k$ is a PS-sequence with respect to $(\mathscr{S}_L, D\mathscr{S}_L)$. Then, Lemma 3.3 shows that $(\gamma_k)_k$ is C^0- bounded. Then, Proposition 3.3 in [3] shows that $(\gamma_k)_k$ has a convergent subsequence. □

3.2. **Construction of a downward pseudo-gradient.** Suppose that $L \in C^\infty(S^1 \times T\mathbb{R}^n)$ satisfies (L1) and (L2). To define a Morse complex of $\mathscr{S}_L$, we need the following condition:

(L0): Every $\gamma \in \mathscr{P}(L)$ is nondegenerate.

The following lemma (basically the same as Theorem 4.1 in [3]) constructs a downward pseudo-gradient vector field for $\mathscr{S}_L$. For the definitions of the terms "Lyapunov function", "Morse vector field", "Morse-Smale condition", see Section 2 of [3].

Lemma 3.5. *If $L \in C^\infty(S^1 \times T\mathbb{R}^n)$ satisfies (L0), (L1), (L2), there exists a smooth vector field X on $\Lambda(\mathbb{R}^n)$ which satisfies the following conditions:*

1. *X is complete.*
2. *$\mathscr{S}_L$ is a Lyapunov function for X.*
3. *X is a Morse vector field. $X(\gamma) = 0$ if and only if $\gamma \in \mathscr{P}(L)$. Every $\gamma \in \mathscr{P}(L)$ has a finite Morse index, which is denoted by* $\mathrm{ind}_{\mathrm{Morse}}(\gamma)$.
4. *The pair $(\mathscr{S}_L, X)$ satisfies the Palais-Smale condition.*
5. *X satisfies the Morse-Smale condition up to every order.*

Proof. In the course of this proof, we use the following abbreviation:

$$\{a < \mathscr{S}_L < b\} := \{\gamma \in \Lambda(\mathbb{R}^n) \mid a < \mathscr{S}_L(\gamma) < b\}.$$

Moreover, $\|\cdot\|_{W^{1,2}}$ is abbreviated as $\|\cdot\|$.

Since $(\mathscr{S}_L, D\mathscr{S}_L)$ satisfies the PS-condition, and all critical points are nondegenerate, for any $a < b$ there exist only finitely many critical points of $\mathscr{S}_L$ on $\{a < \mathscr{S}_L < b\}$. We denote them as $\gamma_1, \ldots, \gamma_m$.

For each $1 \leq j \leq m$, Lemma 4.1 in [3] shows that there exist U_{γ_j}, Y_{γ_j} such that:

- U_{γ_j} is a neighborhood of γ_j in $\{a < \mathscr{S}_L < b\}$.
- Y_{γ_j} is a smooth vector field on U_{γ_j}.
- γ_j is a critical point of Y_{γ_j} with a finite Morse index, and there holds

$$d\mathscr{S}_L(Y_{\gamma_j}(\gamma)) \leq -\lambda(\gamma_j)\|\gamma - \gamma_j\|^2 \qquad (\forall \gamma \in U_{\gamma_j}),$$

  where $\lambda(\gamma_j)$ is a positive constant.

By taking U_{γ_j} sufficiently small, we may asume that $\|Y_{\gamma_j}\| \leq 1$ on U_{γ_j}. We take a smaller neighborhood V_{γ_j} such that $\overline{V_{\gamma_j}} \subset U_{\gamma_j}$.

Since $(\mathscr{S}_L, D\mathscr{S}_L)$ satisfies the PS-condition, there exists $\varepsilon > 0$ such that : for any $\gamma \in \{a < \mathscr{S}_L < b\} \setminus (U_{\gamma_1} \cup \cdots \cup U_{\gamma_m})$, $\|D\mathscr{S}_L(\gamma)\| \geq \varepsilon$. For each $\gamma \notin U_{\gamma_1} \cup \cdots \cup U_{\gamma_m}$, set $Y_\gamma := -D\mathscr{S}_L(\gamma)/\|D\mathscr{S}_L(\gamma)\|$. Then, obviously $\|Y_\gamma\| = 1$. Moreover,

$$d\mathscr{S}_L(\gamma)(Y_\gamma) = \langle D\mathscr{S}_L(\gamma), Y_\gamma \rangle = -\|D\mathscr{S}_L(\gamma)\| \leq -\varepsilon.$$

Since $\mathscr{S}_L$ is C^1 by Lemma 3.1 (1), if U_γ is a sufficiently small neighborhood of γ,

$$\gamma' \in U_\gamma \implies d\mathscr{S}_L(\gamma')(Y_\gamma) \leq -\varepsilon/2.$$

We may also assume that U_γ is disjoint from $\overline{V_{\gamma_1}} \cup \cdots \cup \overline{V_{\gamma_m}}$. Moreover, since $\Lambda(\mathbb{R}^n)$ is paracompact, we can define a locally finite open covering $\{U_\gamma\}_{\gamma \in \Gamma}$ of $\{a < \mathscr{S}_L < b\}$ such that $\gamma_1, \ldots, \gamma_m \in \Gamma$.

Let $\{\chi_\gamma\}_{\gamma \in \Gamma}$ be a partition of unity with respect to $\{U_\gamma\}_{\gamma \in \Gamma}$. Then we define a vector field Y on $\{a < \mathscr{S}_L < b\}$ by $Y := \sum_{\gamma \in \Gamma} \chi_\gamma Y_\gamma$. Since each Y_γ satisfies $\|Y_\gamma\| \leq 1$, it is clear that $\|Y\| \leq 1$. Moreover, there exists $c > 0$ such that

$$\gamma \notin V_{\gamma_1} \cup \cdots \cup V_{\gamma_m} \implies d\mathscr{S}_L(\gamma)(Y(\gamma)) \leq -c.$$

Now we show that $(\mathscr{S}_L, Y)$ satisfies the PS-condition on $\{a < \mathscr{S}_L < b\}$. Let $(x_k)_k$ be a sequence on $\{a < \mathscr{S}_L < b\}$ such that $\lim_{k \to \infty} d\mathscr{S}_L(x_k)(Y(x_k)) = 0$. Then, $x_k \in V_{\gamma_1} \cup \cdots \cup V_{\gamma_m}$ for sufficiently large k. By taking a subsequence, we may assume that $x_k \in V_{\gamma_1}$ for all k. Then, since

$$d\mathscr{S}_L(x_k)(Y(x_k)) = d\mathscr{S}_L(x_k)(Y_{\gamma_1}(x_k)) \leq -\lambda(\gamma_1)\|x_k - \gamma_1\|^2,$$

there holds $\lim_{k \to \infty} \|x_k - \gamma_1\| = 0$. Thus $(\mathscr{S}_L, Y)$ satisfies the PS condition. We have defined a smooth vector field Y on $\{a < \mathscr{S}_L < b\}$, which satisfies (2), (3), (4) and $\|Y\| \leq 1$.

Finally we construct X on $\Lambda(\mathbb{R}^n)$. Take a sequence of closed intervals $(I_m)_{m \in \mathbb{Z}}$ with the following properties:

- $(\min I_m)_m$, $(\max I_m)_m$ are increasing sequences.
- $\bigcup_m I_m = \mathbb{R}$.
- $I_m \cap I_{m'} \neq \emptyset$ if and only if $|m - m'| \leq 1$.
- For any $m \in \mathbb{Z}$, $I_m \cap I_{m+1}$ does not contain critical values of $\mathscr{S}_L$.

For every m, there exists a smooth vector field X_m on $\{\min I_m < \mathscr{S}_L < \max I_m\}$ which satisfies (2), (3), (4) and $\|X_m\| \le 1$. Finally, taking a partition of unity $(\rho_m)_m$ with respect to the open covering $\{\min I_m < \mathscr{S}_L < \max I_m\}_m$ of $\Lambda(\mathbb{R}^n)$, we define a vector field X on $\Lambda(\mathbb{R}^n)$ by $X := \sum_m \rho_m X_m$. Then, it is easy to check that $(\mathscr{S}_L, X)$ satisfies the PS condition. Moreover, since X satisfies $\|X\| \le 1$ everywhere, X is complete.

The vector field X defined above satisfies (1)-(4) in the statement. Since it is of class C^∞, the Sard-Smale theorem shows that (5) is satisfied by a sufficiently small C^∞ perturbation. □

3.3. **Morse complex.** Let X be a downward pseudo-gradient for $\mathscr{S}_L$ on $\Lambda(\mathbb{R}^n)$, which is constructed in Lemma 3.5. Since X is complete, one can define $(\varphi_t^X)_{t\in\mathbb{R}}$, a family of diffeomorphisms on $\Lambda(\mathbb{R}^n)$ so that

$$\varphi_0^X = \mathrm{id}_{\Lambda(\mathbb{R}^n)}, \qquad \partial_t(\varphi_t^X) = X(\varphi_t^X).$$

For each $\gamma \in \mathscr{P}(L)$, its stable and unstable manifolds are defined as

$$W^s(\gamma : X) := \{p \in \Lambda(\mathbb{R}^n) \mid \lim_{t\to\infty} \varphi_t^X(p) = \gamma\},$$
$$W^u(\gamma : X) := \{p \in \Lambda(\mathbb{R}^n) \mid \lim_{t\to-\infty} \varphi_t^X(p) = \gamma\}.$$

For any $\gamma, \gamma' \in \mathscr{P}(L)$, we set $\mathscr{M}_X(\gamma, \gamma') := W^u(\gamma : X) \cap W^s(\gamma', X)$. Since $\mathscr{M}_X(\gamma,\gamma')$ consists of flow lines of X, $\mathscr{M}_X(\gamma,\gamma')$ admits a natural $\mathbb{R}$ action. We denote the quotient by $\bar{\mathscr{M}}_X(\gamma,\gamma')$.

For any $\gamma, \gamma' \in \mathscr{P}(L)$, $W^u(\gamma : X)$ and $W^s(\gamma' : X)$ are transverse, since X satisfies the Morse-Smale condition. Therefore, $\bar{\mathscr{M}}_X(\gamma,\gamma')$ is a smooth manifold with dimension $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{Morse}}(\gamma') - 1$. When $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{Morse}}(\gamma') = 1$, $\bar{\mathscr{M}}_X(\gamma,\gamma')$ consists of finitely many points.

For any $-\infty < a < b < \infty$, $\mathrm{CM}_*^{[a,b)}(L)$ denotes the free $\mathbb{Z}_2$-module generated by $\{\gamma \in \mathscr{P}(L) \mid a \le \mathscr{S}_L(\gamma) < b\}$. We define a differential $\partial_{L,X}$ on $\mathrm{CM}_*^{[a,b)}(L)$ by

$$\partial_{L,X}\big([\gamma]\big) := \sum_{\mathrm{ind}_{\mathrm{Morse}}(\gamma') = \mathrm{ind}_{\mathrm{Morse}}(\gamma) - 1} \sharp\bar{\mathscr{M}}_X(\gamma,\gamma') \cdot [\gamma'].$$

Then $(\mathrm{CM}_*^{[a,b)}(L), \partial_{L,X})$ is a chain complex, and its homology group $\mathrm{HM}_*^{[a,b)}(L, X)$ is isomorphic to $H_*(\{\mathscr{S}_L < b\}, \{\mathscr{S}_L < a\})$. For details, see [1].

Next we discuss functoriality. Consider $L^0, L^1 \in C^\infty(S^1 \times T\mathbb{R}^n)$ which satisfy (L0), (L1), (L2) and $L^0(t,q,v) > L^1(t,q,v)$ for any $(t,q,v) \in S^1 \times T\mathbb{R}^n$. Take vector fields X^0, X^1 on $\Lambda(\mathbb{R}^n)$ such that (L^0, X^0) and (L^1, X^1) satisfy the conditions in Lemma 3.5.

We assume that $\mathscr{P}(L^0) \cap \mathscr{P}(L^1) = \emptyset$ (this can be achieved by slightly perturbing L^0). Then, by a C^∞-small perturbation of X^0, one can assume the following:

> For any $\gamma^0 \in \mathscr{P}(L^0)$ and $\gamma^1 \in \mathscr{P}(L^1)$, $W^u(\gamma^0 : X^0)$ is transverse to $W^s(\gamma^1 : X^1)$.

If this assumption is satisfied, $\mathscr{M}_{X^0,X^1}(\gamma^0,\gamma^1) := W^u(\gamma^0 : X^0) \cap W^s(\gamma^1 : X^1)$ is a smooth manifold with dimension $\mathrm{ind}_{\mathrm{Morse}}(\gamma^0) - \mathrm{ind}_{\mathrm{Morse}}(\gamma^1)$.

We define a chain map $\Phi : \mathrm{CM}^{[a,b)}_*(L^0, X^0) \to \mathrm{CM}^{[a,b)}_*(L^1, X^1)$ by

$$\Phi([\gamma]) := \sum_{\mathrm{ind}_{\mathrm{Morse}}(\gamma')=\mathrm{ind}_{\mathrm{Morse}}(\gamma)} \sharp\mathscr{M}_{X^0,X^1}(\gamma,\gamma') \cdot [\gamma'].$$

Φ induces a homomorphism on homology, which coincides with the homomorphism induced by the inclusion $(\{\mathscr{S}_{L^0} < b\}, \{\mathscr{S}_{L^0} < a\}) \to (\{\mathscr{S}_{L^1} < b\}, \{\mathscr{S}_{L^1} < a\})$.

4. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1, i.e. to compute $\mathrm{SH}^{[a,b)}_*(D^*V)$ for a bounded domain $V \subset \mathbb{R}^n$ with smooth boundary. In Section 4.1, we reduce Theorem 1.1 to Theorem 4.2 and Lemma 4.3. Theorem 4.2 is the main step, and it is proved in Sections 4.2 and 4.3, assuming some C^0- estimates of Floer trajectories: Lemmas 4.8, 4.9, 4.10. These C^0- estimates are proved in Section 5. Lemma 4.3 is a technical lemma on loop space homology, and it is proved in Section 4.4.

4.1. **Outline.** Let us take $(a_m)_m$, an increasing sequence of positive numbers such that $a_m \notin \pi\mathbb{Z}$ for any m, and $\lim_{m\to\infty} a_m = \infty$. We take a sequence $(k_m)_m$ in $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ such that:

- (k1): For every m, $\partial_t k_m(t) > 0$ and $\partial_t^2 k_m(t) \geq 0$ for any $t \geq 0$.
- (k2): For every m, $\partial_t k_m \equiv a_m$ on $\{t \mid k_m(t) \geq 0\}$.
- (k3): $(k_m)_m$ is strictly increasing. Moreover, $\sup_m k_m(t) = \begin{cases} 0 & (0 \leq t \leq 1) \\ \infty & (t > 1) \end{cases}$.

Let us define $K_m \in C^\infty(\mathbb{R}^n, \mathbb{R})$ by $K_m(p) := k_m(|p|^2)$. Then, (k1) implies that K_m is strictly convex. Moreover, (k2) implies that

$$\mathbb{R}^n \to \mathbb{R}^n; p \mapsto v(p) := \partial_p K_m$$

is a diffeomorphism. We denote its inverse by $p(v)$, i.e. $\partial_p K_m(p(v)) = v$. Let $K_m^\vee$ be the Legendre transform of K_m, i.e. $K_m^\vee(v) := p(v) \cdot v - K_m(p(v))$. Then, it is easy to show that $(K_m^\vee)_m$ is strictly decreasing, and $\inf_m K_m^\vee(v) = |v|$ for any $v \in \mathbb{R}^n$.

We take a sequence $(Q_m)_m$ of smooth functions on $\mathbb{R}^n$, such that

- (Q1): There exists a sequence of constants $(c_m)_m$ such that $Q_m(q) - \big(a_m|q|^2 + c_m\big)$ is compactly supported.
- (Q2): $(Q_m)_m$ is strictly increasing. Moreover, $\sup_m Q_m(q) = \begin{cases} 0 & (q \in \bar{V}) \\ \infty & (q \notin \bar{V}) \end{cases}$.

Let $H'_m(q,p) := Q_m(q) + K_m(p)$. Then, for every m, H'_m satisfies (H1). Moreover, $(H'_m)_m$ is strictly increasing, and

$$\sup_m H'_m(q,p) = \begin{cases} 0 & \big((q,p) \in \overline{D^*V}\big) \\ \infty & \big((q,p) \notin \overline{D^*V}\big) \end{cases}.$$

Let L'_m be the fiberwise Legendre transform of H'_m. It is easy to see that $L'_m(q,v) = K^\vee_m(v) - Q_m(q)$. Then, for every m, L'_m satisfies (L1) and (L2). $(L'_m)_m$ is strictly decreasing, and there holds $\inf_m L'_m(q,v) = \begin{cases} |v| & (q \in \bar{V}) \\ -\infty & (q \notin \bar{V}) \end{cases}$.

Since $(H'_m)_m$ is *strictly* increasing, by sufficiently small perturbations of $(H'_m)_m$, one can obtain a sequence $(H^m)_m$ on $C^\infty(S^1 \times T^*\mathbb{R}^n)$ with the following properties:

- For every m, H^m is admissible.
- $(H^m)_m$ is strictly increasing, and $\sup_m H^m(t,q,p) = \begin{cases} 0 & ((q,p) \in \overline{D^*V}) \\ \infty & ((q,p) \notin \overline{D^*V}) \end{cases}$.
- For every m, its Legendre transform L^m is well-defined, and it satisfies (L0), (L1), (L2). $(L^m)_m$ is strictly decreasing, and $\inf_m L^m(t,q,v) = \begin{cases} |v| & (q \in \bar{V}) \\ -\infty & (q \notin \bar{V}) \end{cases}$.

Remark 4.1. For notational reasons, we use superscripts for H^m and L^m.

By the first two properties, $\mathrm{SH}^{[a,b)}_*(D^*V) = \varinjlim_{m\to\infty} \mathrm{HF}^{[a,b)}_*(H^m)$. Now we state the following key result, which is proved in Sections 4.2 and 4.3:

Theorem 4.2. *For any $-\infty < a < b < \infty$ and m, there exists a natural isomorphism $\mathrm{HM}^{[a,b)}_*(L^m) \cong \mathrm{HF}^{[a,b)}_*(H^m)$. The following diagram is commutative for every m:*

$$\begin{array}{ccc} \mathrm{HM}^{[a,b)}_*(L^m) & \longrightarrow & \mathrm{HM}^{[a,b)}_*(L^{m+1}) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{HF}^{[a,b)}_*(H^m) & \longrightarrow & \mathrm{HF}^{[a,b)}_*(H^{m+1}). \end{array}$$

Then we obtain

$$\varinjlim_{m\to\infty} \mathrm{HF}^{[a,b)}_*(H^m) \cong \varinjlim_{m\to\infty} \mathrm{HM}^{[a,b)}_*(L^m) \cong H_*\Big(\bigcup_m \{\mathscr{S}_{L^m} < b\}, \bigcup_m \{\mathscr{S}_{L^m} < a\}\Big).$$

Since $(L^m)_m$ is strictly decreasing and $\inf_m L^m(t,q,v) = \begin{cases} |v| & (q \in \bar{V}) \\ -\infty & (q \notin \bar{V}) \end{cases}$, for any $c \in \mathbb{R}$

$$\inf_m \mathscr{S}_{L^m}(\gamma) < c \iff \gamma(S^1) \not\subset \bar{V} \text{ or (length of } \gamma) < c.$$

Therefore, for any $a < 0$ and $b > 0$,

$$\begin{aligned} \mathrm{SH}^{[a,b)}_*(D^*V) &\cong H_*\big(\Lambda^{<b}(\mathbb{R}^n) \cup (\Lambda(\mathbb{R}^n) \setminus \Lambda(\bar{V})), \Lambda(\mathbb{R}^n) \setminus \Lambda(\bar{V})\big) \\ &\cong H_*\big(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(\bar{V})\big) \\ &\cong H_*(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)), \end{aligned}$$

where the second isomorphism follows from excision, and the third isomorphism follows from the next Lemma 4.3, which is proved in Section 4.4.

Lemma 4.3. *Let V be a bounded domain in $\mathbb{R}^n$ with smooth boundary. For any $0 < b < \infty$, there exists a natural isomorphism*

$$H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(\bar{V})) \cong H_*(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)).$$

Finally, we have to check that for any $b_- < b_+$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SH}_*^{[a,b^-)}(D^*V) & \xrightarrow{\cong} & H_*\big(\Lambda^{<b^-}(\bar{V}), \Lambda^{<b^-}(\bar{V}) \setminus \Lambda(V)\big) \\ \downarrow & & \downarrow \\ \mathrm{SH}_*^{[a,b^+)}(D^*V) & \xrightarrow[\cong]{} & H_*\big(\Lambda^{<b^+}(\bar{V}), \Lambda^{<b^+}(\bar{V}) \setminus \Lambda(V)\big). \end{array}$$

This is clear from the construction, hence omitted.

4.2. **Construction of a chain level isomorphism.** In this and the next subsection, we prove Theorem 4.2. In this subsection, we define an isomorphism

$$\mathrm{HM}_*^{[a,b)}(L^m) \to \mathrm{HF}_*^{[a,b)}(H^m).$$

Following [2], we define this isomorphism by considering so called *hybrid moduli spaces*. Suppose we are given the following data:

- $J^m = (J_t^m)_{t \in S^1}$, which is sufficiently close to the standard one, and $\mathrm{CF}_*^{[a,b)}(H^m, J^m)$ is well-defined.
- Smooth vector field X^m on $\Lambda(\mathbb{R}^n)$, such that $\mathrm{CM}_*^{[a,b)}(L^m, X^m)$ is well-defined.
- $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^m)$.

We consider the following equation for $u \in W^{1,3}(S^1 \times [0, \infty), T^*\mathbb{R}^n)$:

$$\begin{aligned} &\partial_s u - J_t^m\big(\partial_t u - X_{H_t^m}(u)\big) = 0, \\ &\pi(u(0)) \in W^u(\gamma : X^m), \\ &\lim_{s \to \infty} u(s) = x. \end{aligned}$$

π denotes the natural projection $T^*\mathbb{R}^n \to \mathbb{R}^n; (q, p) \mapsto q$. The moduli space of solutions of this equation is denoted by $\mathscr{M}_{X^m,H^m,J^m}(\gamma, x)$.

Remark 4.4. In the definition of $\mathscr{M}_{X^m,H^m,J^m}$, we have used a Sobolev space $W^{1,3}(S^1 \times [0, \infty), T^*\mathbb{R}^n)$. One can replace it with $W^{1,r}(S^1 \times [0, \infty), T^*\mathbb{R}^n)$ for any $2 < r < 4$. The condition $2 < r < 4$ is necessary to carry out Fredholm theory and prove C^0-estimates for Floer trajectories.

To define a homomorphism by counting $\mathscr{M}_{X^m,H^m,J^m}(\gamma, x)$, we need the following results:

Lemma 4.5. *For generic J^m, $\mathscr{M}_{X^m,H^m,J^m}(\gamma, x)$ is a smooth manifold of dimension* $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x)$ *for any $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^m)$.*

Proof. See Section 3.1 in [2]. □

Lemma 4.6. *For any $\gamma \in \mathscr{P}(L^m)$, $x \in \mathscr{P}(H^m)$ and $u \in \mathscr{M}_{X^m,H^m,J^m}(\gamma, x)$, there holds*

$$\mathscr{S}_{L^m}(\gamma) \geq \mathscr{S}_{L^m}(\pi(u(0))) \geq \mathscr{A}_{H^m}(u(0)) \geq \mathscr{A}_{H^m}(x).$$

Proof. See pp.299 in [2]. □

Corollary 4.7. *When $\mathscr{S}_{L^m}(\gamma) < \mathscr{A}_{H^m}(x)$, $\mathscr{M}_{X^m,H^m,J^m}(\gamma,x) = \emptyset$. When $\mathscr{S}_{L^m}(\gamma) = \mathscr{A}_{H^m}(x)$, $\mathscr{M}_{X^m,H^m,J^m}(\gamma,x) \neq \emptyset$ if and only if $\gamma = \pi(x)$. In this case, $\mathscr{M}_{X^m,H^m,J^m}(\gamma,x)$ consists of a single element u such that $u(s,t) := x(t)$.*

We recall that our setup differs from the one of [2] inasmuch as our base manifold is $\mathbb{R}^n$, while the authors of [2] work with compact bases. However, their analysis applies to our situation for all aspects except for the C^0-bounds of Floer moduli spaces.

Now, we state our third C^0 -estimate. It is proved in Section 5.

Lemma 4.8. *There exists $\varepsilon > 0$ such that, if J^m satisfies $\sup_t \|J^m_t - J_{\mathrm{std}}\|_{C^0} < \varepsilon$, $\mathscr{M}_{X^m,H^m,J^m}(\gamma,x)$ is C^0-bounded for any $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^m)$.*

Suppose that J^m satisfies the condition in Lemma 4.5, and it is sufficiently close to J_{std}. By Lemma 4.8, for any $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^m)$ such that $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x) = 0$, $\mathscr{M}_{X^m,H^m,J^m}(\gamma,x)$ is a *compact* 0-dimensional manifold. Then, we can define a homomorphism

$$\Psi^m : \mathrm{CM}^{[a,b)}_*(L^m,X^m) \to \mathrm{CF}^{[a,b)}_*(H^m,J^m);$$
$$[\gamma] \mapsto \sum_{\mathrm{ind}_{\mathrm{CZ}}(x)=\mathrm{ind}_{\mathrm{Morse}}(\gamma)} \sharp\mathscr{M}_{X^m,H^m,J^m}(\gamma,x)\cdot[x].$$

Corollary 4.7 shows that Ψ^m is an isomorphism (for details, see Section 3.5 in [2]). Gluing arguments show that Ψ^m is a chain map (for details, see Section 3.5 in [2]). Hence Ψ^m induces an isomorphism on homology.

4.3. **Chain level commutativity up to homotopy.** In the previous subsection, we constructed a chain level isomorphism

$$\Psi^m : \mathrm{CM}^{[a,b)}_*(L^m,X^m) \to \mathrm{CF}^{[a,b)}_*(H^m,J^m)$$

for every m. In this subsection, we show that

$$\begin{array}{ccc} \mathrm{CM}^{[a,b)}_*(L^m,X^m) & \xrightarrow{\Psi^m} & \mathrm{CF}^{[a,b)}_*(H^m,J^m) \\ \downarrow{\Phi^L} & & \downarrow{\Phi^H} \\ \mathrm{CM}^{[a,b)}_*(L^{m+1},X^{m+1}) & \xrightarrow[\Psi^{m+1}]{} & \mathrm{CF}^{[a,b)}_*(H^{m+1},J^{m+1}) \end{array}$$

commutes up to chain homotopy, where Φ^H and Φ^L are chain maps constructed in Section 2.3 and Section 3.3, respectively.

To prove this, we introduce a chain map

$$\Theta : \mathrm{CM}^{[a,b)}_*(L^m,X^m) \to \mathrm{CF}^{[a,b)}_*(H^{m+1},J^{m+1});$$
$$[\gamma] \mapsto \sum_{\mathrm{ind}_{\mathrm{Morse}}(\gamma)=\mathrm{ind}_{\mathrm{CZ}}(x)} \sharp\mathscr{M}_{X^m,H^{m+1},J^{m+1}}(\gamma,x)\cdot[x].$$

It is enough to show $\Phi^H \circ \Psi^m \sim \Theta \sim \Psi^{m+1} \circ \Phi^L$. ($\sim$ means chain homotopic.)

First we show that $\Psi^{m+1} \circ \Phi^L \sim \Theta$. For any $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^{m+1})$, $\mathscr{N}^0(\gamma, x)$ denotes the set of (α, u, v), where

$$\alpha \in [0, \infty), \quad u : [0, \alpha] \to \Lambda(\mathbb{R}^n), \quad v \in W^{1,3}([0, \infty) \times S^1, T^*\mathbb{R}^n)$$

which satisfy the following conditions:

$$\begin{aligned} &u(0) \in W^u(\gamma : X^m), \qquad u(s) = \varphi_s^{X^{m+1}}(u(0)) \, (0 \le s \le \alpha), \\ &\partial_s v - J_t^{m+1}(\partial_t v - X_{H_t^{m+1}}(v)) = 0, \quad \pi(v(0)) = u(\alpha), \quad \lim_{s \to \infty} v(s) = x. \end{aligned}$$

We state our fourth C^0-estimate. It is proved in Section 5.

Lemma 4.9. *There exists $\varepsilon > 0$ which satisfies the following property:*

> *If J^{m+1} satisfies $\sup_t \|J_t^{m+1} - J_{\mathrm{std}}\|_{C^0} < \varepsilon$, $\|v\|_{C^0}$ is uniformly bounded for any $(\alpha, u, v) \in \mathscr{N}^0(\gamma, x)$, where $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^{m+1})$.*

Suppose that J^m is generic and sufficiently close to J_{std}. Then, due to Lemma 4.9 and gluing arguments, the following holds:

- When $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x) = -1$, $\mathscr{N}^0(\gamma, x)$ is a compact 0-dimensional manifold. Every $(\alpha, u, v) \in \mathscr{N}^0(\gamma, x)$ satisfies $\alpha > 0$.
- When $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x) = 0$, $\mathscr{N}^0(\gamma, x)$ is a 1-dimensional manifold with boundary. Its boundary is $\{\alpha = 0\}$, and its end is compactified by the following moduli spaces (we set $k := \mathrm{ind}_{\mathrm{Morse}}(\gamma) = \mathrm{ind}_{\mathrm{CZ}}(x)$):

$$\begin{aligned} &\bar{\mathscr{M}}_{X^m}(\gamma, \gamma') \times \mathscr{N}^0(\gamma', x) \quad (\gamma' \in \mathscr{P}(L^m), \mathrm{ind}_{\mathrm{Morse}}(\gamma') = k - 1), \\ &\mathscr{M}_{X^m, X^{m+1}}(\gamma, \gamma') \times \mathscr{M}_{X^{m+1}, H^{m+1}, J^{m+1}}(\gamma', x) \\ &\qquad\qquad\qquad\qquad (\gamma' \in \mathscr{P}(L^{m+1}), \mathrm{ind}_{\mathrm{Morse}}(\gamma') = k), \\ &\mathscr{N}^0(\gamma, x') \times \bar{\mathscr{M}}_{H^{m+1}, J^{m+1}}(x', x) \quad (x' \in \mathscr{P}(H^{m+1}), \mathrm{ind}_{\mathrm{CZ}}(x') = k + 1). \end{aligned}$$

Let us define $K^0 : \mathrm{CM}_*^{<a}(L^m, X^m) \to \mathrm{CF}_{*+1}^{<a}(H^{m+1}, J^{m+1})$ by

$$K^0[\gamma] := \sum_{\mathrm{ind}_{\mathrm{CZ}}(x) = \mathrm{ind}_{\mathrm{Morse}}(\gamma) + 1} \sharp \mathscr{N}^0(\gamma, x) \cdot [x].$$

Then, the above results show that $\partial_{H^{m+1}, J^{m+1}} \circ K^0 + K^0 \circ \partial_{L^m, X^m} = \Psi^{m+1} \circ \Phi^L + \Theta$.

Next we show that $\Phi^H \circ \Psi^m \sim \Theta$. Let $H \in C^\infty(\mathbb{R} \times S^1 \times T^*\mathbb{R}^n)$ be a homotopy from H^m to H^{m+1}, and $J = (J_{s,t})_{(s,t) \in \mathbb{R} \times S^1}$ be a homotopy from J^m to J^{m+1}. By (HH1) and (JJ1), there exists $s_0 > 0$ such that

$$(H_{s,t}, J_{s,t}) = \begin{cases} (H_t^m, J_t^m) & (s \le -s_0) \\ (H_t^{m+1}, J_t^{m+1}) & (s \ge s_0) \end{cases}.$$

For any $\gamma \in \mathscr{P}(L^m)$ and $x \in \mathscr{P}(H^{m+1})$, $\mathscr{N}^1(\gamma, x)$ denotes the set of (β, w), where

$$\beta \in (-\infty, s_0], \qquad w \in W^{1,3}([\beta, \infty) \times S^1, T^*\mathbb{R}^n)$$

which satisfy the following properties:

$$\pi(w(\beta)) \in W^u(\gamma : X^m), \qquad \partial_s w - J_{s,t}(\partial_t w - X_{H_{s,t}}(w)) = 0,$$
$$\lim_{s\to\infty} w(s) = x.$$

Now we state our fifth C^0- estimate. It is proved in Section 5.

Lemma 4.10. *There exists $\varepsilon > 0$ which satisfies the following property:*

If J satisfies $\sup_{s,t} \|J_{s,t} - J_{\mathrm{std}}\|_{C^0} < \varepsilon$, $\|w\|_{C^0}$ is uniformly bounded for any $(\beta, w) \in \mathscr{N}_1(\gamma, x)$, where $\gamma \in \mathscr{P}(L^m)$, $x \in \mathscr{P}(H^{m+1})$.

Suppose that J is generic and sufficiently close to J_{std}. Then, by Lemma 4.10 and gluing arguments, the following holds:

- When $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x) = -1$, $\mathscr{N}^1(\gamma, x)$ is a compact 0-dimensional manifold. Every $(\beta, w) \in \mathscr{N}^1(\gamma, x)$ satisfies $\beta < s_0$.
- When $\mathrm{ind}_{\mathrm{Morse}}(\gamma) - \mathrm{ind}_{\mathrm{CZ}}(x) = 0$, $\mathscr{N}^1(\gamma, x)$ is a 1-dimensional manifold with boundary. Its boundary is $\{\beta = s_0\}$, and its ends are compactified by the following moduli spaces (we set $k := \mathrm{ind}_{\mathrm{Morse}}(\gamma) = \mathrm{ind}_{\mathrm{CZ}}(x)$):

$$\bar{\mathscr{M}}_{X^m}(\gamma, \gamma') \times \mathscr{N}^1(\gamma', x) \quad (\gamma' \in \mathscr{P}(L^m), \mathrm{ind}_{\mathrm{Morse}}(\gamma') = k-1),$$
$$\mathscr{M}_{X^m, H^m, J^m}(\gamma, x') \times \mathscr{M}_{H,J}(x', x) \quad (x' \in \mathscr{P}(H^m), \mathrm{ind}_{\mathrm{CZ}}(x') = k),$$
$$\mathscr{N}^1(\gamma, x') \times \bar{\mathscr{M}}_{H^{m+1}, J^{m+1}}(x', x) \quad (x' \in \mathscr{P}(H^{m+1}), \mathrm{ind}_{\mathrm{CZ}}(x') = k+1).$$

Let us define $K^1 : \mathrm{CM}^{[a,b)}_*(L^m, X^m) \to \mathrm{CF}^{[a,b)}_{*+1}(H^{m+1}, J^{m+1})$ by

$$K^1[\gamma] := \sum_{\mathrm{ind}_{\mathrm{CZ}}(x) = \mathrm{ind}_{\mathrm{Morse}}(\gamma)+1} \sharp \mathscr{N}^1(\gamma, x) \cdot [x].$$

Then, the above results show that $\partial_{H^{m+1}, J^{m+1}} \circ K^1 + K^1 \circ \partial_{L^m, X^m} = \Theta + \Phi^H \circ \Psi^m$.

4.4. **Proof of Lemma 4.3.** Finally, we prove Lemma 4.3. Through this section, V denotes a bounded domain in $\mathbb{R}^n$ with smooth boundary. First we need the following lemma:

Lemma 4.11. *For any open neighborhood W of $\bar{V}$ and $b > 0$, the natural homomorphism*

$$H_*(\Lambda^{<b}(W), \Lambda^{<b}(W) \setminus \Lambda(\bar{V})) \to H_*(\Lambda^{<b}(W), \Lambda^{<b}(W) \setminus \Lambda(V))$$

is an isomorphism.

***Proof*.** This is equivalent to showing that $H_*(\Lambda^{<b}(W) \setminus \Lambda(V), \Lambda^{<b}(W) \setminus \Lambda(\bar{V})) = 0$.

Let us take a k-dimensional singular chain $\alpha = \sum_i c_i \alpha_i \in C_k(\Lambda^{<b}(W) \setminus \Lambda(V))$ ($c_i \in \mathbb{Z}_2$, $\alpha_i : \Delta^k \to \Lambda^{<b}(W) \setminus \Lambda(V)$ are continuous maps) such that $\partial\alpha \in C_{k-1}(\Lambda^{<b}(W) \setminus \Lambda(\bar{V}))$. Since Δ^k is compact, there exists $b' < b$ such that $\alpha_i(\Delta^k) \subset \Lambda^{<b'}(W)$ for all i.

Let us take a compactly supported smooth vector field Z on W, which points outwards on ∂V. Let $(\varphi^Z_t)_{t\in\mathbb{R}}$ be the isotopy on W generated by Z, i.e. $\varphi^Z_0 = \mathrm{id}_W$, $\partial_t \varphi^Z_t = Z(\varphi^Z_t)$.

Take $\delta > 0$ and define $\alpha_i^t : \Delta^k \to \Lambda(W)$ by $\alpha_i^t(p) := \varphi_{\delta t}^Z \circ \alpha_i(p)\ (\forall p \in \Delta^k)$. When $\delta > 0$ is sufficiently small, $\alpha_i^t(\Delta^k) \subset \Lambda^{<b}(W)$ for any i and $0 \le t \le 1$.

It is easy to see that α_i^t satisfies the following properties for any i and $0 \le t \le 1$:

- $\alpha_i^0 = \alpha_i$.
- $\alpha^t := \sum_i c_i \alpha_i^t$ satisfies $\alpha^t \in C_k(\Lambda^{<b}(W) \setminus \Lambda(V))$ and $\partial \alpha^t \in C_{k-1}(\Lambda^{<b}(W) \setminus \Lambda(\bar{V}))$ for any $0 \le t \le 1$.
- $\alpha^1 \in C_k(\Lambda^{<b}(W) \setminus \Lambda(\bar{V}))$.

Then we obtain $[\alpha] = [\alpha^0] = [\alpha^1] = 0$ in $H_k(\Lambda^{<b}(W) \setminus \Lambda(V), \Lambda^{<b}(W) \setminus \Lambda(\bar{V}))$. □

Corollary 4.12. *For any open neighborhood W of $\bar{V}$, the natural homomorphism*

$$H_*(\Lambda^{<b}(W), \Lambda^{<b}(W) \setminus \Lambda(V)) \to H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(V))$$

is an isomorphism.

***Proof*.** Consider the following commutative diagram:

$$\begin{array}{ccc} H_*(\Lambda^{<b}(W), \Lambda^{<b}(W) \setminus \Lambda(\bar{V})) & \longrightarrow & H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(\bar{V})) \\ \downarrow & & \downarrow \\ H_*(\Lambda^{<b}(W), \Lambda^{<b}(W) \setminus \Lambda(V)) & \longrightarrow & H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(V)) \end{array}$$

Then, vertical arrows are isomorphism by Lemma 4.11, and the top arrow is an isomorphism by excision. Therefore the bottom arrow is an isomorphism. □

Applying Lemma 4.11 with $W = \mathbb{R}^n$,

$$H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(\bar{V})) \to H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(V))$$

is an isomorphism. Hence, to prove Lemma 4.3 it is enough to show that the natural homomorphism

$$H_*(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)) \to H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(V))$$

is an isomorphism. To show this, we need the following trick: take a sequence $(g^l)_l$ of Riemannian metrics on $\mathbb{R}^n$, with the following properties:

(g-1): For any tangent vector ξ on $\mathbb{R}^n$, $|\xi|_{g^l}$ is decreasing in l: $|\xi|_{g^1} > |\xi|_{g^2} > \cdots$.
(g-2): For any tangent vector ξ on $\mathbb{R}^n$, $\lim_{l \to \infty} |\xi|_{g^l} = |\xi|$, where $|\cdot|$ is the standard metric.
(g-3): For any $l \ge 1$, there exists an embedding $\tau_l : \partial V \times (-\varepsilon_l, \varepsilon_l) \to \mathbb{R}^n$ with the following properties:
- $\tau_l(x, 0) = x$ for any $x \in \partial V$.
- $\tau_l^{-1}(V) = \partial V \times (-\varepsilon_l, 0)$.
- $\tau_l^* g^l$ is a product metric of $g^l|_{\partial V}$ and the standard metric on $(-\varepsilon_l, \varepsilon_l)$.

We set $W_l := V \cup \operatorname{Im} \tau_l$.

For each l we define

$$\Lambda_l^{<b}(\mathbb{R}^n) := \left\{ \gamma \in \Lambda(\mathbb{R}^n) \,\middle|\, \int_{S^1} |\dot{\gamma}(t)|_{g^l}\, dt < b \right\}, \quad \Lambda_l^{<b}(\bar{V}) := \Lambda_l^{<b}(\mathbb{R}^n) \cap \Lambda(\bar{V}).$$

By (g-1), $(\Lambda_l^{<b}(\mathbb{R}^n))_l$, $(\Lambda_l^{<b}(\bar{V}))_l$ are increasing sequences of open sets in $\Lambda^{<b}(\mathbb{R}^n)$, $\Lambda^{<b}(\bar{V})$. By (g-2), $\bigcup_l \Lambda_l^{<b}(\mathbb{R}^n) = \Lambda^{<b}(\mathbb{R}^n)$, $\bigcup_l \Lambda_l^{<b}(\bar{V}) = \Lambda^{<b}(\bar{V})$. Thus there holds

$$H_*(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)) = \varinjlim_{l\to\infty} H_*(\Lambda_l^{<b}(\bar{V}), \Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V)),$$
$$H_*(\Lambda^{<b}(\mathbb{R}^n), \Lambda^{<b}(\mathbb{R}^n) \setminus \Lambda(V)) = \varinjlim_{l\to\infty} H_*(\Lambda_l^{<b}(\mathbb{R}^n), \Lambda_l^{<b}(\mathbb{R}^n) \setminus \Lambda(V)).$$

Therefore Lemma 4.3 is reduced to the following lemma:

Lemma 4.13. *For any $l \geq 1$, the natural homomorphism*

$$H_*(\Lambda_l^{<b}(\bar{V}), \Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V)) \to H_*(\Lambda_l^{<b}(\mathbb{R}^n), \Lambda_l^{<b}(\mathbb{R}^n) \setminus \Lambda(V))$$

is an isomorphism.

Proof. Let us take $W_l \supset \bar{V}$ as in (g-3). Since Corollary 4.12 is valid also for g^l,

$$H_*(\Lambda_l^{<b}(W_l), \Lambda_l^{<b}(W_l) \setminus \Lambda(V)) \to H_*(\Lambda_l^{<b}(\mathbb{R}^n), \Lambda_l^{<b}(\mathbb{R}^n) \setminus \Lambda(V))$$

is an isomorphism. Hence it is enough to show that

$$I : H_*(\Lambda_l^{<b}(\bar{V}), \Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V)) \to H_*(\Lambda_l^{<b}(W_l), \Lambda_l^{<b}(W_l) \setminus \Lambda(V))$$

is an isomorphism. We check surjectivity and injectivity.

We prove surjectivity of I. Take $\alpha = \sum_i c_i\alpha_i \in C_k(\Lambda_l^{<b}(W_l))$ such that $\partial\alpha \in C_{k-1}(\Lambda_l^{<b}(W_l)\setminus \Lambda(V))$. Since Δ^k is compact, there exists $b' < b$ such that

$$\text{length of } \alpha_i(p) \text{ with respect to } g^l < b' \quad (\forall i, \forall p \in \Delta^k).$$

Let us take $\rho \in C^\infty((-\varepsilon_l, \varepsilon_l))$ with the following properties:

- $\rho(s) \equiv 0$ on $[0, \varepsilon_l)$.
- $0 \leq \rho'(s) \leq b/b'$, $-\varepsilon_l < \rho(s) \leq 0$ on $(-\varepsilon_l, 0)$.
- $\rho(s) \equiv s$ near $-\varepsilon_l$.

Then we define a smooth map $\varphi : W_l \times [0,1] \to W_l; (x,t) \mapsto \varphi_t(x)$ such that:

- If $x \notin \operatorname{Im}\tau_l$, $\varphi_t(x) = x$.
- If $x = \tau_l(y,s)$, $\varphi_t(x) = \tau_l(y, (1-t)s + t\rho(s))$.

It is easy to check the following properties of φ:

- $\varphi_0 = \mathrm{id}_{W_l}$, $\varphi_1(W_l) = \bar{V}$.
- For any $0 \leq t \leq 1$, $\varphi_t(W_l \setminus V) \subset W_l \setminus V$, $\varphi_t(\bar{V}) = \bar{V}$.
- For any tangent vector ξ on W_l and $0 \leq t \leq 1$, $|d\varphi_t(\xi)|_{g^l} \leq (b/b')|\xi|_{g^l}$.

We define $\alpha_i^t : \Delta^k \to \Lambda(W_l)$ by $\alpha_i^t(p) := \varphi_t \circ \alpha_i(p) \quad (\forall p \in \Delta^k)$. By the last property of φ, $\alpha_i^t(\Delta^k) \subset \Lambda_l^{<b}(W_l)$. Moreover, $\alpha^t := \sum_i c_i\alpha_i^t$ satisfies the following properties:

- $\alpha^0 = \alpha$.

- $\alpha^t \in C_k(\Lambda_l^{<b}(W_l))$, $\partial\alpha^t \in C_{k-1}(\Lambda_l^{<b}(W_l) \setminus \Lambda(V))$ for any $t \in [0,1]$.
- $\alpha^1 \in C_k(\Lambda_l^{<b}(\bar{V}))$, $\partial\alpha^1 \in C_{k-1}(\Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V))$.

Thus we obtain $[\alpha] = [\alpha^0] = [\alpha^1] \in \operatorname{Im} I$. Hence we have proved surjectivity of I.

We prove injectivity of I. Let $\alpha = \sum_i c_i\alpha_i \in C_k(\Lambda_l^{<b}(\bar{V}))$ such that $\partial\alpha \in C_{k-1}(\Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V))$. We show that if $I([\alpha]) = 0$ then $[\alpha] = 0$. By $I([\alpha]) = 0$, there exists $\beta = \sum_j d_j\beta_j \in C_{k+1}(\Lambda_l^{<b}(W_l))$ such that $\partial\beta - \alpha \in C_k(\Lambda_l^{<b}(W_l) \setminus \Lambda(V))$. Since Δ^k, Δ^{k+1} are compact, there exists $b' < b$ such that

$$\text{length of } \alpha_i(p),\ \beta_j(q) \text{ with respect to } g^l < b' \ (\forall i,\ \forall j,\ \forall p \in \Delta^k,\ \forall q \in \Delta^{k+1}).$$

Taking $\varphi : W_l \times [0,1] \to W_l$ as before, we set

$$\alpha_i^t := \varphi_t \circ \alpha_i, \quad \alpha^t := \sum_i c_i\alpha_i^t, \quad \beta_j^t := \varphi_t \circ \beta_j, \quad \beta^t := \sum_j d_j\beta_j^t.$$

Then, it is easy to confirm the following claims:

- For any $0 \le t \le 1$, $\alpha^t \in C_k(\Lambda_l^{<b}(\bar{V}))$, $\partial\alpha^t \in C_{k-1}(\Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V))$.
- $\beta^1 \in C_{k+1}(\Lambda_l^{<b}(\bar{V}))$.
- $\partial\beta^1 - \alpha^1 \in C_k(\Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V))$.

Thus we obtain $[\alpha] = [\alpha^1] = [\partial\beta^1] = 0$ in $H_k(\Lambda_l^{<b}(\bar{V}), \Lambda_l^{<b}(\bar{V}) \setminus \Lambda(V))$. Hence we have proved injectivity of I. □

5. C^0-estimates

The goal of this section is to prove Lemmas on C^0-estimates for Floer trajectories: Lemma 2.3, 2.4, 4.8, 4.9, 4.10. Our arguments in this section are based on techniques in [2] and [7].

5.1. **$W^{1,2}$-estimate.** The goal of this subsection is to prove the following $W^{1,2}$- estimate. In the following statement, an expression "$c_0(H,M)$" means that c_0 is a constant which depends on H and M.

Proposition 5.1. *For any $H \in C^\infty(\mathbb{R} \times S^1 \times T^*\mathbb{R}^n)$ satisfying (HH2), (HH3) and $M > 0$, there exists a constant $c_0(H,M) > 0$ which satisfies the following property:*

> *Let $I \subset \mathbb{R}$ be a closed interval of length ≤ 3, and $(J_{s,t})_{(s,t)\in I\times S^1}$ be a $I \times S^1$ -family of almost complex structures on $T^*\mathbb{R}^n$, such that every $J_{s,t}$ is compatible with ω_n. Suppose that there holds*
>
> $$\frac{|\xi|^2}{2} \le \omega_n(\xi, J_{s,t}(\xi)) \le 2|\xi|^2$$
>
> *for any $s \in I$, $t \in S^1$ and tangent vector ξ on $T^*\mathbb{R}^n$. Then, for any $W^{1,3}$-map $u : I \times S^1 \to T^*\mathbb{R}^n$ which satisfies*
>
> $$\partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0, \qquad \sup_{s\in I} \left|\mathscr{A}_{H_s}(u(s))\right| \le M,$$

there holds $\|u\|_{W^{1,2}(I\times S^1)} \le c_0$.

Remark 5.2. $H_s \in C^\infty(S^1 \times T^*\mathbb{R}^n)$ is defined as $H_s(t,q,p) := H(s,t,q,p)$.

A crucial step is the following lemma.

Lemma 5.3. *Let H and I be as in Proposition 5.1. Then, there exists a constant $c_1(H) > 0$ such that: for any $x \in C^\infty(S^1, T^*\mathbb{R}^n)$ and $s \in I$, there holds*

$$\|x\|_{L^2}^2 + \|\partial_t x\|_{L^2}^2 \le c_1\left(1 + \int_{S^1} |\partial_t x - X_{H_{s,t}}(x(t))|^2 + \partial_s H_{s,t}(x(t))\, dt\right).$$

Proof. Let us take $c_2(H)$ so that $c_2 > \sup_{s,t} \|\partial_s \Delta_{s,t}\|_{C^0}$ (recall that $\Delta_{s,t}$ was defined in (HH3)). Then we show that there exists a constant $c_3(H) > 0$ such that there holds

$$\|x\|_{L^2}^2 \le c_3\left(c_2 + \int_{S^1} |\partial_t x - X_{H_{s,t}}(x(t))|^2 + \partial_s H_{s,t}(x(t))\, dt\right) \tag{2}$$

for any $x \in C^\infty(S^1, T^*\mathbb{R}^n)$ and $s \in I$. Suppose that this does not hold. Then, there exists a sequence $(x_k)_k$ and $(s_k)_k$ such that

$$\frac{\|x_k\|_{L^2}^2}{c_2 + \int_{S^1} |\partial_t x_k - X_{H_{s_k,t}}(x_k(t))|^2 + \partial_s H_{s_k,t}(x_k(t))\, dt} \to \infty \qquad (k \to \infty). \tag{3}$$

Since $c_2 + \partial_s H_{s,t}(q,p) > 0$ for any (s,t,q,p), there also holds

$$\frac{\|\partial_t x_k - X_{H_{s_k}}(x_k)\|_{L^2}}{\|x_k\|_{L^2}} \to 0 \qquad (k \to \infty).$$

Let us set $m_k := \|x_k\|_{L^2}$, and $v_k := x_k/m_k$. Then, obviously $\|v_k\|_{L^2} = 1$. We show that $(v_k)_k$ is $W^{1,2}$-bounded, i.e. $(\partial_t v_k)_k$ is L^2-bounded. To show this, we set $h^k(t,q,p) := H_{s_k,t}(m_k q, m_k p)/{m_k}^2$, and consider the inequality

$$\|\partial_t v_k\|_{L^2} \le \|\partial_t v_k - X_{h^k}(v_k)\|_{L^2} + \|X_{h^k}(v_k)\|_{L^2}.$$

$\|\partial_t v_k - X_{h^k}(v_k)\|_{L^2}$ is bounded in k, since

$$\|\partial_t v_k - X_{h^k}(v_k)\|_{L^2} = \frac{\|\partial_t x_k - X_{H_{s_k}}(x_k)\|_{L^2}}{m_k} \to 0 \qquad (k \to \infty). \tag{4}$$

To bound $\|X_{h^k}(v_k)\|_{L^2}$, we use the inequality

$$\|X_{Q^{a(s_k)}}(v_k) - X_{h^k}(v_k)\|_{L^2} \le \|X_{Q^{a(s_k)}}(v_k) - X_{h^k}(v_k)\|_{C^0} \le \frac{\sup_{t\in S^1} \|\Delta_{s_k,t}\|_{C^1}}{m_k}. \tag{5}$$

Then, it is easy to see that there exists $c_4(H) > 0$ such that $\|X_{h^k}(v_k)\|_{L^2} \le c_4(1 + \|v_k\|_{L^2})$. Thus we have proved that $(v_k)_k$ is $W^{1,2}$-bounded.

By taking a subsequence of $(v_k)_k$, we may assume that there exists $v \in W^{1,2}(S^1, T^*\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|v - v_k\|_{C^0} = 0$, and $\partial_t v_k$ converges to $\partial_t v$ weakly in L^2. Moreover, we may assume that $(s_k)_k$ converges to $s \in I$.

We show that $\lim_{k\to\infty} \|X_{Q^{a(s)}}(v) - X_{h^k}(v_k)\|_{L^2} = 0$. By the triangle inequality,

$$\|X_{Q^{a(s)}}(v) - X_{h^k}(v_k)\|_{L^2} \le \|X_{Q^{a(s)}}(v) - X_{Q^{a(s)}}(v_k)\|_{L^2} + \|X_{Q^{a(s)}}(v_k) - X_{h^k}(v_k)\|_{L^2}.$$

Then, $\lim_{k\to\infty} \|X_{Q^{a(s)}}(v) - X_{Q^{a(s)}}(v_k)\|_{L^2} = 0$ since $\lim_{k\to\infty} \|v - v_k\|_{L^2} = 0$. On the other hand, (5) shows that $\lim_{k\to\infty} \|X_{Q^{a(s)}}(v_k) - X_{h^k}(v_k)\|_{L^2} = 0$.

Now we show that $\partial_t v - X_{Q^{a(s)}}(v) = 0$ in $L^2(S^1, T^*\mathbb{R}^n)$, i.e.

$$\langle \partial_t v - X_{Q^{a(s)}}(v), \xi\rangle_{L^2} = 0$$

for any $\xi \in C^\infty(S^1, T^*\mathbb{R}^n)$. This follows from

$$\langle \partial_t v - X_{Q^{a(s)}}(v), \xi\rangle_{L^2} = \lim_{k\to\infty} \langle \partial_t v_k - X_{h^k}(v_k), \xi\rangle_{L^2} = 0.$$

The first equality holds since in $L^2(S^1, T^*\mathbb{R}^n)$

$\partial_t v_k$ converges to $\partial_t v$ (weakly), $\quad X_{h^k}(v_k)$ converges to $X_{Q^{a(s)}}(v)$ (in norm).

The second equality follows from (4).

Now we have shown that $\partial_t v - X_{Q^{a(s)}}(v) = 0$ in $L^2(S^1, T^*\mathbb{R}^n)$. Therefore, by a boot strapping argument, we conclude that $v \in C^\infty(S^1, T^*\mathbb{R}^n)$. This implies that $a(s) \in \pi\mathbb{Z}$, hence $a'(s) > 0$ by (HH3). Hence we obtain

$$\begin{aligned} \frac{m_k^2}{c_2 + \int_{S^1} \partial_s H_{s_k,t}(x_k(t))\,dt} &\le \frac{m_k^2}{\int_{S^1} Q^{a'(s_k)}(x_k(t))\,dt} \\ &= \frac{1}{\int_{S^1} Q^{a'(s_k)}(v_k(t))\,dt} \to \frac{1}{a'(s)\|v\|_{L^2}^2} \quad (k\to\infty). \end{aligned}$$

However, this contradicts the assumption that $(x_k)_k$ satisfies (3). Hence we have proved (2). Setting $c_5 := \max\{c_2c_3, c_3\}$, there holds

$$\|x\|_{L^2}^2 \le c_5\left(1 + \int_{S^1} |\partial_t x - X_{H_{s,t}}(x(t))|^2 + \partial_s H_{s,t}(x(t))\,dt\right) \tag{6}$$

for any $x \in C^\infty(S^1, T^*\mathbb{R}^n)$ and $s \in I$. Now, it is enough to show that there exists $c_6(H) > 0$ such that

$$\|\partial_t x\|_{L^2}^2 \le c_6\left(1 + \int_{S^1} |\partial_t x - X_{H_{s,t}}(x(t))|^2 + \partial_s H_{s,t}(x(t))\,dt\right). \tag{7}$$

By using

$$\begin{aligned} \|\partial_t x\|_{L^2} &\le \|\partial_t x - X_{H_s}(x)\|_{L^2} + \|X_{H_s}(x)\|_{L^2} \\ &\le \|\partial_t x - X_{H_s}(x)\|_{L^2} + 2a(s)\|x\|_{L^2} + \sup_{s,t} \|\Delta_{s,t}\|_{C^1}, \end{aligned}$$

(7) follows easily from (6). □

Now we can prove Proposition 5.1.

Proof of Proposition 5.1. Suppose that $u \in W^{1,3}(I \times S^1, T^*\mathbb{R}^n)$ satisfies

$$\partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0, \qquad \sup_{s\in I} \left|\mathscr{A}_{H_s}(u(s))\right| \le M.$$

By elliptic regularity, u is C^∞ on $\mathrm{int} I \times S^1$. By the assumption on $J_{s,t}$, it is easy to see that

$$|J_{s,t}\partial_s u|^2 \le 4|\partial_s u|^2, \qquad |\partial_s u|^2 \le 2\omega_n(\partial_s u, J_{s,t}\partial_s u).$$

By Lemma 5.3, the following inequality holds for any $s \in \mathrm{int} I$:

$$\|u(s)\|_{L^2}^2 + \|\partial_t u(s)\|_{L^2}^2 \le c_1 \biggl(1 + \int_{S^1} 4|\partial_s u(s,t)|^2 + \partial_s H_{s,t}(u(s,t))\, dt \biggr).$$

The RHS is bounded by

$$\begin{aligned}\int_{S^1} 4|\partial_s u(s,t)|^2 + \partial_s H_{s,t}(u(s,t))\, dt &\le \int_{S^1} 8\omega_n(\partial_s u, J_{s,t}\partial_s u) + \partial_s H_{s,t}(u(s,t))\, dt \\ &\le -8\partial_s\bigl(\mathscr{A}_{H_s}(u(s))\bigr).\end{aligned}$$

By similar arguments, it is easy to show that

$$\int_{S^1} |\partial_s u(s,t)|^2\, dt \le -2\partial_s\bigl(\mathscr{A}_{H_s}(u(s))\bigr).$$

Therefore

$$\begin{aligned}&\int_I \|u(s)\|_{L^2}^2 + \|\partial_t u(s)\|_{L^2}^2\, ds \le c_1 \int_I 1 - 8\partial_s(\mathscr{A}_{H_s}(u(s)))\, ds \le c_1(3+16M), \\ &\int_I \|\partial_s u(s)\|_{L^2}^2\, ds \le \int_I -2\partial_s(\mathscr{A}_{H_s}(u(s)))\, ds \le 4M.\end{aligned}$$

Thus we get

$$\int_{I\times S^1} |u(s,t)|^2 + |\partial_t u(s,t)|^2 + |\partial_s u(s,t)|^2\, dsdt \le 3c_1 + (16c_1+4)M.$$

This concludes the proof of Proposition 5.1. □

5.2. **Proof of Lemma 2.3, 2.4.** First notice that Lemma 2.3 is a special case of Lemma 2.4. Hence it is enough to prove Lemma 2.4. First we need the following lemma:

Lemma 5.4. *Suppose that $H \in C^\infty(\mathbb{R}\times S^1 \times T^*\mathbb{R}^n)$ is a homotopy from H^- to H^+. Then, there exists $M > 0$ which depends only on H such that $\bigl|\mathscr{A}_{H_s}(u(s))\bigr| \le M$ for any $s \in \mathbb{R}$ and $u \in \mathscr{M}_{H,J}(x_-, x_+)$, where $x_- \in \mathscr{P}(H^-)$, $x_+ \in \mathscr{P}(H^+)$.*

***Proof*.** Since $\mathscr{P}(H^-)$ and $\mathscr{P}(H^+)$ are finite sets, there exists $M > 0$ such that

$$\mathscr{A}_{H^-}(x), \mathscr{A}_{H^+}(y) \in [-M, M] \qquad \bigl(\forall x \in \mathscr{P}(H^-),\, \forall y \in \mathscr{P}(H^+)\bigr).$$

Since $\mathscr{A}_{H_s}(u(s))$ is decreasing on s, $\mathscr{A}_{H_s}(u(s)) \in [-M, M]$ for any $u \in \mathscr{M}_{H,J}(x_-, x_+)$. □

Now we prove Lemma 2.4. In the course of the proof, constants which we do not need to be specified are denoted as "const".

***Proof of Lemma 2.4*.** To estimate $\|u\|_{C^0}$, it is enough to bound $\|u|_{[j,j+1]\times S^1}\|_{C^0}$ for each integer j. Take a cut-off function χ so that

$$\mathrm{supp}\chi \subset (-1,2), \quad \chi|_{[0,1]} \equiv 1, \quad 0 \le \chi \le 1, \quad -2 \le \chi' \le 2.$$

Setting $v_j(s,t) := \chi(s-j)u(s,t)$, it is enough to bound $\|v_j\|_{C^0}$ (in the following, we omit the subscript j). First notice that

$$\|v\|_{C^0} \le \mathrm{const}\|v\|_{W^{1,3}} \le \mathrm{const}\|\nabla v\|_{L^3},$$

where the first inequality is a Sobolev estimate, and the second one is Poincaré inequality. By the Calderon-Zygmund inequality, there exists $c > 0$ such that

$$\|\nabla v\|_{L^3} \le c\big(\|(\partial_s - J_{\mathrm{std}}\partial_t)v\|_{L^3} + \|v\|_{L^3}\big).$$

We claim that $\varepsilon := 1/2c$ satisfies the requirement in Lemma 2.4. Suppose that $\sup_{s,t} \|J_{\mathrm{std}} - J_{s,t}\|_{C^0} \le 1/2c$. Then

$$\begin{aligned} c\|(\partial_s - J_{\mathrm{std}}\partial_t)v\|_{L^3} &\le c\big(\|J_{\mathrm{std}} - J_{s,t}\|_{C^0}\|\partial_t v\|_{L^3} + \|(\partial_s - J_{s,t}\partial_t)v\|_{L^3}\big) \\ &\le \|\nabla v\|_{L^3}/2 + c\|(\partial_s - J_{s,t}\partial_t)v\|_{L^3}. \end{aligned}$$

Hence we obtain

$$\|\nabla v\|_{L^3} \le 2c\big(\|v\|_{L^3} + \|(\partial_s - J_{s,t}\partial_t)v\|_{L^3}\big).$$

Since $v(s,t) = \chi(s-j)u(s,t)$, it is clear that $\|v\|_{L^3} \le \|u\|_{L^3([j-1,j+2]\times S^1)}$. On the other hand, since

$$(\partial_s - J_{s,t}\partial_t)v(s,t) = \chi'(s-j)u(s,t) + \chi(s-j)J_{s,t}(u)X_{H_{s,t}}(u),$$

and H satisfies (HH3), it is easy to see

$$\|(\partial_s - J_{s,t}\partial_t)v\|_{L^3} \le \mathrm{const}(1 + \|u\|_{L^3([j-1,j+2]\times S^1)}).$$

Then we conclude that

$$\|\nabla v\|_{L^3} \le \mathrm{const}(1 + \|u\|_{L^3([j-1,j+2]\times S^1)}) \le \mathrm{const}(1 + \|u\|_{W^{1,2}([j-1,j+2]\times S^1)}).$$

Then, Lemma 5.4 and Proposition 5.1 shows that the RHS is bounded. □

5.3. **Proof of Lemma 4.8, 4.9, 4.10.** These lemmas are consequences of the following proposition:

Proposition 5.5. *There exists a constant $\varepsilon > 0$ which satisfies the following property:*

> *Suppose we are given the following data:*
> - $H \in C^\infty(\mathbb{R} \times S^1 \times T^*\mathbb{R}^n)$ *which satisfies (HH2), (HH3).*
> - $J = (J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ *which satisfies (JJ2) and* $\sup_{(s,t)} \|J_{s,t} - J_{\mathrm{std}}\|_{C^0} < \varepsilon$.
> - *Constants* $M_0, M_1 > 0$.
>
> *Then, there exists a constant $c(H, M_0, M_1) > 0$ such that, for any $\sigma \in \mathbb{R}$ and $u \in W^{1,3}([\sigma,\infty) \times S^1, T^*\mathbb{R}^n)$ satisfying*
>
> $$\partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0, \quad \sup_{s\ge\sigma} \big|\mathscr{A}_{H_s}(u(s))\big| \le M_0,$$
> $$\|\pi(u(\sigma))\|_{W^{2/3,3}(S^1,\mathbb{R}^n)} \le M_1,$$
>
> *there holds $\|u\|_{C^0} \le c(H, M_0, M_1)$.*

In this subsection, we deduce Lemmas 4.8, 4.9, 4.10 from Proposition 5.5. First notice that Lemma 4.8 is a special case of Lemma 4.10. Hence it is enough to prove Lemma 4.9 and Lemma 4.10.

Proof of Lemma 4.9. Since $\mathscr{P}(L^m)$ and $\mathscr{P}(H^{m+1})$ are finite sets, there exists $M > 0$ such that

$$\mathscr{S}_{L^m}(\gamma),\ \mathscr{A}_{H^{m+1}}(x) \in [-M, M]$$

for any $\gamma \in \mathscr{P}(L^m)$, $x \in \mathscr{P}(H^{m+1})$. For any $(\alpha, u, v) \in \mathscr{N}^0(\gamma, x)$, there holds

$$\mathscr{S}_{L^m}(\gamma) \geq \mathscr{S}_{L^m}(u(0)) \geq \mathscr{S}_{L^{m+1}}(u(0)) \\ \geq \mathscr{S}_{L^{m+1}}(u(\alpha)) \geq \mathscr{A}_{H^{m+1}}(v(0)) \geq \mathscr{A}_{H^{m+1}}(x).$$

In particular, $\mathscr{S}_{L^{m+1}}(u(\alpha))$ is bounded from below. Now we use the following lemma:

Lemma 5.6. *For any $\gamma \in \mathscr{P}(L^m)$ and $d \in \mathbb{R}$, $\varphi^{X^{m+1}}_{[0,\infty)}(W^u(\gamma : X^m)) \cap \{\mathscr{S}_{L^{m+1}} \geq d\}$ is precompact in $\Lambda(\mathbb{R}^n)$.*

Proof. This lemma is an immediate consequence of Proposition 2.2, Corollary 2.3 in [1]. Let $(\gamma_k, t_k)_{k \geq 1}$ be a sequence, where $\gamma_k \in W^u(\gamma : X^m)$ and $t_k \geq 0$, such that, with $\gamma'_k := \varphi^{X^{m+1}}_{t_k}(\gamma_k)$, $\mathscr{S}_{L^{m+1}}(\gamma'_k) \geq d$. Since $\mathscr{S}_{L^m}(\gamma_k) \geq \mathscr{S}_{L^{m+1}}(\gamma'_k) \geq d$, Corollary 2.3 in [1] shows that $(\gamma_k)_k$ has a convergent subsequence. Then, Proposition 2.2 (2) in [1] implies the conclusion. □

Since $\mathscr{S}_{L^{m+1}}(u(\alpha))$ is bounded from below for any $(\alpha, u, v) \in \mathscr{N}^0(\gamma, x)$, Lemma 5.6 shows that $\|u(\alpha)\|_{W^{1,2}}$ is bounded for any (α, u, v). Therefore,

$$\|\pi(v(0))\|_{W^{2/3,3}} \leq \text{const}\|\pi(v(0))\|_{W^{1,2}} = \text{const}\|u(\alpha)\|_{W^{1,2}}$$

is bounded from above (the first inequality is a Sobolev estimate). On the other hand $\sup_{s \geq 0} \left|\mathscr{A}_{H^{m+1}}(v(s))\right| \leq M$. Hence Proposition 5.5 shows that $\|v\|_{C^0}$ is bounded. □

Proof of Lemma 4.10. Suppose that $(\beta, w) \in \mathscr{N}^1(\gamma, x)$. Then, there holds

$$\mathscr{S}_{L^m}(\gamma) \geq \mathscr{S}_{L^m}(\pi(w(\beta))) \geq \mathscr{A}_{H^m}(w(\beta)) \geq \mathscr{A}_{H_\beta}(w(\beta)) \geq \mathscr{A}_{H^{m+1}}(x).$$

Then, $\sup_{s \geq \beta} \left|\mathscr{A}_{H_s}(w(s))\right|$ is bounded. Moreover, since $\mathscr{S}_{L^m}(\pi(w(\beta)))$ is bounded from below, $\|\pi(w(\beta))\|_{W^{1,2}}$ is bounded. Hence $\|w\|_{C^0}$ is bounded. □

5.4. **Proof of Proposition 5.5.** Finally we prove Proposition 5.5.

It is enough to bound $\sup_{(s-\sigma,t) \in [j,j+1] \times S^1} |u(s,t)|$ for each integer $j \geq 0$. The proof for $j \geq 1$ is as the proof of Lemma 2.4. Hence we only consider the case $j = 0$. We denote the q-component and p-component of u by u_q, u_p, i.e. $u(s,t) = \big(u_q(s,t), u_p(s,t)\big)$.

By the theory of Sobolev traces, there exists $\tilde{u}_q(s,t) \in W^{1,3}([\sigma, \infty) \times S^1 : \mathbb{R}^n)$ such that $\tilde{u}_q(\sigma, t) = u_q(\sigma, t)$ for any $t \in S^1$, and there holds

$$\|\tilde{u}_q\|_{W^{1,3}([\sigma,\infty) \times S^1 : \mathbb{R}^n)} \leq \text{const}\|u_q(\sigma)\|_{W^{2/3,3}(S^1 : \mathbb{R}^n)}.$$

Take a cut-off function $\chi \in C^\infty([0,\infty))$ such that

$$\text{supp}\chi \subset [0,2), \qquad \chi|_{[0,1]} \equiv 1, \qquad 0 \leq \chi \leq 1, \qquad -2 \leq \chi' \leq 0.$$

We set $w(s,t) := \chi(s-\sigma)(u_q(s,t) - \tilde{u}_q(s,t), u_p(s,t))$. Since

$$\|u\|_{C^0([\sigma,\sigma+1] \times S^1)} \leq \|w\|_{C^0} + \|\tilde{u}_q\|_{C^0([\sigma,\sigma+1] \times S^1)} \leq \|w\|_{C^0} + \text{const}\|\tilde{u}_q\|_{W^{1,3}},$$

it is enough to bound $\|w\|_{C^0}$. It is easy to see that

$$\|w\|_{C^0} \leq \text{const}\|w\|_{W^{1,3}} \leq \text{const}\|\nabla w\|_{L^3}$$

by the Sobolev estimate and the Poincaré inequality. Since $w_q(\sigma,t)=(0,\dots,0)$, we can use Calderon-Zygmund inequality to obtain

$$\|\nabla w\|_{L^3}\le c\big(\|(\partial_s-J_{\mathrm{std}}\partial_t)w\|_{L^3}+\|w\|_{L^3}\big).$$

We claim that $\varepsilon:=1/2c$ satisfies the requirement in Proposition 5.5.

If $\sup_{s,t}\|J_{s,t}-J_{\mathrm{std}}\|_{C^0}\le 1/2c$, there holds

$$\|\nabla w\|_{L^3}\le 2c\big(\|w\|_{L^3}+\|(\partial_s-J_{s,t}\partial_t)w\|_{L^3}\big).$$

We divide $(\partial_s-J_{s,t}\partial_t)w$ into two parts:

$$(\partial_s-J_{s,t}\partial_t)w=\chi'(s-\sigma)(u_q-\tilde u_q,u_p)+\chi(s-\sigma)(\partial_s-J_{s,t}\partial_t)(u_q-\tilde u_q,u_p).$$

We bound the first and second term on the RHS:

$$\begin{aligned}\|\text{first term}\|_{L^3}&\le \mathrm{const}(\|u\|_{L^3([\sigma,\sigma+2]\times S^1)}+\|\tilde u_q\|_{L^3}),\\ \|\text{second term}\|_{L^3}&\le \mathrm{const}(\|J_{s,t}(u)X_{H_{s,t}}(u)\|_{L^3([\sigma,\sigma+2]\times S^1)}+\|\tilde u_q\|_{W^{1,3}})\\ &\le \mathrm{const}(1+\|u\|_{L^3([\sigma,\sigma+2]\times S^1)}+\|\tilde u_q\|_{W^{1,3}}).\end{aligned}$$

Hence $\|\nabla w\|_{L^3}$ is bounded by

$$\begin{aligned}\|\nabla w\|_{L^3}&\le \mathrm{const}\big(1+\|u\|_{L^3([\sigma,\sigma+2]\times S^1)}+\|\tilde u_q\|_{W^{1,3}}\big)\\ &\le \mathrm{const}\big(1+\|u\|_{W^{1,2}([\sigma,\sigma+2]\times S^1)}+\|u_q(\sigma)\|_{W^{2/3,3}(S^1)}\big).\end{aligned}$$

Since $\sup_{s\ge\sigma}\big|\mathscr{A}_{H_s}(u(s))\big|$ is bounded by assumption, Proposition 5.1 shows that $\|u\|_{W^{1,2}([\sigma,\sigma+2]\times S^1)}$ is bounded. On the other hand, $\|u_q(\sigma)\|_{W^{2/3,3}(S^1)}$ is bounded by assumption. Hence the RHS is bounded. ☐

6. Floer-Hofer-Wysocki capacity and periodic billiard trajectory

The goal of this section is to prove Proposition 1.2 and Corollary 1.4.

6.1. **Symplectic homology of RCT-domains.**

In this subsection, we collect some results on symplectic homology of RCT (restricted contact type) domains, which are essentially established in [10].

Definition 6.1. *Let U be a bounded domain in $T^*\mathbb{R}^n$ with a smooth boundary. U is called RCT (restricted contact type), when there exists a vector field Z on $T^*\mathbb{R}^n$ such that $L_Z\omega_n=\omega_n$ and Z points strictly outwards on ∂U.*

Let U be an RCT-domain in $T^*\mathbb{R}^n$. Then, $\mathscr{R}_{\partial U}:=\ker(\omega|_{\partial U})$ is a 1-dimensional foliation on ∂U, which is called *characteristic foliation*. $\mathscr{R}_{\partial U}$ has a canonical orientation: for any $p\in\partial U$, $\xi\in\mathscr{R}_{\partial U}(p)$ is positive if and only if $\omega_n(Z(p),\xi)>0$. $\mathscr{P}_{\partial U}$ denotes the set of m-fold coverings of closed leaves of $\mathscr{R}_{\partial U}$, where $m\ge 1$.

For each $\gamma\in\mathscr{P}_{\partial U}$, $\mathscr{A}(\gamma):=\int_\gamma i_Z\omega_n$ is called the *action* of γ. By our definition of orientation of $\mathscr{R}_{\partial U}$, $\mathscr{A}(\gamma)>0$ for any $\gamma\in\mathscr{P}_{\partial U}$.

One can also define the Conley-Zehnder index $\operatorname{ind}_{\mathrm{CZ}}(\gamma)$ for any $\gamma \in \mathscr{P}_{\partial U}$, even when γ is degenerate. For details, see Section 3.2 in [11]. For each integer k, we set

$$\Sigma_k(\partial U) := \{\mathscr{A}(\gamma) \mid \gamma \in \mathscr{P}_{\partial U}, \operatorname{ind}_{\mathrm{CZ}}(\gamma) \le k\}, \qquad \Sigma(\partial U) := \bigcup_{k \in \mathbb{Z}} \Sigma_k(\partial U).$$

$\Sigma(\partial U) \subset \mathbb{R}$ is called the *action spectrum*.

Lemma 6.2. *For any RCT-domain U in $T^*\mathbb{R}^n$, the following statements hold:*

1. *For any $0 < a < \min \Sigma_{n+1}(\partial U)$, $\mathrm{SH}_n^{[-1,a)}(U) \cong \mathbb{Z}_2$.*
2. *Let V be another RCT-domain in $T^*\mathbb{R}^n$ such that $V \subset U$. Then, for any a satisfying*
$$0 < a < \min \Sigma_{n+1}(\partial U), \min \Sigma_{n+1}(\partial V),$$
the natural homomorphism $\mathrm{SH}_n^{[-1,a)}(U) \to \mathrm{SH}_n^{[-1,a)}(V)$ is an isomorphism.
3. *For any $0 < \varepsilon < \min \Sigma_{n+1}(\partial U)$,*
$$c_{\mathrm{FHW}}(U) = \inf\{a \mid \mathrm{SH}_n^{[-1,\varepsilon)}(U) \to \mathrm{SH}_n^{[-1,a)}(U) \text{ vanishes}\}.$$
4. *$c_{\mathrm{FHW}}(U) \in \Sigma_{n+1}(\partial U)$.*

Proof. In Proposition 4.7 in [10], the following statement is proved:

> Let U be an RCT-domain, and $0 < a < \min \Sigma(\partial U)$. Then, $\mathrm{SH}_*^{[-1,a)}(U) \cong H_{n+*}(U, \partial U)$.

(1) in our Lemma 6.2 can be proved in the same way as this statement in [10], although our assumption $a < \min \Sigma_{n+1}(\partial U)$ is weaker. (2) also follows directly from the proof of Proposition 4.7 in [10]. For details, see [10] pp.360 – 361. (3) is Proposition 5.7 in [10]. (4) is proved in exactly the same way as Theorem 8 in [11]. □

6.2. **Periodic billiard trajectory.** The goal of this subsection is to prove Proposition 1.2. Throughout this subsection, V denotes a bounded domain in $\mathbb{R}^n$ with smooth boundary. First we clarify the definition of periodic billiard trajectory.

Definition 6.3. A continuous map $\gamma : \mathbb{R}/T\mathbb{Z} \to \bar{V}$ is called a *periodic billiard trajecotory* if there exists a finite set $\mathscr{B} \subset \mathbb{R}/T\mathbb{Z}$ such that the following holds:

- On $(\mathbb{R}/T\mathbb{Z}) \setminus \mathscr{B}$, there holds $\ddot{\gamma} \equiv 0$ and $|\dot{\gamma}| \equiv 1$.
- For any $t \in \mathscr{B}$, $\dot{\gamma}_\pm(t) := \lim_{h \to \pm 0} \dot{\gamma}(t+h)$ satisfies the law of reflection:
$$\dot{\gamma}_+(t) + \dot{\gamma}_-(t) \in T_{\gamma(t)}\partial V, \qquad \dot{\gamma}_+(t) - \dot{\gamma}_-(t) \in (T_{\gamma(t)}\partial V)^\perp \setminus \{0\}.$$

Elements of $\mathscr{B}$ are called as *bounce times*, and T is called the *length* of γ.

First we construct a sequence of RCT-domains which approximates D^*V. Fix a positive smooth function $h : V \to \mathbb{R}_{>0}$ and a compactly supported vector field Z on $\mathbb{R}^n$ so that:

- $h(q) = \operatorname{dist}(q, \partial V)^{-2}$ when q is sufficiently close to ∂V.
- Z points strictly outwards on ∂V.
- $dh(Z) \ge 0$ everywhere on V.

- Setting $Z = \sum_j Z_j \partial_{q_j}$, $\sup_{q \in V} |\partial_{q_i} Z_j(q)| \le 1/2n$.

For any $\varepsilon > 0$, we set $H_\varepsilon(q,p) := |p|^2/2 + \varepsilon h(q)$, and $U_\varepsilon := \{H_\varepsilon < 1/2\} \subset D^*V$.

We show that U_ε is an RCT-domain. We define $H_Z \in C^\infty(T^*\mathbb{R}^n)$ by $H_Z(q,p) := p \cdot Z(q)$. We define a vector field $\bar{Z}$ on $T^*\mathbb{R}^n$ by

$$\bar{Z} := \sum_i p_i \partial_{p_i} + X_{H_Z}.$$

It is easy to check that $L_{\bar{Z}} \omega_n = \omega_n$. $(\varphi^{\bar{Z}}_t)_t$ denotes the flow generated by $\bar{Z}$, i.e. $\varphi^{\bar{Z}}_0 = \mathrm{id}_{T^*\mathbb{R}^n}$, and $\partial_t \varphi^{\bar{Z}}_t = \bar{Z}(\varphi^{\bar{Z}}_t)$.

Lemma 6.4. *When $\varepsilon > 0$ is sufficiently small, $dH_\varepsilon(\bar{Z}) > 0$ on $\{H_\varepsilon = 1/2\}$. In particular, $U^-_\varepsilon = \{H_\varepsilon < 1/2\}$ is an RCT-domain. There exists $T_\varepsilon > 0$ such that $\varphi^{\bar{Z}}_{T_\varepsilon}(U^-_\varepsilon) \supset D^*V$. Moreover, we can take T_ε so that $\lim_{\varepsilon \to 0} T_\varepsilon = 0$.*

Proof. By simple computations,

$$\begin{aligned}
\sum_i p_i dp_i(\bar{Z})(q,p) &= |p|^2 - \sum_{i,j} p_i p_j \partial_{q_i} Z_j(q) \\
&\ge |p|^2 - \sum_{i,j} \frac{p_i^2 + p_j^2}{2} \cdot \sup_{q \in V} |\partial_{q_i} Z_j(q)| \ge |p|^2/2, \\
dH_\varepsilon(\bar{Z})(q,p) &= \sum_i p_i dp_i(\bar{Z})(q,p) + \varepsilon dh(Z(q)) \ge |p|^2/2.
\end{aligned}$$

Hence there holds the following claims:

- $dH_\varepsilon(\bar{Z}) > 0$ everywhere on $D^*V \setminus \{p = 0\}$.
- $\sum_i p_i dp_i(\bar{Z}) > 0$ on $\{|p| = 1\}$.
- $\bar{Z}$ points outwards on $\{(q,p) \mid q \in \partial V\}$.

Since Z points outwards on ∂V, for sufficiently small $\varepsilon > 0$, $dh(Z) > 0$ on $\{h = 1/2\varepsilon\}$. Hence the first property implies that $dH_\varepsilon(\bar{Z}) > 0$ on $\{H_\varepsilon = 1/2\}$. By the second and third properties, $\varphi^{\bar{Z}}_{-T}(D^*V) \subset D^*V$ for any $T > 0$. Hence, for sufficiently small $\varepsilon > 0$, there holds $\varphi^{\bar{Z}}_{-T}(D^*V) \subset U^-_\varepsilon$. This means that $D^*V \subset \varphi^{\bar{Z}}_T(U^-_\varepsilon)$. □

By Lemma 6.4, there exist sequences $\varepsilon_1 > \varepsilon_2 > \cdots$, $T_1 > T_2 > \cdots$ such that:

- $U^-_k := \{H_{\varepsilon_k} < 1/2\}$ is an RCT-domain with respect to $\bar{Z}$.
- Setting $U^+_k := \varphi^{\bar{Z}}_{T_k}(U^-_k)$, there holds $U^+_1 \supset U^+_2 \supset \cdots$ and $\bigcap_k U^+_k = \overline{D^*V}$.
- $\lim_{k \to \infty} \varepsilon_k = \lim_{k \to \infty} T_k = 0$.

Since $U_k^+ = \varphi_{T_k}^{\bar{Z}}(U_k^-)$ and $L_{\bar{Z}}\omega_n = \omega_n$, $\lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^+)/c_{\mathrm{FHW}}(U_k^-) = \lim_{k\to\infty} e^{T_k} = 1$. On the other hand, $c_{\mathrm{FHW}}(U_k^-) \le c_{\mathrm{FHW}}(D^*V) \le c_{\mathrm{FHW}}(U_k^+)$. Therefore there holds

$$\lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^-) = \lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^+) = c_{\mathrm{FHW}}(D^*V).$$

Lemma 6.5. *Suppose that there exists a sequence $(\gamma_k)_k$ such that $\gamma_k \in \mathscr{P}_{\partial U_k^-}$, which satisfies $\sup_k \mathrm{ind}_{\mathrm{CZ}}(\gamma_k) \le m$ and $\lim_{k\to\infty} \mathscr{A}(\gamma_k) = a$, where m is an integer and $a \ge 0$. Then, there exists a periodic billiard trajectory with at most m bounce times and length equal to a. In particular, $a > 0$.*

Proof. By our assumption, there exists $\Gamma_k : \mathbb{R}/\tau_k\mathbb{Z} \to \{H_{\varepsilon_k} = 1/2\}$ such that

$$\dot{\Gamma}_k = X_{H_{\varepsilon_k}}(\Gamma_k), \qquad \int_{\Gamma_k} \sum_i p_i dq_i = \mathscr{A}(\gamma_k), \qquad \mathrm{ind}_{\mathrm{CZ}}(\Gamma_k) \le m.$$

For the last estimate, see Lemma 8 in [11]. Let $q_k : \mathbb{R}/\tau_k\mathbb{Z} \to \mathbb{R}^n$ be the q-component of Γ_k. Then, by simple computations

$$\ddot{q}_k + \varepsilon_k \nabla h(q_k) \equiv 0, \qquad \frac{|\dot{q}_k|^2}{2} + \varepsilon_k h(q_k) \equiv 1/2, \qquad \int_0^{\tau_k} |\dot{q}_k|^2 dt = \mathscr{A}(\gamma_k).$$

Moreover, the following identity is well-known (see Theorem 7.3.1 in [12]):

$$\mathrm{ind}_{\mathrm{Morse}}(q_k) = \mathrm{ind}_{\mathrm{CZ}}(\Gamma_k) \le m.$$

To show that $(q_k)_k$ converges to a periodic billiard trajectory on V, first we show that $\liminf_k \tau_k > 0$. If this is not the case, by taking a subsequence, we may assume that $\lim_{k\to\infty} \tau_k = 0$. Then, according to Proposition 2.3 in [4], there exists $q_\infty \in \bar{V}$ such that $(q_k)_k$ converges to the constant loop at q_∞ in C^0-norm. However, this leads to a contradiction by the following arguments:

- Suppose $q_\infty \in V$. Let K be a compact neighborhood of q in V. Then, for sufficiently large k, $\mathrm{Im}\, q_k \subset K$. On the other hand, $\lim_{k\to\infty} \|\varepsilon_k h|_K\|_{C^1} = 0$, and q_k satisfies $\ddot{q}_k + \varepsilon_k \nabla h(q_k) \equiv 0$, $|\dot{q}_k|^2/2 + \varepsilon_k h(q_k) \equiv 1/2$. This is a contradiction.
- Suppose $q_\infty \in \partial V$. Let ν be the inward normal vector of ∂V at q_∞. For any k, there exists $\theta_k \in S^1$ such that $\ddot{q}_k(\theta_k) \cdot \nu \le 0$. On the other hand, for any $x \in V$ sufficiently close to q_∞, there holds $\nabla h(x) \cdot \nu < 0$. This contradicts our assumption that q_k satisfies $\ddot{q}_k + \varepsilon_k \nabla h(q_k) \equiv 0$ for any k.

Thus we have proved $\liminf_k \tau_k > 0$. On the other hand, there also holds $\limsup_k \tau_k < \infty$ by exactly the same arguments as on pp. 3312 in [4] (see also Lemma 15 in [11]).

Since $(\tau_k)_k$ satisfies $0 < \liminf_k \tau_k \le \limsup_k \tau_k < \infty$ and $\mathrm{ind}_{\mathrm{Morse}}(q_k) \le m$, Proposition 2.1 and 2.2 in [4] show that a certain subsequence of $(q_k)_k$ converges to a periodic billiard trajectory on V with at most m bounce times, and length $\lim_{k\to\infty} \mathscr{A}(\gamma_k) = a$. □

Now, the proof of Proposition 1.2 is immediate.

Proof. By Lemma 6.2 (4), there exists $\gamma_k \in \mathscr{P}_{\partial U_k^-}$ such that $\mathscr{A}(\gamma_k) = c_{\mathrm{FHW}}(U_k^-)$ and $\mathrm{ind}_{\mathrm{CZ}}(\gamma_k) \leq n+1$. Since $\lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^-) = c_{\mathrm{FHW}}(D^*V)$, Lemma 6.5 concludes the proof. □

6.3. **Floer-Hofer-Wysocki capacity.** In this subsection, we prove Corollary 1.4. First we need the following lemma:

Lemma 6.6. *For sufficiently small $\varepsilon > 0$, the following holds:*

(1) *For sufficiently large k, the natural homomorphisms*

$$\mathrm{SH}_n^{[-1,\varepsilon)}(U_k^+) \to \mathrm{SH}_n^{[-1,\varepsilon)}(D^*V), \qquad \mathrm{SH}_n^{[-1,\varepsilon)}(D^*V) \to \mathrm{SH}_n^{[-1,\varepsilon)}(U_k^-)$$

are isomorphisms, and the above homology groups are isomorphic to $\mathbb{Z}_2$.

(2) *The natural homomorphism $H_n(\bar{V}, \partial V) \to H_n\big(\Lambda^{<\varepsilon}(\bar{V}), \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)\big)$ is an isomorphism.*

Proof. Lemma 6.5 shows that $\liminf_k \min \Sigma_{n+1}(\partial U_k^-) > 0$. Then, for any $0 < \varepsilon < \liminf_k \min \Sigma_{n+1}(\partial U_k^-)$, (1) follows from Lemma 6.2 (1), (2). In particular, Theorem 1.1 shows that $H_n\big(\Lambda^{<\varepsilon}(\bar{V}), \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)\big) \cong \mathbb{Z}_2$. Hence (2) holds when $H_n(\bar{V}, \partial V) \to H_n\big(\Lambda^{<\varepsilon}(\bar{V}), \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)\big)$ is injective.

Let us fix $p \in V$. Then, if $0 < \varepsilon < 2\mathrm{dist}(p, \partial V)$, any $\gamma \in \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)$ satisfies $\gamma(S^1) \subset \bar{V} \setminus \{p\}$. Hence we get a commutative diagram

$$\begin{array}{ccc} H_n(\bar{V}, \partial V) & \longrightarrow & H_n(\Lambda^{<\varepsilon}(\bar{V}), \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)) \\ & \searrow & \downarrow (\mathrm{ev})_* \\ & & H_n(\bar{V}, \bar{V} \setminus \{p\}) \end{array}$$

where the vertical arrow is induced by the evaluation map $\mathrm{ev} : \gamma \mapsto \gamma(0)$. Since the diagonal map is an isomorphism, the horizontal arrow is injective. □

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. Our goal is to show that $c_{\mathrm{FHW}}(D^*V)$ is equal to

$$b_* := \inf\{b \mid H_n(\bar{V}, \partial V) \to H_n\big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big) \text{ vanishes}\}.$$

Take $\varepsilon > 0$ so that it satisfies the conditions in Lemma 6.6. Then,

$$\begin{aligned} & H_n(\bar{V}, \partial V) \to H_n\big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big) \text{ vanishes} \\ \iff & H_n\big(\Lambda^{<\varepsilon}(\bar{V}), \Lambda^{<\varepsilon}(\bar{V}) \setminus \Lambda(V)\big) \to H_n\big(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)\big) \text{ vanishes} \\ \iff & \mathrm{SH}_n^{[-1,\varepsilon)}(D^*V) \to \mathrm{SH}_n^{[-1,b)}(D^*V) \text{ vanishes.} \end{aligned}$$

The first equivalence follows from Lemma 6.6 (2), and the second equivalence follows from Theorem 1.1. Since $c_{\mathrm{FHW}}(D^*V) = \lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^+) = \lim_{k\to\infty} c_{\mathrm{FHW}}(U_k^-)$, it is enough to show that $c_{\mathrm{FHW}}(U_k^+) \geq b_*$ and $c_{\mathrm{FHW}}(U_k^-) \leq b_*$ for any k. Since U_k^- and U_k^+ are RCT-domains, Lemma 6.2 (3) implies that

$$c_{\mathrm{FHW}}(U_k^{\pm}) = \inf\{b \mid \mathrm{SH}_n^{[-1,\varepsilon)}(U_k^{\pm}) \to \mathrm{SH}_n^{[-1,b)}(U_k^{\pm}) \text{ vanishes}\}.$$

Therefore, $c_{\mathrm{FHW}}(U_k^+) \geq b_*$ follows from the commutativity of

$$\begin{array}{ccc} \mathrm{SH}_n^{[-1,\varepsilon)}(U_k^+) & \longrightarrow & \mathrm{SH}_n^{[-1,b)}(U_k^+) \\ \cong\downarrow & & \downarrow \\ \mathrm{SH}_n^{[-1,\varepsilon)}(D^*V) & \longrightarrow & \mathrm{SH}_n^{[-1,b)}(D^*V). \end{array}$$

On the other hand, $c_{\mathrm{FHW}}(U_k^-) \leq b_*$ follows from the commutativity of

$$\begin{array}{ccc} \mathrm{SH}_n^{[-1,\varepsilon)}(D^*V) & \longrightarrow & \mathrm{SH}_n^{[-1,b)}(D^*V) \\ \cong\downarrow & & \downarrow \\ \mathrm{SH}_n^{[-1,\varepsilon)}(U_k^-) & \longrightarrow & \mathrm{SH}_n^{[-1,b)}(U_k^-). \end{array}$$

□

7. Floer-Hofer-Wysocki capacity and inradius

The goal of this section is to prove Theorem 1.6. First of all, the lower bound $2r(V) \leq c_{\mathrm{FHW}}(D^*V)$ is immediate from the proof of Lemma 6.6 (2). The upper bound $c_{\mathrm{FHW}}(D^*V) \leq 2(n+1)r(V)$ is a consequence of the following lemma:

Lemma 7.1. *For any $b > 2(n+1)r(V)$, there exists a continuous map $C : \bar{V} \times [0,1] \to \Lambda^{<b}(\bar{V})$ which satisfies the following properties:*

- (a): *For any $x \in \bar{V}$, $C(x,0) = c_x :=$ constant loop at x.*
- (b): *Setting $\tilde{V} := \partial V \times [0,1] \cup V \times \{1\}$, $C(\tilde{V}) \subset \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)$.*

Let us check that Lemma 7.1 implies the upper bound. By Corollary 1.4, it is enough to show that

$$(\iota^b)_* : H_n(\bar{V}, \partial V) \to H_n(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V))$$

vanishes when $b > 2(n+1)r(V)$. Take $C : \bar{V} \times [0,1] \to \Lambda^{<b}(\bar{V})$ as in Lemma 7.1. Setting $I : \bar{V} \to \bar{V} \times [0,1]$ by $I(x) := (x,0)$, consider the following diagram:

$$\begin{array}{ccc} H_*(\bar{V}, \partial V) & \xrightarrow{(\iota^b)_*} & H_*(\Lambda^{<b}(\bar{V}), \Lambda^{<b}(\bar{V}) \setminus \Lambda(V)). \\ I_*\downarrow & \nearrow C_* & \\ H_*(\bar{V} \times [0,1], \tilde{V}) & & \end{array}$$

(a) in Lemma 7.1 implies that the above diagram is commutative. It is easy to see that $H_*(\bar{V} \times [0,1], \tilde{V}) = 0$. Therefore $(\iota^b)_* = 0$. This completes the proof of $c_{\mathrm{FHW}}(D^*V) \leq 2(n+1)r(V)$ modulo Lemma 7.1.

Now our task is to prove Lemma 7.1. Since $b > 2(n+1)r(V)$, one can take ρ so that $\rho > 2r(V)$ and $(n+1)\rho < b$. We fix ρ in the following argument.

Lemma 7.2. *For any $p \in \bar{V}$, there exists U_p, a neighborhood of p in $\bar{V}$ and a continuous map $\sigma_p : U_p \times [0,1] \to \Lambda^{<\rho}(\bar{V})$ so that the following holds for any $x \in U_p$:*

- *For any $0 \le t \le 1$, $\sigma_p(x,t)$ maps $0 \in S^1$ to x.*
- *$\sigma_p(x,0) = c_x$.*
- *$\sigma_p(x,1) \notin \Lambda(V)$.*

Proof. Since $\rho/2 > r(V)$, there exists a smooth path $\gamma : [0,1] \to \bar{V}$ such that $\gamma(0) = p$, $\gamma(1) \in \partial V$ and length of γ is less than $\rho/2$. There exists a neighborhood U_p of p and a continuous map $\Gamma : U_p \to W^{1,2}([0,1], \bar{V})$ such that

- $\Gamma(p) = \gamma$.
- For any $x \in U_p$, $\Gamma(x)(0) = x$, $\Gamma(x)(1) \in \partial V$.
- For any $x \in U_p$, length of $\Gamma(x)$ is less than $\rho/2$.

Now define $\sigma_p : U_p \times [0,1] \to \Lambda^{<\rho}(\bar{V})$ by

$$\sigma_p(x,t)(\tau) = \begin{cases} \Gamma(x)(2t\tau) & (0 \le \tau \le 1/2) \\ \Gamma(x)(2t(1-\tau)) & (1/2 \le \tau \le 1) \end{cases}.$$

Then, it is immediate to see that σ_p satisfies the required conditions. □

Lemma 7.3. *Let $(U_p)_{p \in \bar{V}}$ be an open covering of $\bar{V}$ as in Lemma 7.2. Then, there exists $(W_j)_{1 \le j \le m}$, which is a refinement of $(U_p)_{p \in \bar{V}}$ and such that:*

For any $x \in \bar{V}$, the number of j such that $x \in W_j$ is at most $n+1$.

Proof. Actually this lemma is valid for any covering of $\bar{V}$. By Lebesgue's number lemma, one can take $\delta > 0$ so that any subset of $\bar{V}$ with diameter less than δ is contained in some U_p. We fix such δ, and take a (smooth) triangulation Δ of $\bar{V}$ so that every simplex has diameter less than $\delta/2$. For each vertex v of Δ, $\mathrm{Star}(v)$ denotes the union of all open faces of Δ (we include v itself), which contain v in their closures.

Let $v_1, \ldots, v_m$ be vertices of Δ, and set $W_j := \mathrm{Star}(v_j)$ for $j = 1, \ldots, m$. Since each W_j has diameter less than δ, $(W_j)_{1 \le j \le m}$ is a refinement of $(U_p)_{p \in \bar{V}}$. Moreover, if $x \in \bar{V}$ is contained in a k-dimensional open face of Δ, the number of j such that $x \in W_j$ is exactly $k+1$. Hence $(W_j)_{1 \le j \le m}$ satisfies the required condition. □

Remark 7.4. The above proof of Lemma 7.3 is the same as the standard proof of the fact that any n-dimensional polyhedron has Lebesgue covering dimension $\le n$ (see Section 2 in [6]).

Take $(W_j)_{1 \le j \le m}$ as in Lemma 7.3. Since it is a refinement of $(U_p)_{p \in \bar{V}}$, one can define a continuous map $\sigma_j : W_j \times [0,1] \to \Lambda^{<\rho}(\bar{V})$ so that the following holds for any $x \in W_j$:

- For any $0 \le t \le 1$, $\sigma_j(x,t) \in \Lambda^{<\rho}(\bar{V})$ maps $0 \in S^1$ to x.
- $\sigma_j(x,0) = c_x$.
- $\sigma_j(x,1) \notin \Lambda(V)$.

For each $1 \le j \le m$, let us take $\chi_j \in C^0(\bar{V})$ so that $0 \le \chi_j \le 1$, $\mathrm{supp}\chi_j \subset W_j$, and $K_j := \{x \in V \mid \chi_j(x) = 1\}$ satisfies $\bigcup_{1 \le j \le m} K_j = \bar{V}$. We define $\tilde{\sigma}_j : \bar{V} \times [0,1] \to \Lambda^{<\rho}(\bar{V})$ by

$$\tilde{\sigma}_j(x,t) = \begin{cases} c_x & (x \notin W_j) \\ \sigma_j(x, \chi_j(x)t) & (x \in W_j) \end{cases}.$$

Then, it is immediate that $\tilde{\sigma}_j$ satisfies the following properties:

- For any $x \in \bar{V}$ and $0 \le t \le 1$, $\tilde{\sigma}_j(x,t)$ maps $0 \in S^1$ to x.
- For any $x \in \bar{V}$, $\tilde{\sigma}_j(x,0) = c_x$.
- For any $x \in K_j$, $\tilde{\sigma}_j(x,1) \notin \Lambda(V)$.

To finish the proof of Lemma 7.1, we introduce the following notation:

Definition 7.5. For any $\gamma_1, \ldots, \gamma_m \in \Lambda(\bar{V})$ such that $\gamma_1(0) = \cdots = \gamma_m(0)$, we define their *concatenation* $\mathrm{con}(\gamma_1, \ldots, \gamma_m) \in \Lambda(\bar{V})$ by

$$\mathrm{con}(\gamma_1, \ldots, \gamma_m)(t) := \gamma_{j+1}\biggl(m\Bigl(t - \frac{j}{m}\Bigr)\biggr) \ \biggl(\frac{j}{m} \le t \le \frac{j+1}{m},\ j = 0, \ldots, m-1\biggr).$$

Proof of Lemma 7.1. We define $C : \bar{V} \times [0,1] \to \Lambda(\bar{V})$ by

$$C(x,t) := \mathrm{con}\bigl(\tilde{\sigma}_1(x,t), \ldots, \tilde{\sigma}_m(x,t)\bigr).$$

Since $\tilde{\sigma}_1(x,t), \ldots, \tilde{\sigma}_m(x,t)$ maps $0 \in S^1$ to x, the above definition makes sense. We claim that this map C satisfies all requirements in Lemma 7.1.

First we have to check that length of $C(x,t)$ is less than b. Obviously, length of $C(x,t)$ is a sum of the lengths of $\tilde{\sigma}_j(x,t)$ for $j = 1, \ldots, m$. If $x \notin W_j$, $\tilde{\sigma}_j(x,t) = c_x$ by definition. Hence $\tilde{\sigma}_j(x,t)$ has length 0. Moreover, the number of j such that $x \in W_j$ is at most $n+1$, by Lemma 7.3. Hence length of $C(x,t)$ is less than $(n+1)\rho < b$. Finally we verify conditions (a), (b). (a) follows from:

$$C(x,0) = \mathrm{con}\bigl(\tilde{\sigma}_1(x,0), \ldots, \tilde{\sigma}_m(x,0)\bigr) = \mathrm{con}(c_x, \ldots, c_x) = c_x.$$

To verify (b), we have to check the following two claims:

(b-1): For any $x \in \bar{V}$, $C(x,1) \notin \Lambda(V)$.
(b-2): For any $x \in \partial V$ and $0 \le t \le 1$, $C(x,t) \notin \Lambda(V)$.

We check (b-1). Since $(K_j)_{1 \le j \le m}$ is a covering of $\bar{V}$, there exists j such that $x \in K_j$. Then $\tilde{\sigma}_j(x,1) \notin \Lambda(V)$, therefore $C(x,1) \notin \Lambda(V)$. (b-2) is clear since $C(x,t)$ maps $0 \in S^1$ to $x \notin V$. □

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