Mathematische annalen manuscript No.
(will be inserted by the editor)

Ideal-adic semi-continuity problem for minimal log discrepancies

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Received: date / Accepted: date

Abstract We discuss the ideal-adic semi-continuity problem for minimal log discrepancies by Mustaţă. We study the purely log terminal case, and prove the semi-continuity of minimal log discrepancies when a Kawamata log terminal triple deforms in the ideal-adic topology.

Mathematics Subject Classification (2010) Primary 14E18, 14E30 · Secondary 14B07

Introduction

In the minimal model program, singularities are measured in terms of log discrepancies. The log discrepancy is attached to each divisor on an extraction of the singularity, and their infimum is called the *minimal log discrepancy*. Recently, de Fernex, Ein and Mustaţă in [3] after Kollár in [13] proved the ideal-adic semi-continuity of log canonicity effectively to obtain Shokurov's ACC conjecture [20] for log canonical thresholds on l.c.i. varieties. This paper discusses its generalisation to minimal log discrepancies, proposed by Mustaţă.

Conjecture (Mustaţă) *Let (X,Δ) be a pair, Z a closed subset of X and $\mathscr{I}_Z$ its ideal sheaf. Let $\mathfrak{a}$ be an ideal sheaf and r a positive real number. Then there exists an integer l such that the following holds: if an ideal sheaf $\mathfrak{b}$ satisfies $\mathfrak{a}+\mathscr{I}_Z^l=\mathfrak{b}+\mathscr{I}_Z^l$, then*

$$\mathrm{mld}_Z(X,\Delta,\mathfrak{a}^r)=\mathrm{mld}_Z(X,\Delta,\mathfrak{b}^r).$$

Partial support was provided by Grant-in-Aid for Young Scientists (A) 20684002.

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The mld above denotes the minimal log discrepancy. Mustaţă observed that an effective form of the conjecture implies the ACC for minimal log discrepancies on a fixed germ by the argument of generic limits of ideals.

The conjecture is not difficult to prove in the Kawamata log terminal case, stated in Theorem 1.6. It is however inevitable to deal with log canonical singularities in the study of limits. As its first extension, we treat a purely log terminal triple $(X, F + \Delta, \mathfrak{a}^r)$ with a Cartier divisor F and control the minimal log discrepancy of $(X, G + \Delta, \mathfrak{b}^r)$ for $G, \mathfrak{b}$ close to $F, \mathfrak{a}$. Our main theorem compares minimal log discrepancies on F, G rather than those on X. We adopt the weaker condition $\mathfrak{a} \approx_l \mathfrak{b}$ defined by $\mathfrak{a}^n + \mathscr{I}_Z^{nl} = \mathfrak{b}^n + \mathscr{I}_Z^{nl}$ for some n.

Theorem (full form in Theorem 1.9) *Let (X,Δ), Z, $\mathfrak{a}$ and r be as in* Conjecture*. Let F be a reduced Cartier divisor such that $(X, F+\Delta, \mathfrak{a}^r)$ is plt about Z. Then there exists an integer l such that the following holds: if an effective Cartier divisor G and an ideal sheaf $\mathfrak{b}$ satisfy $\mathscr{O}_X(-F) \approx_l \mathscr{O}_X(-G)$ and $\mathfrak{a} \approx_l \mathfrak{b}$, then G is reduced about Z and with the normalisation $\nu\colon G^\nu \to G$,*

$$\mathrm{mld}_{F\cap Z}(F, \Delta_F, \mathfrak{a}^r\mathscr{O}_F) = \mathrm{mld}_{\nu^{-1}(G\cap Z)}(G^\nu, \Delta_{G^\nu}, \mathfrak{b}^r\mathscr{O}_{G^\nu}).$$

The theorem can be regarded as an extension to the case when a variety as well as a boundary deforms, so it might provide a perspective in the study of the behaviour of minimal log discrepancies under deformations. It should be related to Shokurov's reduction [21] of the termination of flips. The equality $\mathrm{mld}_Z(X, F + \Delta, \mathfrak{a}^r) = \mathrm{mld}_Z(X, G+\Delta, \mathfrak{b}^r)$ is recovered if the precise inversion of adjunction in [14] holds on X such as l.c.i. varieties in [7], [8].

We prove the theorem by using motivic integration due to Kontsevich in [16] and Denef and Loeser in [6]. Take a divisor E on an extraction of X whose restriction computes the minimal log discrepancy on G. By the plt assumption, the order of (the inverse image of) the Jacobian $\mathscr{J}'_G$ of G along E should be small in contrast to those of F, G, hence it coincides with that of the Jacobian $\mathscr{J}'_F$ of F. This provides further the equality of the orders of the ideal sheaves $\mathscr{J}_{r,F}, \mathscr{J}_{r,G}$, and we derive the theorem by the descriptions of minimal log discrepancies involving $\mathscr{J}_{r,F}, \mathscr{J}_{r,G}$ by Ein, Mustaţă and Yasuda in [8].

We work over an algebraically closed field k of characteristic zero throughout. $\mathbb{Z}_{>0}, \mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}$ denote the sets of positive/non-negative integers/real numbers.

1 The $\mathscr{I}$-adic semi-continuity problem

In this section we discuss general aspects of Mustaţă's $\mathscr{I}$-adic semi-continuity problem for minimal log discrepancies.

For the study of limits, we formulate the notion of $\mathbb{R}$-ideal sheaves by extending that of $\mathbb{Q}$-ideal sheaves in [11, Section 2]. Usual ideal sheaves are assumed to be quasi-coherent. On a scheme X we let $\mathfrak{R}_X$ denote the free semi-group generated by the family $\mathfrak{I}_X$ of all ideal sheaves on X, with coefficients in the semi-group $\mathbb{R}_{\geq 0}$. An element of $\mathfrak{R}_X$ is written multiplicatively as $\mathfrak{a}_1^{r_1}\cdots\mathfrak{a}_k^{r_k}$ with $\mathfrak{a}_i \in \mathfrak{I}_X, r_i \in \mathbb{R}_{\geq 0}$. We say that $\mathfrak{a}, \mathfrak{b} \in \mathfrak{R}_X$ are *adhered* if they are written as $\mathfrak{a} = \prod_{i,j}\mathfrak{a}_{ij}^{r_i m_{ij}} \cdot \mathscr{O}_X^a \cdot 0^{a'}$, $\mathfrak{b} = \prod_{i,k}\mathfrak{b}_{ik}^{r_i n_{ik}} \cdot$

$\mathscr{O}_X^b \cdot 0^{b'}$ in $\mathfrak{R}_X$ with $\mathfrak{a}_{ij}, \mathfrak{b}_{ik} \in \mathfrak{I}_X$, $r_i, a, a', b, b' \in \mathbb{R}_{\geq 0}$, $m_{ij}, n_{ik} \in \mathbb{Z}_{\geq 0}$, such that $\prod_j \mathfrak{a}_{ij}^{m_{ij}}$ equals $\prod_k \mathfrak{b}_{ik}^{n_{ik}}$ as ideal sheaves for each i, or $a', b' > 0$. We say that $\mathfrak{a}, \mathfrak{b} \in \mathfrak{R}_X$ are *equivalent* if there exist $\mathfrak{c}_0, \dots, \mathfrak{c}_i \in \mathfrak{R}_X$ with $\mathfrak{c}_0 = \mathfrak{a}, \mathfrak{c}_i = \mathfrak{b}$ such that each $\mathfrak{c}_{j-1}$ is adhered to $\mathfrak{c}_j$.

Definition 1.1 An $\mathbb{R}$-*ideal sheaf* on X is an equivalence class of the above relation in $\mathfrak{R}_X$.

We let $\mathfrak{I}_X^{\mathbb{R}}$ denote the family of $\mathbb{R}$-ideal sheaves on X. By an *expression* of $\mathfrak{a} \in \mathfrak{I}_X^{\mathbb{R}}$ we mean an element $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_k^{r_k} \in \mathfrak{R}_X$ with $\mathfrak{a}_i \in \mathfrak{I}_X, r_i \in \mathbb{R}_{>0}$ in the class of $\mathfrak{a}$.

Remark 1.1.1 While in some places in the literature one defines an $\mathbb{R}$-ideal sheaf as an element of $\mathfrak{R}_X$, we define it as an element of $\mathfrak{I}_X^{\mathbb{R}}$, from the viewpoint that for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_X$, one should identify for example the product of $\mathfrak{a}^{\sqrt{2}+1}, \mathfrak{b}$ and that of $\mathfrak{a}^{\sqrt{2}}, \mathfrak{a}\mathfrak{b}$, which remain different in $\mathfrak{R}_X$.

One can extend the notion of orders to $\mathbb{R}$-ideal sheaves. Recall that for an ideal sheaf $\mathfrak{a}$ on a scheme X, and for a prime divisor E on a normal variety X' equipped with a morphism $X' \to X$, one can define the *order* of $\mathfrak{a}$ along E by taking that of $\mathfrak{a}\mathscr{O}_{X'}$. Mostly we work on a normal variety X and consider an *extraction* of X, which is by definition a normal variety X' equipped with a proper birational morphism $X' \to X$. A divisor on an extraction of X is called a divisor *over* X.

Remark 1.1.2 Two ideal sheaves on a normal variety X have the same order along every divisor over X if and only if they have the same integral closure. This gives an equivalence relation in $\mathfrak{I}_X$. However, we will not pursue this, because the relation does not seem to be compatible with the notion of $\mathscr{I}$-adic topology.

One can also extend the notion of resolutions to $\mathbb{R}$-ideal sheaves.

Lemma-Definition 1.2 *Let $\mathfrak{f}_1^{r_1} \cdots \mathfrak{f}_k^{r_k}$, $\mathfrak{g}_1^{s_l} \cdots \mathfrak{g}_l^{s_l}$ be two expressions of the same $\mathbb{R}$-ideal sheaf $\mathfrak{a}$ on a variety X. Suppose that for every i, we have $\mathfrak{f}_i = \mathscr{O}_X(-F_i)$ for a Cartier divisor F_i. Then $\mathfrak{g}_j = \mathscr{O}_X(-G_j)$ for some Cartier divisor G_j, and $\sum_i r_i F_i = \sum_j s_j G_j$. Such $\mathfrak{a}$ is called a locally principal $\mathbb{R}$-ideal sheaf. In particular, the notion of resolutions of $\mathbb{R}$-ideal sheaves makes sense.*

Proof It suffices to prove that if $\mathfrak{a}_1, \mathfrak{a}_2$ are non-zero ideals on a local domain R such that $\mathfrak{a}_1\mathfrak{a}_2$ is principal, then so are $\mathfrak{a}_1, \mathfrak{a}_2$. Set $\mathfrak{a}_1\mathfrak{a}_2 = (f)$ with $f \in R \setminus \{0\}$. Then one can write $f = \sum_j f_{1j} f_{2j}$ and $f_{1j} f_{2j} = c_j f$ with $f_{ij} \in \mathfrak{a}_i, c_j \in R$. Thus $1 = \sum_j c_j$, so there exists j such that c_j is a unit, that is, $(f_{1j} f_{2j}) = (f)$. In this case $\mathfrak{a}_1 = (f_{1j})$, $\mathfrak{a}_2 = (f_{2j})$. Indeed, if $u \in \mathfrak{a}_1$, then $u f_{2j} \in \mathfrak{a}_1\mathfrak{a}_2 = (f) = (f_{1j} f_{2j})$, whence $u f_{2j} = c f_{1j} f_{2j}$ for some $c \in R$, that is, $u = c f_{1j}$. q.e.d.

We introduce the notion of $\mathscr{I}$-adic topology for $\mathbb{R}$-ideal sheaves.

Definition 1.3 Fix a closed subscheme Z of a scheme X and let $\mathscr{I}_Z$ denote its ideal sheaf.

(i) For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_X$ and $l \in \mathbb{Z}_{\geq 0}$, we write $\mathfrak{a} \equiv_l \mathfrak{b}$ if

$$\mathfrak{a} + \mathscr{I}_Z^l = \mathfrak{b} + \mathscr{I}_Z^l.$$

(ii) For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_X$ and $l \in \mathbb{R}$, we write $\mathfrak{a} \approx_l \mathfrak{b}$ if there exist $m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}$ such that

$$\mathfrak{a}^n \equiv_m \mathfrak{b}^n, \qquad m/n \geq l.$$

(iii) For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_X^{\mathbb{R}}$ and $l \in \mathbb{R}$, we write $\mathfrak{a} \sim_l \mathfrak{b}$ if there exist expressions $\mathfrak{a} = \mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_k^{r_k}$, $\mathfrak{b} = \mathfrak{b}_1^{r_1} \cdots \mathfrak{b}_k^{r_k}$ such that for each i

$$\mathfrak{a}_i \approx_{l/r_i} \mathfrak{b}_i.$$

Remark 1.3.1 One may replace the condition $\mathfrak{a}_i \approx_{l/r_i} \mathfrak{b}_i$ in (iii) above with $\mathfrak{a}_i \equiv_{l_i} \mathfrak{b}_i$, $l_i \geq l/r_i$.

Remark 1.3.2 From the point of view of orders in Remark 1.1.2, one can consider a relation $\sim_l'$ in $\mathfrak{I}_X^{\mathbb{R}}$ for a normal variety X, by setting $\mathfrak{a} \sim_l' \mathfrak{b}$ if $\min\{\operatorname{ord}_E \mathfrak{a}, l \operatorname{ord}_E \mathscr{I}_Z\} = \min\{\operatorname{ord}_E \mathfrak{b}, l \operatorname{ord}_E \mathscr{I}_Z\}$ for all divisors E over X. This condition is weaker than that of (iii) above, but Lemma 2.5(i) does not seem to work with $\sim_l'$.

The following basic fact will be used repeatedly.

Remark 1.3.3 If $\mathfrak{a} \sim_l \mathfrak{b}$ *and* $l \operatorname{ord}_E \mathscr{I}_Z > \operatorname{ord}_E \mathfrak{a}$ *along a divisor E on X' with $X' \to X$, then* $\operatorname{ord}_E \mathfrak{a} = \operatorname{ord}_E \mathfrak{b}$. This follows from the inequality $\operatorname{ord}_E \mathfrak{a}_i \leq r_i^{-1} \operatorname{ord}_E \mathfrak{a} < r_i^{-1} l \operatorname{ord}_E \mathscr{I}_Z \leq \operatorname{ord}_E \mathscr{I}_Z^{l_i}$ in the context $\mathfrak{a}_i + \mathscr{I}_Z^{l_i} = \mathfrak{b}_i + \mathscr{I}_Z^{l_i}$ of Remark 1.3.1.

We recall the theory of singularities in the minimal model program. A *pair* (X, Δ) consists of a normal variety X and a *boundary* Δ, that is, an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We treat a *triple* $(X, \Delta, \mathfrak{a})$ by attaching an $\mathbb{R}$-ideal sheaf $\mathfrak{a}$. For a prime divisor E on an extraction $\varphi \colon X' \to X$, its *log discrepancy* is

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \operatorname{ord}_E(K_{X'} - \varphi^*(K_X + \Delta)) - \operatorname{ord}_E \mathfrak{a}.$$

The image $\varphi(E)$ is called the *centre* of E on X. The triple $(X, \Delta, \mathfrak{a})$ is said to be *log canonical* (*lc*), *purely log terminal* (*plt*), *Kawamata log terminal* (*klt*) respectively if $a_E(X, \Delta, \mathfrak{a}) \geq 0$ ($\forall E$), > 0 ($\forall$exceptional E), > 0 ($\forall E$). For a closed subset Z of X, the *minimal log discrepancy*

$$\operatorname{mld}_Z(X, \Delta, \mathfrak{a})$$

over Z is the infimum of $a_E(X, \Delta, \mathfrak{a})$ for all E with centre in Z. It is either a non-negative real number or $-\infty$. The log canonicity of $(X, \Delta, \mathfrak{a})$ about Z is equivalent to $\operatorname{mld}_Z(X, \Delta, \mathfrak{a}) \geq 0$. See [12, Section 1], [15] for details.

De Fernex, Ein and Mustaţă in [3] after Kollár in [13] proved the $\mathscr{I}$-adic semi-continuity of log canonicity effectively to obtain with [5] the ACC for log canonical thresholds on l.c.i. varieties. We state its direct extension to the case with boundaries here.

Theorem 1.4 ([3, Theorem 1.4]) *Let (X,Δ) be a pair and Z a closed subset of X. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf such that*

$$\mathrm{mld}_Z(X,\Delta,\mathfrak{a})=0.$$

Then there exists a real number l such that the following holds: if an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ satisfies $\mathfrak{a}\sim_l\mathfrak{b}$, then

$$\mathrm{mld}_Z(X,\Delta,\mathfrak{b})=0.$$

Remark 1.4.1 The l is given effectively in terms of a divisor E with centre in Z such that $a_E(X,\Delta,\mathfrak{a})=0$. One may take an arbitrary l such that $l\,\mathrm{ord}_E\mathscr{I}_Z>\mathrm{ord}_E\mathfrak{a}$ by [3, Theorem 1.4] and Remark 1.3.3.

We will consider its generalisation to minimal log discrepancies, proposed by Mustaţă.

Conjecture 1.5 (Mustaţă) *Let (X,Δ) be a pair and Z a closed subset of X. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf. Then there exists a real number l such that the following holds: if an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ satisfies $\mathfrak{a}\sim_l\mathfrak{b}$, then*

$$\mathrm{mld}_Z(X,\Delta,\mathfrak{a})=\mathrm{mld}_Z(X,\Delta,\mathfrak{b}).$$

This conjecture is related to Shokurov's ACC conjecture [18], [20, Conjecture 4.2] for minimal log discrepancies. In fact, Conjecture 1.5 has originated in Mustaţă's following observation parallel to [3], [4] by generic limits of ideals.

Remark 1.5.1 (Mustaţă) If Conjecture 1.5 *holds effectively, then for a klt singularity $x\in X$ and a set R of positive real numbers which satisfies the descending chain condition, the set*

$$\{\mathrm{mld}_x(X,\mathfrak{a}_1^{r_1}\cdots\mathfrak{a}_k^{r_k})\mid\mathfrak{a}_i\in\mathfrak{I}_X,r_i\in R\}$$

satisfies the ascending chain condition.

We shall sketch the strategy, but it is still an observation since we do not know the correct formulation of the effective version of Conjecture 1.5.

We want the stability of an arbitrary non-decreasing sequence of elements $c_i=\mathrm{mld}_x(X,\mathfrak{a}_{i1}^{r_{i1}}\cdots\mathfrak{a}_{ik_i}^{r_{ik_i}})\ge 0$. We may assume that each $\mathfrak{a}_{ij}$ is non-trivial at x, then for a fixed divisor F with centre x, we have $\sum_j r_{ij}\le\sum_j r_{ij}\,\mathrm{ord}_F\mathfrak{a}_{ij}\le a_F(X)$. R has a minimum r, whence $k_i\le r^{-1}a_F(X)$. Thus by passing to a subsequence, we may assume the constancy $k=k_i$. We may further assume that $\{r_{ij}\}_i$ is a non-decreasing sequence for each j. Then $\{r_{ij}\}_i$ has a limit r_j by $r_{ij}\le a_F(X)$.

Take generic limits $\mathfrak{a}_j$ of $\mathfrak{a}_{ij}$ following [3, Section 4], [4, Section 3], [13]. We have $\mathfrak{a}_j$ on $\mathscr{X}=\mathrm{Spec}\,\widehat{\mathscr{O}_{X,x}\otimes_k}K$ with an extension K of the ground field k. Set $c:=\mathrm{mld}_o(\mathscr{X},\mathfrak{a}_1^{r_1}\cdots\mathfrak{a}_k^{r_k})$ with $o=x\times_X\mathscr{X}$. Fix an integer l and a divisor $\mathscr{E}$ over $\mathscr{X}$ with centre o such that

$$\begin{aligned}c&=\mathrm{mld}_o(\mathscr{X},(\mathfrak{a}_1+\mathfrak{m}^l\mathscr{O}_{\mathscr{X}})^{r_1}\cdots(\mathfrak{a}_k+\mathfrak{m}^l\mathscr{O}_{\mathscr{X}})^{r_k})\\&=a_{\mathscr{E}}(\mathscr{X},(\mathfrak{a}_1+\mathfrak{m}^l\mathscr{O}_{\mathscr{X}})^{r_1}\cdots(\mathfrak{a}_k+\mathfrak{m}^l\mathscr{O}_{\mathscr{X}})^{r_k}),\end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal sheaf of $x \in X$. By the same argument as in [3, Proposition 4.4], [4, Proposition 3.3], taking large l, we have for infinitely many i,

$$c = \mathrm{mld}_x(X, (\mathfrak{a}_{i1} + \mathfrak{m}^l)^{r_1} \cdots (\mathfrak{a}_{ik} + \mathfrak{m}^l)^{r_k}) = a_{E_i}(X, (\mathfrak{a}_{i1} + \mathfrak{m}^l)^{r_1} \cdots (\mathfrak{a}_{ik} + \mathfrak{m}^l)^{r_k}),$$
$$\mathrm{ord}_{\mathscr{E}} \mathfrak{a}_j = \mathrm{ord}_{\mathscr{E}}(\mathfrak{a}_j + \mathfrak{m}^l \mathscr{O}_{\mathscr{X}}) = \mathrm{ord}_{E_i}(\mathfrak{a}_{ij} + \mathfrak{m}^l) = \mathrm{ord}_{E_i} \mathfrak{a}_{ij},$$

with a suitable divisor E_i over X with centre x.

Now we suppose an effective form of Conjecture 1.5 which guarantees *the existence of l such that* $\mathrm{mld}_x(X, (\mathfrak{a}_{i1} + \mathfrak{m}^l)^{r_1} \cdots (\mathfrak{a}_{ik} + \mathfrak{m}^l)^{r_k}) = \mathrm{mld}_x(X, \mathfrak{a}_{i1}^{r_1} \cdots \mathfrak{a}_{ik}^{r_k})$ *for infinitely many i above*. Then

$$c \le c_i \le a_{E_i}(X, \mathfrak{a}_{i1}^{r_{i1}} \cdots \mathfrak{a}_{ik}^{r_{ik}}) = c + \sum_j (r_j - r_{ij}) \, \mathrm{ord}_{\mathscr{E}} \mathfrak{a}_j,$$

and the right-hand side converges to c. Thus $c_i = c$.

We expect an effective form of Conjecture 1.5, but the naive generalisation of Remark 1.4.1 never holds.

Remark-Example 1.5.2 Set $X = \mathbb{A}^2$ with coordinates x, y and $\mathfrak{a} = (x^2 + y^3)\mathscr{O}_X$, $\mathfrak{b} = x^2 \mathscr{O}_X$. The pair $(X, \mathfrak{a}^{2/3})$ has minimal log discrepancy $2/3 = a_E(X, \mathfrak{a}^{2/3})$ over the origin o, computed by the divisor E obtained by the blow-up at o. We have $\mathfrak{a} + \mathscr{I}_o^3 = \mathfrak{b} + \mathscr{I}_o^3$ and $\mathrm{ord}_E \mathfrak{a} = 2 < 3$, but $(X, \mathfrak{b}^{2/3})$ is not log canonical.

We provide a few reductions of the conjecture.

Remark 1.5.3 One inequality $\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) \ge \mathrm{mld}_Z(X, \Delta, \mathfrak{b})$ is obvious. Indeed, take a divisor E with centre in Z such that $a_E(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{a})$, or negative in the non-lc case. Any l such that $l \, \mathrm{ord}_E \mathscr{I}_Z > \mathrm{ord}_E \mathfrak{a}$ will work by Remark 1.3.3.

Remark 1.5.4 Conjecture 1.5 is reduced to the case when X *is affine and has $\mathbb{Q}$-factorial terminal singularities, Δ is zero and Z is irreducible*. Indeed, by [2] one can construct an extraction $\varphi \colon X' \to X$ such that X' has $\mathbb{Q}$-factorial terminal singularities with effective Δ' defined by $K_{X'} + \Delta' = \varphi^*(K_X + \Delta)$. Then $\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \mathrm{mld}_{\varphi^{-1}(Z)}(X', \Delta', \mathfrak{a}\mathscr{O}_{X'})$, so the conjecture is reduced to that on X'. We may further assume $\Delta = 0$ by forcing $\mathfrak{a}$ to absorb Δ. It is permissible to reduce to the affine case with irreducible Z by the property $\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \min_{i,j}\{\mathrm{mld}_{Z_j \cap U_i}(U_i, \Delta|_{U_i}, \mathfrak{a}|_{U_i})\}$ for a covering $\bigcup_i U_i$ of X and a decomposition $\bigcup_j Z_j$ of Z.

Remark 1.5.5 Mostly, we need just a weaker form of Conjecture 1.5 in which *an expression $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_k^{r_k}$ of $\mathfrak{a}$ is fixed and only those $\mathfrak{b} = \mathfrak{b}_1^{r_1/n_1} \cdots \mathfrak{b}_k^{r_k/n_k}$ with $\mathfrak{a}_i^{n_i} \equiv_{l_i} \mathfrak{b}_i$, $l_i \ge l n_i / r_i$ are considered*. This is reduced to the case when $\mathfrak{a}_i, \mathfrak{b}_i$ *are locally principal $\mathbb{R}$-ideal sheaves*. Indeed, first we reduce to the affine case by Remark 1.5.4, then after replacing $\mathfrak{a}_i^{r_i}$ with the s-uple of $\mathfrak{a}_i^{r_i/s}$ for some s, we may assume that $\mathrm{mld}_Z(X, \Delta, \mathfrak{a})$ equals $\mathrm{mld}_Z(X, \Delta, \mathfrak{f})$ for some $\mathfrak{f} = \prod_i (f_i \mathscr{O}_X)^{r_i}$ with $f_i \in \mathfrak{a}_i$ as in the argument in [17, Proposition 9.2.26]. By $\mathfrak{a}_i^{n_i} \equiv_{l_i} \mathfrak{b}_i$ one can write $f_i^{n_i} = g_i + h_i$ with $g_i \in \mathfrak{b}_i$, $h_i \in \mathscr{I}_Z^{l_i}$, so $f_i^{n_i} \mathscr{O}_X \equiv_{l_i} g_i \mathscr{O}_X$. For $\mathfrak{g} = \prod_i (g_i \mathscr{O}_X)^{r_i/n_i}$ the weaker conjecture for locally principal $\mathbb{R}$-ideal sheaves provides

$$\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{f}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{g}) \le \mathrm{mld}_Z(X, \Delta, \mathfrak{b}),$$

and we have the equality by Remark 1.5.3.

In the klt case, it is not difficult to prove our conjecture.

Theorem 1.6 *Conjecture* 1.5 *holds for a klt triple* $(X,\Delta,\mathfrak{a})$.

Proof By Remark 1.5.3, it suffices to prove $\mathrm{mld}_Z(X,\Delta,\mathfrak{a}) \le \mathrm{mld}_Z(X,\Delta,\mathfrak{b})$. Since $(X,\Delta,\mathfrak{a})$ is klt, we can fix $t,t' > 0$ such that $\mathrm{mld}_Z(X,\Delta,\mathfrak{a}^{1+t}\mathscr{I}_Z^{t'}) = 0$. Then by Theorem 1.4 there exists

$$l \ge t^{-1}\,\mathrm{mld}_Z(X,\Delta,\mathfrak{a})$$

such that $\mathfrak{a} \sim_l \mathfrak{b}$ implies $\mathrm{mld}_Z(X,\Delta,\mathfrak{b}^{1+t}\mathscr{I}_Z^{t'}) = 0$. Thus every divisor E with centre in Z satisfies

$$a_E(X,\Delta,\mathfrak{b}) > t\,\mathrm{ord}_E\,\mathfrak{b}.$$

Suppose $a_E(X,\Delta,\mathfrak{a}) \neq a_E(X,\Delta,\mathfrak{b})$, equivalently $\mathrm{ord}_E\,\mathfrak{a} \neq \mathrm{ord}_E\,\mathfrak{b}$. Then by Remark 1.3.3,

$$\mathrm{ord}_E\,\mathfrak{b} \ge l\,\mathrm{ord}_E\,\mathscr{I}_Z \ge l.$$

The above three inequalities give $a_E(X,\Delta,\mathfrak{b}) > \mathrm{mld}_Z(X,\Delta,\mathfrak{a})$, which completes the theorem. q.e.d.

Even if we start with klt singularities, it is inevitable to deal with log canonical singularities in the study of limits of them.

Example 1.7 Set $X = \mathbb{A}^2$ with coordinates x,y and $\mathfrak{a}_n = x(x+y^n)\mathscr{O}_X$. The limit of these $\mathfrak{a}_n$ is $\mathfrak{a}_\infty = x^2\mathscr{O}_X$, so the limit of klt pairs $(X,\mathfrak{a}_n^{1/2})$ is a plt pair $(X,\mathfrak{a}_\infty^{1/2}) = (X,x\mathscr{O}_X)$.

It is standard to reduce to lower dimensions by the restriction of pairs to subvarieties. For a pair $(X,G+\Delta)$ such that G is a reduced divisor which has no component in the support of effective Δ, one can construct the *different* Δ_{G^ν} on the normalisation $\nu\colon G^\nu \to G$ as in [14, Chapter 16], [19, §3]. It is a boundary which satisfies the equality $K_{G^\nu} + \Delta_{G^\nu} = \nu^*((K_X+G+\Delta)|_G)$.

As the first extension of Theorem 1.6, we study the plt case in which the boundary involves a Cartier divisor F. Let F be a Cartier divisor on a triple $(X,\Delta,\mathfrak{a})$ such that $(X,F+\Delta,\mathfrak{a})$ is plt. Then F is normal by the connectedness lemma [14, Theorem 17.4], [19, 5.7], and the induced triple $(F,\Delta_F,\mathfrak{a}\mathscr{O}_F)$ is klt. In this setting, we control $\mathrm{mld}_Z(X,G+\Delta,\mathfrak{b})$ for $G,\mathfrak{b}$ close to $F,\mathfrak{a}$. We adopt the notation

$$F \sim_l G$$

for the condition $\mathscr{O}_X(-F) \sim_l \mathscr{O}_X(-G)$, and $(F,\mathfrak{a}) \sim_l (G,\mathfrak{b})$ for $F \sim_l G$, $\mathfrak{a} \sim_l \mathfrak{b}$. We compare minimal log discrepancies on F,G rather than those on X, so G should be a divisor of the following type.

Definition 1.8 A *transversal* divisor on a triple $(X,\Delta,\mathfrak{b})$ is a reduced Cartier divisor which has no component in the support of Δ or the zero locus of $\mathfrak{b}$.

For example, an effective Cartier divisor G is transversal if $(X, G+\Delta, \mathfrak{b})$ is log canonical.

We state our theorem in the plt case, which will be proved in Section 2.

Theorem 1.9 *Let (X, Δ) be a pair and Z a closed subset of X. Let F be a reduced Cartier divisor and $\mathfrak{a}$ an $\mathbb{R}$-ideal sheaf such that $(X, F+\Delta, \mathfrak{a})$ is plt about Z. Then there exists a real number l such that the following holds: if an effective Cartier divisor G and an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ satisfy $(F, \mathfrak{a}) \sim_l (G, \mathfrak{b})$, then G is transversal on $(X, \Delta, \mathfrak{b})$ about Z and*

$$\mathrm{mld}_{F\cap Z}(F, \Delta_F, \mathfrak{a}\mathscr{O}_F) = \mathrm{mld}_{\nu^{-1}(G\cap Z)}(G^\nu, \Delta_{G^\nu}, \mathfrak{b}\mathscr{O}_{G^\nu}).$$

Theorem 1.9 compares minimal log discrepancies on different varieties, so it might provide a perspective in the study of the behaviour of these invariants under deformations. One can interpret it as an extension of Theorem 1.6 to the case when a variety as well as a boundary deforms. Theorem 1.9 can be related with Conjecture 1.5 via the *precise inversion of adjunction* in [14, Chapter 17].

Conjecture 1.10 (precise inversion of adjunction) *Let $(X, G+\Delta)$ be a pair such that G is a reduced divisor which has no component in the support of effective Δ, and Z a closed subset of G. Let Δ_{G^ν} be the different on the normalisation $\nu\colon G^\nu \to G$. Then*

$$\mathrm{mld}_Z(X, G+\Delta) = \mathrm{mld}_{\nu^{-1}(Z)}(G, \Delta_{G^\nu}).$$

The equality of minimal log discrepancies on X follows if the precise inversion of adjunction holds on X, such as l.c.i. varieties in [7], [8].

Corollary 1.11 *Let $(X, \Delta, \mathfrak{a})$, Z and F be as in Theorem 1.9. Suppose that the precise inversion of adjunction holds on X. Then there exists a real number l such that the following holds: if effective Cartier divisors G_i and an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ satisfy $F \sim_l G_i$, $\mathfrak{a} \sim_l \mathfrak{b}$, then for $G = \sum_i g_i G_i$ with $1 = \sum_i g_i$, $g_i \in \mathbb{R}_{\geq 0}$,*

$$\mathrm{mld}_Z(X, F+\Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, G+\Delta, \mathfrak{b}).$$

The proof uses the result of Lemma 2.2 that $F \sim_l G$ for large l implies $F \cap Z = G \cap Z$. However we believe that it is best to give the proof here.

Proof By Remark 1.5.3, we want $\mathrm{mld}_Z(X, F+\Delta, \mathfrak{a}) \leq \mathrm{mld}_Z(X, G+\Delta, \mathfrak{b})$. Since $\mathrm{mld}_Z(X, G+\Delta, \mathfrak{b}) \geq \sum_i g_i \mathrm{mld}_Z(X, G_i+\Delta, \mathfrak{b})$ by $K_X+G+\Delta = \sum_i g_i(K_X+G_i+\Delta)$, we are reduced to the case with one Cartier divisor G. We may assume $Z \subset F$ by Theorem 1.6, and $Z \subset G$ by the mentioned result of Lemma 2.2. Then the statement follows from Theorem 1.9. Note that the precise inversion of adjunction for triples is reduced to that for pairs. q.e.d.

We close this section with one observation related to Conjecture 1.5.

Proposition 1.12 *Let (X, Δ) be a pair and Z a closed subset of X. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf. Then there exist real numbers l and $0 < t \leq 1$ such that the following holds: if an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ satisfies $\mathfrak{a} \sim_l \mathfrak{b}$, then*

$$\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{a}^{1-t}\mathfrak{b}^t).$$

Proof By Remark 1.5.3, it suffices to prove $\mathrm{mld}_Z(X,\Delta,\mathfrak{a}) \le \mathrm{mld}_Z(X,\Delta,\mathfrak{a}^{1-t}\mathfrak{b}^t)$. We may assume the log canonicity of $(X,\Delta,\mathfrak{a})$. Fix a log resolution $\varphi\colon X' \to X$ of $(X,\Delta,\mathfrak{a}\mathscr{I}_Z)$ and set $K_{X'}+\Delta' := \varphi^*(K_X+\Delta)$. Let A denote the effective $\mathbb{R}$-divisor on X' defined by the locally principal $\mathbb{R}$-ideal sheaf $\mathfrak{a}\mathscr{O}_{X'}$, and S the reduced divisor whose support is the union of the exceptional locus, $\mathrm{Supp}\,\Delta'$ and $\mathrm{Supp}\,A$. We take $0 < t \le 1$ such that $tA \le S$. By Theorem 1.4 we have l such that $\mathfrak{a} \sim_l \mathfrak{b}$ implies the log canonicity of $(X',S-tA,\mathfrak{b}^t\mathscr{O}_{X'})$. In particular, for a divisor E on an extraction $\psi\colon Y \to X'$ with $(\varphi\circ\psi)(E) \subset Z$,

$$\begin{aligned}
a_E(X,\Delta,\mathfrak{a}^{1-t}\mathfrak{b}^t) &= a_E(X',(1-t)A,\mathfrak{b}^t\mathscr{O}_{X'}) - \mathrm{ord}_E\,\Delta' \\
&= a_E(X',S-tA,\mathfrak{b}^t\mathscr{O}_{X'}) + \mathrm{ord}_E(S-A-\Delta') \\
&\ge \mathrm{ord}_E(S-A-\Delta').
\end{aligned}$$

$S-A-\Delta' = K_X' + S - (\varphi^*(K_X+\Delta)+A) \ge 0$, and if we take a divisor F such that $\psi(E) \subset F \subset \varphi^{-1}(Z)$, then

$$\mathrm{ord}_E(S-A-\Delta') \ge \mathrm{ord}_F(S-A-\Delta') = a_F(X,\Delta,\mathfrak{a}).$$

These two inequalities prove the proposition. q.e.d.

2 The purely log terminal case

The purpose of this section is to prove Theorem 1.9; see Lemmata 2.4 and 2.9.

As (X,Δ) is klt, by [2] there exists a $\mathbb{Q}$-factorialisation $\varphi\colon X' \to X$, that is, an extraction with X' $\mathbb{Q}$-factorial which is isomorphic in codimension one. Then as in Remark 1.5.4 we can reduce the theorem to that on X', and hence we may assume that X *is* $\mathbb{Q}$*-factorial and* $\Delta = 0$. *We shall discuss on the germ at a closed point of* X.

We shall define some ideal sheaves for the theory of motivic integration. Let d denote the dimension of X. We fix a positive integer r such that rK_X is a Cartier divisor. We extend the construction in [11, Section 2] to transversal divisors. A general l.c.i. subscheme Y of dimension d of a smooth ambient space A which contains X is the union

$$Y = X \cup C^Y \tag{1}$$

of X and another variety C^Y by Bertini's theorem. The subscheme $D^Y := C^Y|_X$ of X is defined by the conductor ideal sheaf $\mathscr{C}_{X/Y} := \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_X,\mathscr{O}_Y)$, and is a divisor such that $\mathscr{O}_X(rK_X) = \mathscr{O}_X(-rD^Y)\omega_Y^{\otimes r}$. The summation $\mathscr{D}'_X := \sum_Y \mathscr{C}_{X/Y}$ over all general Y is called the *l.c.i. defect ideal sheaf* of X, which one can define for reduced schemes of pure dimension. We also consider the summation $\mathscr{D}_{r,X} := \sum_Y \mathscr{O}_X(-rD^Y)$. For a reduced Cartier divisor G on X, the above $Y = X \cup C^Y$ has a Cartier divisor $Y_G = G \cup C^Y|_{Y_G}$. Indeed, by taking a general hypersurface H in A such that $X \cap H = G$, we set Y_G as $Y \cap H = (X\cap H) \cup (C^Y \cap H) = G \cup C^Y|_{Y_G}$. Thus G has its l.c.i. defect ideal sheaf

$$\mathscr{D}'_G = \mathscr{D}'_X\mathscr{O}_G, \tag{2}$$

and we have $\mathscr{O}_X(r(K_X+G))\mathscr{O}_G = \mathscr{O}_X(-rD^Y)\mathscr{O}_G \cdot \omega_{Y_G}^{\otimes r}$.

Let $\mathscr{J}'_G$ be the Jacobian ideal sheaf of G, and $\mathscr{J}_{r,G}$ the image of the natural map $(\Omega_G^{d-1})^{\otimes r} \otimes \mathscr{O}_X(-r(K_X+G)) \to \mathscr{O}_G$. Let $\tilde{\mathscr{J}}'_G, \tilde{\mathscr{J}}_{r,G}$ be the inverse images of $\mathscr{J}'_G, \mathscr{J}_{r,G}$ respectively by the natural map $\mathscr{O}_X \to \mathscr{O}_G$. The argument in [11] provides the equality $\sum_Y {\mathscr{J}'_{Y_G}}^r \mathscr{O}_G = \mathscr{J}_{r,G} \cdot \mathscr{D}_{r,X}\mathscr{O}_G$ similar to [11, (2.4)], where $\mathscr{J}'_{Y_G}$ is the Jacobian of Y_G. The left-hand side is nothing but ${\mathscr{J}'_G}^r$. In order to see this, set local coordinates $x_1,\dots,x_k$ of A, denote by $\mathscr{I}_X, \mathscr{I}_Y$ the ideal sheaves of X, Y on A, and take $f_1,\dots,f_c \in \mathscr{O}_A$, $c = k-d+1$, such that $f_1|_X$ defines G and $f_2,\dots,f_c$ generate $\mathscr{I}_Y$. Then for arbitrary $g_2,\dots,g_c \in \mathscr{I}_X$ and general $t_2,\dots,t_c \in k$, the subscheme defined by $f_i + t_i g_i$, $2 \le i \le c$, is an l.c.i. Y' which has a decomposition $X \cup C^{Y'}$ as in (1). Thus with $g_1 := f_1$ and $t_1 \in k$, the r-th powers of determinants of $c \times c$ minors of the matrix $(\partial(f_i+t_ig_i)/\partial x_j)_{ij}|_G$ are contained in $\sum_Y {\mathscr{J}'_{Y_G}}^r \mathscr{O}_G$, whence so are those of $(\partial g_i/\partial x_j)_{ij}|_G$. This means $\sum_Y {\mathscr{J}'_{Y_G}}^r \mathscr{O}_G = \sum_{j \in \mathscr{J}'_G} j^r \mathscr{O}_G$, and the right-hand side equals ${\mathscr{J}'_G}^r$ by the same trick. Hence we obtain

$$\begin{aligned} {\mathscr{J}'_G}^r &= \mathscr{J}_{r,G} \cdot \mathscr{D}_{r,X}\mathscr{O}_G, \\ {\tilde{\mathscr{J}}'_G}{}^r + \mathscr{O}_X(-G) &= \tilde{\mathscr{J}}_{r,G} \cdot \mathscr{D}_{r,X} + \mathscr{O}_X(-G). \end{aligned} \tag{3}$$

We set

$$c := \mathrm{mld}_{F\cap Z}(F, \mathfrak{a}\mathscr{O}_F).$$

As $(X, F, \mathfrak{a})$ is plt, we can fix $t > 0, t' \ge 0$ such that

$$\mathrm{mld}_Z(X, F, \mathfrak{a}^{1+t}\, {\tilde{\mathscr{J}}'_F}{}^{rt} {\mathscr{D}'_X}^t \mathscr{I}_Z^{t'}) = 0.$$

We fix a log resolution $\bar{\varphi}\colon \bar{X} \to X$ of $(X, F, \mathfrak{a}\mathscr{I}_Z \tilde{\mathscr{J}}'_F \tilde{\mathscr{J}}_{r,F} \mathscr{D}'_X \mathscr{D}_{r,X})$. Let $\bar{F}$ be the strict transform of F. By blowing up $\bar{X}$ further, we may assume the existence of a prime divisor $E_F \subset \bar{\varphi}^{-1}(F\cap Z)$ which intersects $\bar{F}$ properly and satisfies

$$a_{E_F}(X, F, \mathfrak{a}) = a_{E_F|_{\bar{F}}}(F, \mathfrak{a}\mathscr{O}_F) = c. \tag{4}$$

Take the decomposition $\bar{\varphi}^*F = V_F + H_F$, where V_F consists of prime divisors in $\bar{\varphi}^{-1}(Z)$ and H_F those not in $\bar{\varphi}^{-1}(Z)$. By blowing up $\bar{X}$ further, we may assume that every divisor $\bar{E}$ with $\bar{E} \subset \mathrm{Supp}\, V_F$, $\bar{E} \cap \mathrm{Supp}\, H_F \ne \emptyset$ satisfies

$$\mathrm{ord}_{\bar{E}} V_F > t^{-1}c. \tag{5}$$

We take an integer l_1 such that

$$l_1 > \mathrm{ord}_{\bar{E}} V_F, \qquad l_1 > \mathrm{ord}_{\bar{E}} \mathfrak{a} \tag{6}$$

for all divisors $\bar{E}$ on $\bar{X}$ with $\bar{\varphi}(\bar{E}) \subset Z$. Note that

$$l_1 > t^{-1}c + 1 \tag{7}$$

unless $F \subset Z$.

The next lemma is a direct application of Theorem 1.4 with Remark 1.4.1 by (6).

Lemma 2.1 *For $\mathbb{R}$-ideal sheaves $\mathfrak{g},\mathfrak{b}$ such that $\mathscr{O}_X(-F)\sim_{l_1}\mathfrak{g}$, $\mathfrak{a}\sim_{l_1}\mathfrak{b}$, we have* $\mathrm{mld}_Z(X,\mathfrak{g}\mathfrak{b}^{1+t}\tilde{\mathscr{J}}_F'^{\,rt}\mathscr{D}_X'^{\,t}\mathscr{I}_Z^{t'})=0$. *Especially, if $(F,\mathfrak{a})\sim_{l_1}(G,\mathfrak{b})$, then G is a transversal divisor on $(X,\mathfrak{b})$.*

We can replace the condition $F\sim_l G$ with the stronger one $F\approx_l G$ defined by $\mathscr{O}_X(-F)\approx_l\mathscr{O}_X(-G)$.

Lemma 2.2 *If $F\sim_l G$ with $l\ge l_1$, then $F\approx_l G$.*

Proof G is reduced by Lemma 2.1. By Lemma-Definition 1.2 and the definition of $F\sim_l G$, there exist decompositions $1=\sum_j f_jn_j$, $G=\sum_j f_jH_j$ with $f_j\in\mathbb{R}_{>0}$, $n_j\in\mathbb{Z}_{>0}$ and effective Cartier divisors H_j such that $\mathscr{O}_X(-n_jF)\equiv_{m_j}\mathscr{O}_X(-H_j)$ with $m_j\ge l/f_j$. Note that $\mathscr{O}_X(-F)\approx_{m_j/n_j}\mathscr{O}_X(-H_i)^{1/n_j}$ and $m_j/n_j\ge l/f_jn_j\ge l$. Hence all coefficients in $n_j^{-1}H_j$ are at most one by Lemma 2.1. Thus each component G_i of G has $\mathrm{ord}_{G_i}H_j\le n_j$, so $1=\sum_j f_j\,\mathrm{ord}_{G_i}H_j\le\sum_j f_jn_j=1$ and $\mathrm{ord}_{G_i}H_j=n_j$, $H_j=n_jG$. Now the lemma follows from $\mathscr{O}_X(-n_jF)\equiv_{m_j}\mathscr{O}_X(-n_jG)$ and $m_j/n_j\ge l$. q.e.d.

Now we may assume that *Z is an irreducible proper subset of F, and is contained in G also*. Indeed, since $F\approx_1 G$ implies $F\cap Z=G\cap Z$ as sets, we may assume $Z\subset F,G$ by replacing Z with $F\cap Z$. If $Z=F$, then $G\ge F$ and $F\approx_2 G$ means $\mathscr{O}_X(-nF)=\mathscr{O}_X(-nF)(\mathscr{O}_X(-n(G-F))+\mathscr{O}_X(-nF))$ for some n, so $F=G$, $\mathfrak{a}\mathscr{O}_F=\mathfrak{b}\mathscr{O}_G$ and the statement is trivial.

We write $(F,\mathfrak{a})\approx_l(G,\mathfrak{b})$ for the condition $F\approx_l G$, $\mathfrak{a}\sim_l\mathfrak{b}$. G is transversal if $(F,\mathfrak{a})\approx_{l_1}(G,\mathfrak{b})$ by Lemma 2.1. We then consider a log resolution $G'\to G$ embedded into some log resolution $\varphi\colon X'\to X$ of $(X,F+G,\mathfrak{a}\mathfrak{b}\tilde{\mathscr{J}}_G'\mathscr{J}_{r,G})$ which factors through $\bar{X}$. Set $\varphi'\colon X'\to\bar{X}$. Let I denote the set of all φ-exceptional prime divisors E on X' intersecting G', and I_Z the subset of I consisting of all E with $\varphi(E)\subset Z$. By blowing up X' further, we may assume that G' does not intersect the strict transform of the divisorial part of the zero locus of $\mathfrak{b}$, and that for all $E\in I$

$$\varphi'(E)=\varphi'(E|_{G'}).\tag{8}$$

Then $\mathrm{mld}_{\nu^{-1}(Z)}(G^\nu,\mathfrak{b}\mathscr{O}_{G^\nu})$ equals the minimum of $a_E(X,G,\mathfrak{b})=a_{E|_{G'}}(G^\nu,\mathfrak{b}\mathscr{O}_{G^\nu})$ over all $E\in I_Z$, or $-\infty$ if the minimum is negative.

Lemma 2.3 *If $(F,\mathfrak{a})\approx_{l_1}(G,\mathfrak{b})$, then for $E\in I_Z$*

(i) $rt\,\mathrm{ord}_E\tilde{\mathscr{J}}_F'+t\,\mathrm{ord}_E\mathscr{D}_X'+t\,\mathrm{ord}_E\mathfrak{b}\le a_{E|_{G'}}(G^\nu,\mathfrak{b}\mathscr{O}_{G^\nu})$.

(ii) $\mathrm{ord}_E F>t^{-1}c$ *and* $\mathrm{ord}_E G>t^{-1}c$.

Proof (i) This follows from Lemma 2.1.

(ii) If we write $\mathscr{I}_Z\mathscr{O}_{\bar{X}}=\mathscr{O}_{\bar{X}}(-V_Z)$, then by (6) the divisor $l_1V_Z-V_F$ is effective with support $\bar{\varphi}^{-1}(Z)$. By $F\approx_{l_1}G$ we have the decomposition $\bar{\varphi}^*G=V_F+H_G$ in which H_G consists of divisors not in $\bar{\varphi}^{-1}(Z)$, and moreover

$$\begin{aligned}&\mathscr{O}_{\bar{X}}(-nV_F)(\mathscr{O}_{\bar{X}}(-nH_F)+\mathscr{O}_{\bar{X}}(-n(l_1V_Z-V_F))\\&=\mathscr{O}_{\bar{X}}(-nV_F)(\mathscr{O}_{\bar{X}}(-nH_G)+\mathscr{O}_{\bar{X}}(-n(l_1V_Z-V_F))\end{aligned}$$

for some n. Hence on the reduced divisor $\bar{\varphi}^{-1}(Z)$,

$$nH_F \cap \bar{\varphi}^{-1}(Z) = nH_G \cap \bar{\varphi}^{-1}(Z) \tag{9}$$

scheme-theoretically, and its support contains $\varphi'(E)$ by (8). Thus there exists a prime divisor $\bar{E}$ on $\bar{X}$ with $\varphi'(E) \subset \bar{E} \subset \bar{\varphi}^{-1}(Z)$ and $\bar{E} \cap \operatorname{Supp} H_F \neq \emptyset$. $\bar{E}$ has $\operatorname{ord}_{\bar{E}} G = \operatorname{ord}_{\bar{E}} F > t^{-1}c$ by (5), so $\operatorname{ord}_E F \geq \operatorname{ord}_{\bar{E}} F > t^{-1}c$, $\operatorname{ord}_E G \geq \operatorname{ord}_{\bar{E}} G > t^{-1}c$. q.e.d.

We obtain one inequality in Theorem 1.9 as in Remark 1.5.3.

Lemma 2.4 *If* $(F,\mathfrak{a}) \approx_{l_1} (G,\mathfrak{b})$*, then* $\operatorname{mld}_Z(F,\mathfrak{a}\mathscr{O}_F) \geq \operatorname{mld}_{\nu^{-1}(Z)}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu})$.

Proof We have the divisor $E_F \subset \bar{\varphi}^{-1}(Z)$ in (4). $W := \bar{F} \cap E_F$ is contained in the support of the locus (9), whence $W \subset \operatorname{Supp} H_G \cap E_F$. This implies $W \subset \bar{G} \cap E_F$ for the strict transform $\bar{G}$ of G by the s.n.c. property of $\bar{F} + E_F + \operatorname{Supp}(H_G - \bar{G})$. Moreover by (9), $nW = n\bar{G}|_{E_F}$ as divisors on E_F at the generic point η_W of W. Hence $W = \bar{G} \cap E_F$ scheme-theoretically at η_W, and its strict transform W' on G' is defined. With (6) we obtain

$$\operatorname{mld}_{\nu^{-1}(Z)}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \leq a_{W'}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) = a_{E_F}(X,G,\mathfrak{b}) = a_{E_F}(X,F,\mathfrak{a}) = c.$$

q.e.d.

We shall prove the other inequality $\operatorname{mld}_{\nu^{-1}(Z)}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \geq c$ in Theorem 1.9 by studying $E \in I_Z$ with $a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \leq c$. We fix a prime divisor E_Z on $\bar{X}$ such that $\bar{\varphi}(E_Z) = Z$. Then we can fix an integer $l_2 \geq l_1$ such that

$$\bar{\varphi}_* \mathscr{O}_{\bar{X}}(-l_2 E_Z) \subset \mathscr{I}_Z^{l_1}, \tag{10}$$

as in the proof of [10, Lemma 3]. Indeed, we apply Zariski's subspace theorem [1, (10.6)] to the natural map $(\mathscr{O}_{X,Z}, \mathfrak{m}_Z) \to (\mathscr{O}_{\bar{X},E_Z}, \mathfrak{m}_{E_Z})$ of local rings, where $\mathfrak{m}_Z, \mathfrak{m}_{E_Z}$ denote the maximal ideals of $\mathscr{O}_{X,Z}, \mathscr{O}_{\bar{X},E_Z}$. Then we have an integer l_2 such that $\mathscr{O}_{X,Z} \cap \mathfrak{m}_{E_Z}^{l_2} \subset \mathfrak{m}_Z^{l_1}$, which means the inclusion (10) at the generic point of Z. We repeat this application to the specialisations of the map $\mathscr{O}_{X,Z} \to \mathscr{O}_{\bar{X},E_Z}$, and finally obtain (10) by Noetherian induction.

Lemma 2.5 *If* $(F,\mathfrak{a}) \approx_{l_2} (G,\mathfrak{b})$ *and* $E \in I_Z$ *satisfies* $a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \leq c$*, then*

(i) $\operatorname{ord}_E \tilde{\mathscr{J}}'_F = \operatorname{ord}_E \tilde{\mathscr{J}}'_G \leq (rt)^{-1}c$.
(ii) $\operatorname{ord}_E \mathscr{J}_{r,F} = \operatorname{ord}_E \mathscr{J}_{r,G} \leq t^{-1}c$.
(iii) $\operatorname{ord}_E \mathscr{D}'_X \leq t^{-1}c$.
(iv) $\operatorname{ord}_E \mathfrak{a} = \operatorname{ord}_E \mathfrak{b} \leq t^{-1}c$.

Proof (i) We use the explicit descriptions of $\tilde{\mathscr{J}}'_F, \tilde{\mathscr{J}}'_G$ in terms of Jacobian matrices. Embed X into a smooth ambient space A with local coordinates $x_1,\dots,x_k$ and take $f,g \in \mathscr{O}_A$ such that $f|_X, g|_X$ define F,G. By $F \approx_{l_2} G$, $f^n\mathscr{O}_X + \mathscr{I}_Z^{nl_2} = g^n\mathscr{O}_X + \mathscr{I}_Z^{nl_2}$ for some n. Note that $f^n|_X \notin \mathscr{I}_Z^{nl_2}$ by $\operatorname{ord}_{E_Z} f|_X < l_1$ from (6). If we choose $u,v \in \mathscr{O}_A$ so that $f^n - ug^n|_X, g^n - vf^n|_X \in \mathscr{I}_Z^{nl_2}$, then $(1-uv)f^n|_X \in \mathscr{I}_Z^{nl_2}$ so uv should be a unit. We take an etale cover $\tilde{X} \to X$ by adding a function y with $y^n = u$ to produce the factorisation $f^n - ug^n = \prod_i (f - \mu^i yg)$ with a primitive n-th root μ of unity, and discuss on the germ $\tilde{U}$ at some closed point of $\tilde{X}$. Set the prime divisor $\tilde{E}_Z := E_Z \times_X \tilde{U}$ on $\tilde{\varphi}\colon \bar{X} \times_X \tilde{U} \to \tilde{U}$. Since $\prod_i (f - \mu^i yg)|_{\tilde{U}} \in \tilde{\varphi}_* \mathscr{O}_{\bar{X}\times_X \tilde{U}}(-nl_2\tilde{E}_Z)$, with (10) there exists i such that

$$f - \mu^i yg|_{\tilde{U}} \in \tilde{\varphi}_* \mathscr{O}_{\bar{X}\times_X \tilde{U}}(-l_2\tilde{E}_Z) = \bar{\varphi}_* \mathscr{O}_{\bar{X}}(-l_2 E_Z) \otimes_{\mathscr{O}_X} \mathscr{O}_{\tilde{U}} \subset \mathscr{I}_Z^{l_1} \mathscr{O}_{\tilde{U}}.$$

$F \times_X \tilde{U}, G \times_X \tilde{U}$ are given by $f|_{\tilde{U}}, \mu^i yg|_{\tilde{U}}$. By the description of $\tilde{\mathscr{J}}'_F \mathscr{O}_{\tilde{U}}, \tilde{\mathscr{J}}'_G \mathscr{O}_{\tilde{U}}$ in terms of Jacobian matrices, we have

$$\tilde{\mathscr{J}}'_F \mathscr{O}_{\tilde{U}} + \mathscr{C} = \tilde{\mathscr{J}}'_G \mathscr{O}_{\tilde{U}} + \mathscr{C}$$

for $\mathscr{C} := \sum_j (\partial(f - \mu^i yg)/\partial x_j \cdot \mathscr{O}_{\tilde{U}}) \subset \mathscr{I}_Z^{l_1 - 1} \mathscr{O}_{\tilde{U}}$. By Lemma 2.3(i) and (7), for $\tilde{E} := E \times_X \tilde{U}$

$$\begin{aligned} \operatorname{ord}_{\tilde{E}} \tilde{\mathscr{J}}'_F \mathscr{O}_{\tilde{U}} &= \operatorname{ord}_E \tilde{\mathscr{J}}'_F \leq (rt)^{-1} c < l_1 - 1, \\ \operatorname{ord}_{\tilde{E}} \tilde{\mathscr{J}}'_G \mathscr{O}_{\tilde{U}} &= \operatorname{ord}_E \tilde{\mathscr{J}}'_G, \end{aligned}$$

which prove (i).

(ii) Lemma 2.3 implies $\operatorname{ord}_E \tilde{\mathscr{J}}'^{\,r}_F \leq t^{-1} c < \operatorname{ord}_E F, \operatorname{ord}_E G$. Thus (ii) follows from (i) and (3) for F, G.

(iii) This follows from Lemma 2.3(i).

(iv) This follows from Lemma 2.3(i), (7) and Remark 1.3.3. q.e.d.

We shall apply the theory of motivic integration due to Kontsevich in [16] and Denef and Loeser in [6] to transversal divisors. We fix notation following [11, Section 3]. For a scheme X of finite type of dimension d, we let $J_n X$ denote its *jet scheme* of order n, $J_\infty X$ its *arc space*, and set $\pi_n^X : J_\infty X \to J_n X$ and $\pi_{nm}^X : J_m X \to J_n X$ for $m \geq n$. One has the *motivic measure* $\mu_X : \mathscr{B}_X \to \widehat{\mathscr{M}}$ from the family $\mathscr{B}_X$ of *measurable* subsets of $J_\infty X$ to an extension $\widehat{\mathscr{M}}$ of the Grothendieck ring. $\mathscr{B}_X$ is an extension of the family of stable subsets. A subset S of $J_\infty X$ is said to be *stable* at level n if $\pi_n^X(S)$ is constructible, $S = (\pi_n^X)^{-1}(\pi_n^X(S))$, and $\pi_{m+1}^X(S) \to \pi_m^X(S)$ is piecewise trivial with fibres $\mathbb{A}^d$ for $m \geq n$. Such a set S has measure

$$\mu_X(S) = [\pi_n^X(S)] \mathbb{L}^{-(n+1)d}$$

with $\mathbb{L} = [\mathbb{A}^1]$.

For a morphism $\varphi\colon X \to Y$, we write $\varphi_n\colon J_n X \to J_n Y$, $\varphi_\infty\colon J_\infty X \to J_\infty Y$ for the induced morphisms. For a closed subset Z, we let $J_n X|_Z, J_\infty X|_Z$ denote the inverse images of Z by $J_n X, J_\infty X \to X$. Finally for an $\mathbb{R}$-ideal sheaf $\mathfrak{a}$, the *order* $\operatorname{ord}_{\mathfrak{a}} \gamma$ along $\mathfrak{a}$ is defined for $\gamma \in J_\infty X$. It makes sense to say $\operatorname{ord}_{\mathscr{I}} \gamma_n = m$ for $\gamma_n \in J_n X$ and an ideal sheaf $\mathscr{I}$ as long as $m \leq n$.

Back to the theorem, we fix an expression

$$\mathfrak{a} = \mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_k^{r_k}.$$

We fix an integer c_1 such that

$$c_1 \geq t^{-1} c, \qquad c_1 \geq (r_i t)^{-1} c \tag{11}$$

for all i. Applying Greenberg's result [9] to F, one can find $c_2 \geq c_1$ such that

$$\pi^F_{c_1c_2}(J_{c_2}F) = \pi^F_{c_1}(J_\infty F). \tag{12}$$

We take an integer $l_3 \geq l_2$ such that

$$l_3 > c_2. \tag{13}$$

From now on we fix an arbitrary $E \in I_Z$ for $(G,\mathfrak{b}) \approx_{l_3} (F,\mathfrak{a})$ such that

$$a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \leq c, \tag{14}$$

and will derive the opposite inequality $a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}) \geq c$. To avoid confusion we set $\psi := \varphi|_{G'} \colon G' \to G$. By blowing up X' further, we may assume that $E'|_{G'}$ is ψ-exceptional for all $E' \in I \setminus \{E\}$ with $E|_{G'} \cap E'|_{G'} \neq \emptyset$. Take the subset T' of $J_\infty G'$ which consists of all arcs γ such that

$$\mathrm{ord}_{E'|_{G'}} \gamma = \begin{cases} 1 & \text{if } E' = E, \\ 0 & \text{if } E' \in I \setminus \{E\},\, E'|_{G'} \cap E|_{G'} \neq \emptyset. \end{cases}$$

T' is stable at level one. Set $T := \psi_\infty(T') \subset J_\infty G$, $T'_n := \pi^{G'}_n(T') \subset J_nG'$ and $T_n := \pi^G_n(T) = \psi_n(T'_n) \subset J_nG$ as

$$\begin{array}{ccccccc}
J_\infty G' & \supset & T' & \xrightarrow{\pi^{G'}_n} & T'_n & \subset & J_nG' \\
\downarrow & & \downarrow{\scriptstyle \psi_\infty} & & \downarrow{\scriptstyle \psi_n} & & \downarrow \\
J_\infty G & \supset & T & \xrightarrow{\pi^G_n} & T_n & \subset & J_nG.
\end{array}$$

One can regard $J_nF, J_nG \subset J_nX$. Then $F \approx_{l_3} G$ implies $J_{c_2}F|_Z = J_{c_2}G|_Z$ by (13). Hence by (12)

$$T_{c_1} \subset \pi^G_{c_1c_2}(J_{c_2}G|_Z) = \pi^F_{c_1c_2}(J_{c_2}F|_Z) = \pi^F_{c_1}(J_\infty F|_Z).$$

Thus if we set

$$S := (\pi^F_{c_1})^{-1}(T_{c_1}) \subset J_\infty F$$

and $S_n := \pi^F_n(S) \subset J_nF$, then $S_{c_1} = T_{c_1}$ as

$$J_\infty F \supset S \xrightarrow{\pi^F_n} S_n \xrightarrow{\pi^F_{c_1n}} S_{c_1} = T_{c_1}. \tag{15}$$

We translate Lemma 2.5 into the language of arcs.

Lemma 2.6 (i) *On S and T,* $\mathrm{ord}_{\tilde{\mathscr{J}}'_F} = \mathrm{ord}_{\tilde{\mathscr{J}}'_G}$ *and takes constant value* $\mathrm{ord}_E \tilde{\mathscr{J}}'_F = \mathrm{ord}_E \tilde{\mathscr{J}}'_G \leq c_1$.

(ii) *On S and T,* $\mathrm{ord}_{\tilde{\mathscr{J}}_{r,F}} = \mathrm{ord}_{\tilde{\mathscr{J}}_{r,G}}$ *and takes constant value* $\mathrm{ord}_E \tilde{\mathscr{J}}_{r,F} = \mathrm{ord}_E \tilde{\mathscr{J}}_{r,G} \leq c_1$.

(iii) *On S and T,* $\mathrm{ord}_{\mathscr{D}'_X}$ *takes constant value* $\mathrm{ord}_E \mathscr{D}'_X \leq c_1$.

(iv) *On T,* $\mathrm{ord}_{\mathfrak{a}} = \mathrm{ord}_{\mathfrak{b}}$ *and takes constant value* $\mathrm{ord}_E \mathfrak{a} = \mathrm{ord}_E \mathfrak{b} \le c_1$. *On S,* $\mathrm{ord}_{\mathfrak{a}}$ *takes constant value* $\mathrm{ord}_E \mathfrak{a} = \mathrm{ord}_E \mathfrak{b}$.

Proof It is clear by Lemma 2.5, (11) and the construction of T'. Note that $\mathrm{ord}_E \mathfrak{a}_i \le r_i^{-1} \mathrm{ord}_E \mathfrak{a} \le c_1$. q.e.d.

Let $\mathscr{J}_\psi$ be the image of the natural map $\psi^*\Omega_G^{d-1} \otimes \omega_{G'}^{-1} \to \mathscr{O}_{G'}$. By definition we obtain the equality

$$\mathscr{J}_\psi^r = \tilde{\mathscr{J}}_{r,G}\mathscr{O}_{G'}\big(-r\sum_{E'\in I}(a_{E'|_{G'}}(G^\nu)-1)E'|_{G'}\big).$$

Hence $\mathscr{J}_\psi$ is resolved on G', and on T' the order along $\mathscr{J}_\psi$ takes constant value

$$e := \mathrm{ord}_{E|_{G'}} \mathscr{J}_\psi = r^{-1}\,\mathrm{ord}_E\, \tilde{\mathscr{J}}_{r,G} + a_{E|_{G'}}(G^\nu) - 1.$$

We use the following form of [6, Lemma 4.1] to estimate $\mu_F(S)$.

Proposition 2.7 *Let X be a reduced scheme of pure dimension, and L_n^X the locus of $J_\infty X$ on which the orders along the Jacobian ideal sheaf $\mathscr{J}_X'$ and the l.c.i. defect ideal sheaf $\mathscr{D}_X'$ are at most n. Then L_n^X is stable at level n.*

Proof For a l.c.i. scheme, the proposition follows from the proof of [6, Lemma 4.1] directly. Note that the l.c.i. defect ideal sheaf of a l.c.i. scheme is trivial.

For general X, we fix a jet $\gamma_n \in \pi_n^X(L_n^X)$. By the definitions of $\mathscr{J}_X', \mathscr{D}_X'$, one can embed X into a l.c.i. scheme $Y = X \cup C^Y$ as in (1) so that on a neighbourhood U_{γ_n} of γ_n in J_nY, $\mathrm{ord}_{\mathscr{J}_Y'} \le \mathrm{ord}_{\mathscr{J}_X'}(\gamma_n)$ and $\mathrm{ord}_{\mathscr{C}_{X/Y}} \le \mathrm{ord}_{\mathscr{D}_X'}(\gamma_n)$ for the Jacobian $\mathscr{J}_Y'$ and the conductor $\mathscr{C}_{X/Y}$. Then $(\pi_n^X)^{-1}(U_{\gamma_n}) \subset L_n^X$ and $(\pi_n^Y)^{-1}(U_{\gamma_n}) \subset L_n^Y$. By $\mathscr{C}_{X/Y}\mathscr{I}_{X/Y} = 0$ for the ideal sheaf $\mathscr{I}_{X/Y}$ of X on Y, we have $J_\infty Y \setminus (\mathrm{ord}_{\mathscr{C}_{X/Y}})^{-1}(\infty) \subset J_\infty X$. Hence $(\pi_n^X)^{-1}(U_{\gamma_n}) = (\pi_n^Y)^{-1}(U_{\gamma_n})$, and the statement is reduced to that of the l.c.i. scheme Y. q.e.d.

Lemma 2.8 $\mu_F(S) = \mu_G(T) = \mu_{G'}(T')\mathbb{L}^{-e}$.

Proof We apply Proposition 2.7 to $S \subset L_{c_1}^F$, $T \subset L_{c_1}^G$ by Lemma 2.6(i), (iii) and (2), to obtain their stabilities at level c_1 and by $S_{c_1} = T_{c_1}$ in (15)

$$\mu_F(S) = \mu_G(T).$$

By applying [6, Lemma 3.4] for $T \subset \mathscr{L}^{(c_1)}(G) := J_\infty G \setminus (\pi_{c_1}^G)^{-1}(J_{c_1}G_{\mathrm{sing}})$, where G_{sing} is the singular locus of G, we have $n \ge c_1, e, 1$ such that $\mathrm{ord}_{\mathscr{J}_\psi}$ takes constant value e on $\psi_n^{-1}(T_n)$, and that $\psi_n^{-1}(T_n) \to T_n$ is piecewise trivial with fibres $\mathbb{A}^e$. If the equality $T_n' = \psi_n^{-1}(T_n)$ holds, then

$$\mu_G(T) = [T_n]\mathbb{L}^{-(n+1)(d-1)} = [T_n']\mathbb{L}^{-(n+1)(d-1)-e} = \mu_{G'}(T')\mathbb{L}^{-e}.$$

Thus it suffices to prove $\psi_n^{-1}(T_n) \subset T_n'$.

Take a variety U_n dense in T_n such that $\psi_n^{-1}(U_n)$ is irreducible. The closure C_n of $\psi_n^{-1}(U_n)$ in J_nG' contains the closure $J_nG'|_{E|_{G'}}$ of T_n', which is a prime divisor. Thus

$C_n = J_nG'|_{E|_{G'}}$ by the irreducibility of C_n, so the image of the restricted morphism $\chi_n\colon J_nG'|_{E|_{G'}} \to J_nG$ contains T_n. Its fibre $\chi_n^{-1}(t)$ at $t \in T_n$ has dimension at least e and is contained in $\psi_n^{-1}(t) \simeq \mathbb{A}^e$. Hence $\chi_n^{-1}(t) = \psi_n^{-1}(t)$ as $\chi_n^{-1}(t)$ is closed. This means $\psi_n^{-1}(T_n) \subset J_nG'|_{E|_{G'}}$.

Consider on $\psi_n^{-1}(T_n)$ the constant function

$$e = \operatorname{ord}_{\mathscr{J}_\psi} = \sum_{E' \in I} (\operatorname{ord}_{E'|_{G'}} \mathscr{J}_\psi) \cdot \operatorname{ord}_{E'|_{G'}} .$$

Note that

$$\operatorname{ord}_{E|_{G'}} \mathscr{J}_\psi = e, \qquad \operatorname{ord}_{E'|_{G'}} \mathscr{J}_\psi > 0 \text{ for } E' \in I \setminus \{E\},\ E'|_{G'} \cap E|_{G'} \neq \emptyset,$$

because such $E'|_{G'}$ is ψ-exceptional and $\mathscr{J}_\psi$ vanishes on the support of $\Omega_{G'/G}$. Moreover $\operatorname{ord}_{E|_{G'}}$ is positive on $\psi_n^{-1}(T_n) \subset J_nG'|_{E|_{G'}}$. Hence $\psi_n^{-1}(T_n) \subset T'_n$ by the definition of T'. q.e.d.

Remark 2.8.1 We need only the inequality $\dim \mu_F(S) \geq \dim \mu_{G'}(T')\mathbb{L}^{-e}$ for the proof of Theorem 1.9.

We shall complete the proof by using the description below of $c = \operatorname{mld}_Z(F, \mathfrak{a}\mathscr{O}_F)$ in terms of motivic integration; see [8] and also [11, Remark 3.3].

$$c = -\dim \int_{J_\infty F|_Z} \mathbb{L}^{r^{-1}\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}} + \operatorname{ord}_\mathfrak{a}} d\mu_F. \tag{16}$$

We explain the above formula briefly. The space $J_\infty F|_Z$ is realised, up to a subset of measure zero, as a countable disjoint union of stable subsets B_i on which $r^{-1}\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}} + \operatorname{ord}_\mathfrak{a}$ takes constant value b_i. Then the motivic integration in (16) is by definition $\sum_i \mu_F(B_i)\mathbb{L}^{b_i}$ in the complete ring $\widehat{\mathscr{M}}$, and its dimension is the maximum of $\dim \mu_F(B_i)\mathbb{L}^{b_i}$ as the convergence of the summation is known. Note that $\dim \mu_F(B_i)\mathbb{L}^{b_i} = \dim[\pi^F_{n_i}(B_i)]\mathbb{L}^{-(n_i+1)(d-1)+b_i} = \dim \pi^F_{n_i}(B_i) - (n_i+1)(d-1) + b_i$ when B_i is stable at level n_i.

Lemma 2.9 *If* $(F, \mathfrak{a}) \approx_{l_3} (G, \mathfrak{b})$*, then* $\operatorname{mld}_Z(F, \mathfrak{a}\mathscr{O}_F) \leq \operatorname{mld}_{\nu^{-1}(Z)}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu})$.

Proof We have fixed an arbitrary $E \in I_Z$ which satisfies (14). By Lemma 2.6(ii), (iv), $\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}}$ and $\operatorname{ord}_\mathfrak{a}$ take constant values $\operatorname{ord}_E \tilde{\mathscr{J}}_{r,G}$ and respectively $\operatorname{ord}_E \mathfrak{b}$ on S. Thus with Lemma 2.8,

$$\begin{aligned}\int_S \mathbb{L}^{r^{-1}\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}} + \operatorname{ord}_\mathfrak{a}} d\mu_F &= \mu_F(S)\mathbb{L}^{r^{-1}\operatorname{ord}_E \tilde{\mathscr{J}}_{r,G} + \operatorname{ord}_E \mathfrak{b}} \\ &= \mu_{G'}(T')\mathbb{L}^{r^{-1}\operatorname{ord}_E \tilde{\mathscr{J}}_{r,G} + \operatorname{ord}_E \mathfrak{b} - e},\end{aligned}$$

and

$$\begin{aligned}\dim \int_{J_\infty F|_Z} \mathbb{L}^{r^{-1}\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}} + \operatorname{ord}_\mathfrak{a}} d\mu_F &\geq \dim \int_S \mathbb{L}^{r^{-1}\operatorname{ord}_{\tilde{\mathscr{J}}_{r,F}} + \operatorname{ord}_\mathfrak{a}} d\mu_F \\ &= -1 + r^{-1}\operatorname{ord}_E \tilde{\mathscr{J}}_{r,G} + \operatorname{ord}_E \mathfrak{b} - e \\ &= -a_{E|_{G'}}(G^\nu) + \operatorname{ord}_E \mathfrak{b} \\ &= -a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathscr{O}_{G^\nu}).\end{aligned}$$

Hence $a_{E|_{G'}}(G^\nu, \mathfrak{b}\mathcal{O}_{G^\nu}) \geq c$ by (16), which proves the lemma. q.e.d.

Theorem 1.9 is therefore proved.

Acknowledgements This research has originated in the problem of Professor M. Mustaţă. I was asked it at the workshop at MSRI in 2007, and discussed with him during my visit at University of Michigan in 2009. I really appreciate his approval of introducing his problem and observation. I am also grateful to him for valuable discussions and cordial hospitality. Finally, I should like to acknowledge the detailed comments of the referee.

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