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A Note on Polylinking Flow Networks

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Abstract This is a supplementary note on M. X. Goemans, S. Iwata, and R. Zenklusen's paper that proposes a flow model based on polylinking systems. Their flow model is a series (or tandem) connection of polylinking systems. We can consider an apparently more general model of a polylinking flow network which consists of an ordinary arc-capacitated network endowed with polylinking systems on the vertex set, one for each vertex of the network. This is a natural, apparent generalization of polymatroidal flow model of E. L. Lawler and C. U. Martel and of generalized-polymatroidal flow model of R. Hassin. We give a max-flow min-cut formula for the polylinking network flow problem and discuss some acyclic flow property of polylinking flows.

Keywords Linking systems · Polylinking flows · Submodular functions

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1 Introduction

M. X. Goemans, S. Iwata, and R. Zenklusen [6] proposed a flow model based on polylinking systems of A. Schrijver [9]. The present note is supplementary to their paper and points out an apparent generalization of their model, which is also a natural, apparent generalization of polymatroidal flow model of E. L. Lawler and C. U. Martel [8] and of generalized-polymatroidal flow model of R. Hassin [7]. We give a max-flow min-cut formula for the polylinking network flow problem and discuss some acyclic flow property of polylinking flows. The results are easy consequences of those in the theory of submodular functions but it may be worth noting and useful for wireless information networks [1], which motivated the work of [6]

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2 Preliminaries: Base polyhedra and polylinking systems

Let W be a nonempty set and $f : 2^W \to \mathbf{R}$ be a submodular function, i.e., f satisfies

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (\forall X,\ Y \subseteq W). \tag{1}$$

We assume $f(\emptyset) = 0$. The *base polyhedron* associated with f is defined by

$$\mathrm{B}(f) = \{x \in \mathbf{R}^W \mid \forall X \subseteq W : x(X) \leq f(X),\ x(W) = f(W)\}. \tag{2}$$

Here for simplicity we consider submodular functions on power sets (or Boolean lattices) but we can easily adapt the arguments in this note to submodular functions on ring families (or distributive lattices). A vector $x \in \mathrm{B}(f)$ is called a *base*. For any base $x \in \mathrm{B}(f)$ and $u \in W$ we define $\mathrm{dep}(x,u)$ by

$$\mathrm{dep}(x,u) = \{v \in W \mid \exists \alpha > 0 : x + \alpha(\chi_u - \chi_v) \in \mathrm{B}(f)\}, \tag{3}$$

where for any $w \in W$ χ_w is the unit vector such that $\chi_w(w) = 1$ and $\chi_w(s) = 0$ for all $s \in W \setminus \{w\}$. In other words, when $v \in \mathrm{dep}(x,u) \setminus \{u\}$, we can increase $x(u)$ and at the same time decrease $x(v)$ by some positive amount without leaving the base polyhedron $\mathrm{B}(f)$. The function $\mathrm{dep} : \mathrm{B}(f) \times W \to 2^W$ is called the *dependence function*. Moreover, for any $v \in \mathrm{dep}(x,u) \setminus \{u\}$ define

$$\tilde{\mathrm{c}}(x,u,v) = \max\{\alpha \in \mathbf{R} \mid x + \alpha(\chi_u - \chi_v) \in \mathrm{B}(f)\}, \tag{4}$$

which is called the *exchange capacity* from v to u associated with base x. Dependence functions and exchange capacities will appear only in Section 4.3. For more details about the theory of submodular functions see [5].

For any vector $x \in \mathbf{R}^W$ and any subset U of W define x^U to be the vector in $\mathbf{R}^U$ such that $x^U(u) = x(u)$ for all $u \in U$, which is the *restriction* of x to U. For any disjoint nonempty subsets $U_1, U_2 \subset W$ and any vectors $x \in \mathbf{R}^{U_1}$ and $y \in \mathbf{R}^{U_2}$ denote by $x \oplus y$ the vector in $\mathbf{R}^{U_1 \cup U_2}$ such that $(x \oplus y)(u) = x(u)$ for all $u \in U_1$ and $(x \oplus y)(u) = y(u)$ for all $u \in U_2$.

Suppose that $f(W) = 0$ and $f(X) \geq 0$ for all $X \subseteq W$, which implies $\mathbf{0} \in \mathrm{B}(f)$. We assume this property for all submodular functions appearing in the sequel. Let (U_1, U_2) be an ordered pair of nonempty subsets of W such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = W$. We call it an *ordered proper bisection* of W. Consider a *reflection by* U_1 of the base polyhedron given by

$$\mathrm{B}_{(U_1,U_2)}(f) = \{y \mid x \in \mathrm{B}(f),\ y^{U_1} = -x^{U_1},\ y^{U_2} = x^{U_2}\}. \tag{5}$$

Then the triple $(U_1, U_2, \mathrm{B}_{(U_1,U_2)}(f))$ is a *polylinking system* and $\mathrm{B}_{(U_1,U_2)}(f)$ is the associated *polylinking polyhedron*. We call f the *submodular function associated with the polylinking system*. Here we define a polylinking system by means of a submodular function (cf. [5, Sec. 3.5(b)]). (The original polylinking system introduced by Schrijver [9] is the restriction of $\mathrm{B}_{(U_1,U_2)}(f)$ to the nonnegative orthant $\mathbf{R}^W_+$.) For any $y \in \mathrm{B}_{(U_1,U_2)}(f)$ we say y^{U_1} is *linked to* y^{U_2}, and (y^{U_1}, y^{U_2}) is called a *pair of linked vectors*. Note that $y(U_1) = y(U_2)$ since $f(W) = 0$ by definition.

3 The Polylinking Flow Model of Goemans, Iwata, and Zenklusen

Now let us give a description of the polylinking flow model of Goemans, Iwata, and Zenklusen [6] for completeness of the presentation. Consider nonempty disjoint sets V_i $(i = 1,\cdots,r)$ with an integer $r \geq 2$ and polylinking systems (V_i, V_{i+1}, L_i) $(i = 1,\cdots,r-1)$. The pair (V,L), where $V = (V_1,\cdots,V_r)$ and $L = (L_1,\cdots,L_{r-1})$, is called a *polylinking flow model* in [6]. It is a series (or tandem) connection of polylinking systems. A *flow* in the polylinking flow model (V,L) is a tuple $x = (x_1,\cdots,x_r)$ such that (x_i, x_{i+1}) is a pair of linked vectors in L_i for all $i = 1,\cdots,r-1$ and x_i is nonnegative for all $i = 1,\cdots,r$. Note that we always have a feasible flow consisting of zero linked vectors. We have a common value $x_1(V_1) = \cdots = x_r(V_r)$, which is called the *value* of flow $x = (x_1,\cdots,x_r)$.

Goemans, Iwata, and Zenklusen [6] considered a problem of finding a flow of maximum value in the polylinking flow model, showed a min-max formula, and gave an efficient algorithm for finding a maximum flow in the polylinking flow model by reducing the problem to a submodular flow problem of J. Edmonds and R. Giles [2] and by employing an efficient algorithm for submodular flows such as L. Fleischer and Iwata's [3] together with the fast Fourier transformation on finite fields.

4 Polylinking Flow Networks

Goemans, Iwata, and Zenklusen [6] considered a series (or tandem) connection of polylinking systems. We can consider an apparently more general model which consists of an ordinary arc-capacitated network endowed with polylinking systems, one for each vertex of the network. This is a natural, apparent generalization of a polymatroidal flow model of Lawler and Martel [8] and that of a generalized-polymatroidal flow of Hassin [7].

4.1 Definition of a (general) polylinking flow network

Let $G = (V,A)$ be a directed graph with a vertex set V and an arc set A, and let $\underline{c} : A \to \mathbf{R} \cup \{-\infty\}$ and $\bar{c} : A \to \mathbf{R} \cup \{+\infty\}$ be lower and upper capacity functions on arc set A such that $\underline{c}(a) \leq \bar{c}(a)$ for all $a \in A$. For each vertex $v \in V$ we are given a polylinking system $(\delta^- v, \delta^+ v, L_v)$, where let $f_v : 2^{\delta^- v \cup \delta^+ v} \to \mathbf{R}$ be the submodular function associated with the polylinking polyhedron L_v. (For any vertex $v \in V$, $\delta^- v$ denotes the set of arcs in G whose terminal vertices are v, and $\delta^+ v$ the set of arcs in G whose initial vertices are v.) We call $\mathscr{N} = (G, c, \boldsymbol{L})$ a *polylinking flow network*, where $\boldsymbol{L} = (L_v \mid v \in V)$.

A *feasible flow* (or a *polylinking flow*) in the polylinking network $\mathscr{N} = (G, c, \boldsymbol{L})$ is a function $\varphi : A \to \mathbf{R}$ that satisfies the following.

$$\underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \qquad (\forall a \in A), \tag{6}$$

$$\varphi^{\delta^- v \cup \delta^+ v} \in L_v \qquad (\forall v \in V), \tag{7}$$

where recall that φ^F for $F \subseteq A$ is the restriction of φ to F.

Remark 1 A polymatroidal flow network of Lawler and Martel [8] is a special case of a polylinking flow network where each polylinking polyhedron L_v for $v \in V$ is a composition of polymatroids on $\delta^- v$ and on $\delta^+ v$, which is defined as follows. For two polymatroid polyhedra $P_1 \subset \mathbf{R}^{S_1}$ and $P_2 \subset \mathbf{R}^{S_2}$ with $S_1 \cap S_2 = \emptyset$ define a polytope $L(P_1, P_2) \subset \mathbf{R}^{S_1 \cup S_2}$ by

$$L(P_1, P_2) = \{x_1 \oplus x_2 \mid x_1 \in P_1,\ x_2 \in P_2,\ x_1(S_1) = x_2(S_2)\}. \tag{8}$$

We can see that the reflection of $L(P_1, P_2)$ by S_1 is a base polyhedron, and hence $L(P_1, P_2)$ gives a polylinking polyhedron. Also Hassin [7] considered a polylinking flow network when each polylinking polyhedron L_v is a composition of generalized polymatroids [4] on $\delta^- v$ and on $\delta^+ v$ for $v \in V$, which is defined similarly as above by replacing polymatroids by generalized polymatroids.

These facts can be understood as follows. Let $P_1 \subset \mathbf{R}^{S_1}$ and $P_2 \subset \mathbf{R}^{S_2}$ be generalized polymatroids. We embed P_1 (resp. P_2) in $\mathbf{R}^{S_1 \cup S_2}$ by taking the direct sum of P_1 (resp. P_2) with the zero vector in $\mathbf{R}^{S_2}$ (resp. $\mathbf{R}^{S_1}$). Then consider a new element e_0 commonly used for P_1 and P_2, put $T = S_1 \cup S_2 \cup \{e_0\}$, and let $B_1 \subset \mathbf{R}^T$ and $B_2 \subset \mathbf{R}^T$ be, respectively, the base polyhedra lying in the hyperplane $x(T) = 0$ such that the projection along the axis e_0 into the hyperplane $x(e_0) = 0$ are P_1 and P_2 (after being restricted on S_1 and S_2) [5, Sec. 3.5(a)]. Then, the Minkowski sum of $-B_1$ and B_2 is a base polyhedron, denoted by $B_{1,2}$, in $\mathbf{R}^T$, where $-B_1 = \{-x \mid x \in B_1\}$ is also a base polyhedron. Taking a section of $B_{1,2}$ by the hyperplane $x(e_0) = 0$ and restricting it to $T \setminus \{e_0\} = S_1 \cup S_2$, we get a base polyhedron $\hat{B}$ in $\mathbf{R}^{S_1 \cup S_2}$. Finally, by the reflection of $\hat{B}$ by S_1 we obtain the polylinking polyhedron $L(P_1, P_2)$ defined by (8) for generalized polymatroids $P_1 \subset \mathbf{R}^{S_1}$ and $P_2 \subset \mathbf{R}^{S_2}$. □

4.2 Equivalence between polylinking flows and submodular flows

Now we show that any polylinking flow network can be reduced to a submodular flow network. The reduction technique given below is the same as the one shown in [5, Sec. 5.2(c)], where polymatroids with the flow conservation are considered instead of polylinking systems.

Given a graph $G = (V, A)$, lower and upper capacity functions $\underline{c} : A \to \mathbf{R}$ and $\bar{c} : A \to \mathbf{R}$ with $\underline{c}(a) \le \bar{c}(a)$ for all $a \in A$, and a submodular function $f : 2^V \to \mathbf{R}$ with $f(\emptyset) = f(V) = 0$, a *submodular flow* is a function $\varphi : A \to \mathbf{R}$ that satisfies

$$\underline{c}(a) \le \varphi(a) \le \bar{c}(a) \qquad (\forall a \in A), \tag{9}$$

$$\partial \varphi \in \mathrm{B}(f), \tag{10}$$

where $\partial \varphi$ is the *boundary* of flow φ defined by $\partial \varphi(v) = \sum_{a \in \delta^+ v} \varphi(a) - \sum_{a \in \delta^- v} \varphi(a)$ for all $v \in V$.

For any polylinking flow network $\mathscr{N} = (G = (V, A), \underline{c}, \bar{c}, \boldsymbol{L} = (L_v \mid v \in V))$ with associated submodular functions $f_v : 2^{\delta^- v \cup \delta^+ v} \to \mathbf{R}$ for all $v \in V$, construct a submodular flow network $\mathscr{N}_0 = (G_0 = (V_0, A), \underline{c}, \bar{c}, f_0)$ as follows.

$$V_0 = \bigcup_{v \in V} (W_v^- \cup W_v^+), \tag{11}$$

$$W_v^- = \{u_a^- \mid a \in \delta^- v\}, \quad W_v^+ = \{u_a^+ \mid a \in \delta^+ v\} \quad (\forall v \in V), \tag{12}$$

where $\delta^{\pm}$ are those defined with respect to G. Each arc $a \in A$ of G_0 has its head u_a^- and tail u_a^+. (Note that the arc set A is common to G and G_0 while the vertex sets V and V_0 are different as well as the vertex-arc incidence relations.) Moreover,

$$f(U) = \sum_{v \in V} \bar{f}_v((W_v^- \cup W_v^+) \cap U) \quad (\forall U \subseteq V_0), \tag{13}$$

where $\bar{f}_v$ is the submodular function on $2^{W_v^- \cup W_v^+}$ that is identified with f_v by the natural correspondence between $\delta^- v \cup \delta^+ v$ and $W_v^- \cup W_v^+$. It is easy to see that $\varphi : A \to \mathbf{R}$ is a polylinking flow in $\mathscr{N}$ if and only if φ is a submodular flow in $\mathscr{N}_0$.

Remark 2 Similarly as in [5, Sec. 5.2] we can show that any submodular flow network can be reduced to a polylinking flow network. Hence these two models are equivalent. That is, the polylinking flow problem is what is called a *neoflow problem* in [5, Sec. 5]. From now on we consider both networks $\mathscr{N}$ and $\mathscr{N}_0$ and identify a polylinking flow φ in $\mathscr{N}$ with its corresponding submodular flow φ in $\mathscr{N}_0$. Note that the two flows are the same function on A. □

Suppose that we are given a reference arc $a_0 \in A$. Then we have a max-flow min-cut theorem as follows (see [5, Theorem 5.11]).

Theorem 1 *Suppose that there exists a feasible polylinking flow in $\mathscr{N}$ (or equivalently a feasible submodular flow in $\mathscr{N}_0$). Then we have*

$$\begin{aligned} &\max\{\varphi(a_0) \mid \varphi : \text{ a feasible polylinking flow in } \mathscr{N}\} \\ &= \min[\, \bar{c}(a_0), \\ &\qquad \min\{\bar{c}(\Delta^+ X) - \underline{c}(\Delta^- X \setminus \{a_0\}) + f(V \setminus X) \mid X \subseteq V_0,\ a_0 \in \Delta^- X\}\,], \end{aligned} \tag{14}$$

where operators $\Delta^{\pm}$ appearing in the right-hand side are defined with respect to graph $G_0 = (V_0, A)$ for network $\mathscr{N}_0$ ($\Delta^+ X$ is the set of arcs leaving X and $\Delta^- X$ the set of arcs entering X). Moreover, if $\underline{c}$, $\bar{c}$, and f are integer-valued, then there exists an integral maximum polylinking flow in $\mathscr{N}$ (with respect to reference arc a_0). □

4.3 Existence of acyclic polylinking flows of given flow value

It is well-known that for any two-terminal flow φ in a classical flow network there exists a two-terminal flow ψ such that the two flow values are the same, flow ψ is φ-equisignum (i.e. $\psi(a) > 0$ implies $\varphi(a) > 0$ and $\psi(a) < 0$ implies $\varphi(a) < 0$), and the network restricted on the support of ψ is acyclic. We consider such a property for polylinking flows.

For simplicity let us assume $\underline{c}(a) = 0$ for all $a \in A$. Let φ be a feasible flow in network $\mathscr{N}_0$. Define an *auxilliary graph* $G_\varphi = (V_0, A_\varphi)$ as follows.

$$A_\varphi = A_\varphi^+ \cup \bigcup_{v \in V} D_\varphi^v, \tag{15}$$

$$A_\varphi^+ = \{a \mid a \in A,\ \varphi(a) > 0\}, \tag{16}$$

$$D_\varphi^v = \{(u_a^-, u_b^+) \mid a \in \delta^- v,\ b \in \delta^+ v,\ b \in \mathrm{dep}_v((-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v}, a)\} \quad (\forall v \in V), \tag{17}$$

where $\delta^{\pm}$ are those defined with respect to graph $G = (V, A)$ for network $\mathscr{N}$ and dep_v for $v \in V$ is the *dependence function* associated with f_v and a base $(-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v} \in \mathrm{B}(f_v)$. (Recall that $b \in \mathrm{dep}_v((-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v}, a)$ means that we can decrease $\varphi(a)$ and $\varphi(b)$ by some (and the same) amount $\alpha > 0$ while keeping $\varphi^{\delta^- v \cup \delta^+ v} \in L_v$. The maximum of such values α is called the *exchange capacity* from b to a with respect to base $(-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v} \in \mathrm{B}(f_v)$ and is denoted by $\tilde{c}_v((-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v}, a, b)$.)

The following algorithmic property is well-known [5, Sec. 5.5].

- If there exists a directed cycle in the auxiliary graph G_φ, let Q be one of such directed cycles, regarded as a subset of A_φ, that do not have any short-cuts. Then we can obtain a new feasible flow φ' by

$$\varphi'(a) = \begin{cases} \varphi(a) - \alpha & (a \in Q) \\ \varphi(a) & (a \in A \setminus Q), \end{cases} \tag{18}$$

where α is a positive number less than or equal to

$$\min\{\min\{\varphi(a) \mid a \in Q \cap A_\varphi^+\},$$
$$\min\{\tilde{c}_v((-\varphi^{\delta^- v}) \oplus \varphi^{\delta^+ v}, u_a^-, u_b^+) \mid (u_a^-, u_b^+) \in Q \cap D_\varphi^v, v \in V\}\}.$$

Hence, given a feasible flow φ in $\mathscr{N}_0$, reducing flows along directed cycles not containing reference arc a_0, we can obtain a feasible flow ψ of the same flow value as φ such that ψ is φ-equisignum and the auxiliary graph G_ψ has no directed cycle Q with $a_0 \notin Q$, as is shown in the next theorem.

We have the following acyclic reduction property of polylinking flows in general polylinking networks.

Theorem 2 *Given a feasible flow φ in polylinking network $\mathscr{N}$ with a reference arc a_0 and $\underline{c} = \mathbf{0}$, there exists a feasible flow ψ that has the same flow value as φ and is φ-equisignum and there exists no directed cycle Q with $a_0 \notin Q$ in the auxiliary graph G_ψ.*

(Proof) Re-define

$$\bar{c}(a) = \varphi(a) \qquad (\forall a \in A), \tag{19}$$

$$\underline{c}(a_0) = \varphi(a_0). \tag{20}$$

Moreover, consider a cost function $\gamma : A \to \mathbf{R}$ such that $\gamma(a) = 1$ for all $a \in A$. Then let ψ be the minimum-cost submodular flow in $\mathscr{N}_0$ with the upper and lower capacity functions and the cost function defined as above. The optimality of ψ [5, Sec. 5.4] implies that there does not exist any directed cycle in the auxiliary graph G_ψ for the re-defined network $\mathscr{N}_0$. It follows that the flow ψ satisfies the condition of the statement in the present theorem. □

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