<table>
<thead>
<tr>
<th>Title</th>
<th>Samelson products in p-regular exceptional Lie groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hasui, Sho; Kishimoto, Daisuke; Ohsita, Akihiro</td>
</tr>
<tr>
<td>Citation</td>
<td>Topology and its Applications (2014), 178: 17-29</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/189893">http://hdl.handle.net/2433/189893</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 2014 Elsevier B.V.; This is not the published version. Please cite only the published version. この論文は出版社版でありません。引用の際には出版社版をご確認ご利用ください。</td>
</tr>
<tr>
<td>Type</td>
<td>Journal Article</td>
</tr>
<tr>
<td>TextVersion</td>
<td>author</td>
</tr>
<tr>
<td>Metadata</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
Samelson products in $p$-regular exceptional Lie groups

Sho Hasui$^{a,*}$, Daisuke Kishimoto$^{a,1}$, Akihiro Ohsita$^{b}$

$^a$Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
$^b$Faculty of Economics, Osaka University of Economics, Osaka 533-8533, Japan

Abstract

The (non)triviality of Samelson products of the inclusions of the spheres into $p$-regular exceptional Lie groups is completely determined, where a connected Lie group is called $p$-regular if it has the $p$-local homotopy type of a product of spheres.

Keywords: exceptional Lie group, Samelson product, Weyl group invariant

2010 MSC: Primary 55Q15; Secondary 57T10

1. Introduction and statement of the result

For a homotopy associative H-space with inverse $X$, the correspondence $X \wedge X \to X$, $(x, y) \mapsto x y x^{-1} y^{-1}$ induces a binary operation

$$\langle -,- \rangle : \pi_i(X) \otimes \pi_j(X) \to \pi_{i+j}(X)$$

called the Samelson product in $X$. We consider the basic Samelson products in $p$-regular Lie groups. Let $G$ be a compact simply connected Lie group. By the Hopf theorem, $G$ has the rational homotopy type of the product $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$, where $n_1 \leq \cdots \leq n_\ell$. The sequence $n_1, \ldots, n_\ell$ is called the type of $G$ and is denoted by $t(G)$. We here list the types of exceptional Lie groups.
We say that $G$ is $p$-regular if it has the $p$-local homotopy type of a product of spheres. By the classical result of Serre, it is known that $G$ is $p$-regular if and only if $p \geq n_\ell$, in which case

$$G_{(p)} \simeq S_{(p)}^{2n_1-1} \times \cdots \times S_{(p)}^{2n\ell-1}.$$  

Suppose that $G$ is $p$-regular, and let $\epsilon_{2n_i-1}$ be the composite

$$S^{2n_i-1} \xrightarrow{\text{incl}} S_{(p)}^{2n_i-1} \times \cdots \times S_{(p)}^{2n\ell-1} \simeq G_{(p)}$$

where if there are more than one $i$ in $\mathfrak{t}(G)$, we distinguish the corresponding $\epsilon_{2i-1}$ but not write it explicitly. The Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ are fundamental in studying the homotopy (non)commutativity of $G_{(p)}$ as in [KK] and its applications (See [KKTh, KKTs, Th], for example). So we would like to determine their (non)triviality. In [B], Bott computes the Samelson products in the classical groups $U(n)$ and $Sp(n)$. Then by combining with the information of the $p$-primary component of the homotopy groups of spheres [To], the (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is completely determined when $G = SU(n), Sp(n), Spin(2n+1)$, where $Sp(n)_{(p)} \simeq Spin(2n + 1)_{(p)}$ as loop spaces by [F] since $p$ is odd. For example, when $G = SU(n)$ and $p \geq n$, the type of $G$ is given by $2, \ldots, n$ and

$$\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle \neq 0 \quad \text{if and only if} \quad i + j > p.$$  

So apart from $\text{Spin}(2n)$, all we have to consider is the exceptional Lie groups. The (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is known only in a few cases, and the most general result so far is:

**Theorem 1.1** (Hamanaka and Kono [HK]). Let $G$ be a $p$-regular exceptional Lie group. If $i, j \in \mathfrak{t}(G)$ satisfy $i + j = p + 1$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.

**Remark 1.2.** The Samelson products in $G_2$ are first computed in [O], and some more Samelson products in $E_7$ and $E_8$ are computed in [KK].

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathfrak{t}(G)$</th>
<th>$G$</th>
<th>$\mathfrak{t}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2$</td>
<td>2, 6</td>
<td>$E_6$</td>
<td>2, 5, 6, 8, 9, 12</td>
</tr>
<tr>
<td>$F_4$</td>
<td>2, 6, 8, 12</td>
<td>$E_7$</td>
<td>2, 6, 8, 10, 12, 14, 18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_8$</td>
<td>2, 8, 12, 14, 18, 20, 24, 30</td>
</tr>
</tbody>
</table>
Based on this result, Kono posed the following conjecture (in a private communication).

**Conjecture 1.3.** Let $G$ be a $p$-regular exceptional Lie group. For $i, j \in \mathfrak{t}(G)$, there exists $k \in \mathfrak{t}(G)$ satisfying $i + j = k + p - 1$ if and only if $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.

Notice that the only if part of the conjecture follows immediately from the information of the $p$-primary component of the homotopy groups of spheres $[To]$ (cf. [KK]). We will prove the if part and obtain:

**Theorem 1.4.** Conjecture 1.3 is true.

The paper is structured as follows. In \S 2, we reduce the nontriviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ in the $p$-regular Lie group $G$ to a certain condition of the Steenrod operation $P^1$ on the mod $p$ cohomology of the classifying space $BG$. Then for a $p$-regular exceptional Lie group $G$, we compute the mod $p$ cohomology of $BG$ as the ring of invariants of the Weyl group of $G$. With this description of the mod $p$ cohomology of $BG$, we compute the action of $P^1$ on it. In \S 3, we prove that the above condition on $P^1$ is satisfied to complete the proof of Theorem 1.4.

2. **Mod $p$ cohomology of $BG$**

2.1. **Reduction**

Let $G$ be a compact simply connected Lie group. We first reduce Theorem 1.4 to the action of the Steenrod operation $P^1$ on the mod $p$ cohomology of the classifying space $BG$ as in [HK, KK]. Recall that if the integral homology of $G$ has no $p$-torsion, the mod $p$ cohomology of the classifying space $BG$ is given by

$$H^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[x_{2i} | i \in \mathfrak{t}(G)], \quad |x_j| = j. \quad (1)$$

When there are more than one $i$ in $\mathfrak{t}(G)$, we distinguish corresponding $x_{2i}$ but do not write it explicitly as in the case of $\epsilon_{2i-1}$ in the preceding section.

**Lemma 2.1.** Suppose that $G$ is $p$-regular. For $i, j \in \mathfrak{t}(G)$, if there is $k \in \mathfrak{t}(G)$ such that $P^1x_{2k}$ involves $\lambda x_{2i}x_{2j}$ with $\lambda \neq 0$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.
Proof. Let $\bar{\epsilon}_{2i} : S^{2i} \to BG(p)$ be the adjoint of $\epsilon_{2i-1}$ for $i \in t(G)$, and so we may assume that $\bar{\epsilon}_{2i}^*(x_{2i}) = u_{2i}$ for a generator $u_{2i}$ of $H^{2i}(S^{2i};\mathbb{Z}/p)$. Assume that the Samelson product $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is trivial, which is equivalent to the triviality of the Whitehead product $[\bar{\epsilon}_{2i}, \bar{\epsilon}_{2j}]$ by the adjointness of Samelson products and Whitehead products. Then the map $\bar{\epsilon}_{2i} \lor \bar{\epsilon}_{2j} : S^{2i} \lor S^{2j} \to BG(p)$ extends to a map $\mu : S^{2i} \times S^{2j} \to BG(p)$, up to homotopy. Hence since $P^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$, we have

$$\mu^*(P^1 x_{2k}) = \mu^*(\lambda x_{2i} x_{2j}) = \lambda u_{2i} \times u_{2j} \neq 0.$$ 

On the other hand, by the naturality of $P^1$, we also have

$$\mu^*(P^1 x_{2k}) = P^1 \mu^*(x_{2k}) = 0$$

since $P^1$ is trivial on $H^*(S^{2i} \times S^{2j};\mathbb{Z}/p)$, which is a contradiction. Therefore the proof is completed.

By Lemma 2.1, we obtain the if part of Theorem 1.4 by the following.

**Theorem 2.2.** Let $G$ be a $p$-regular exceptional Lie group. If $i, j, k \in t(G)$ satisfy $i + j = k + p - 1$, $P^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$.

The rest of this paper is devoted to prove Theorem 2.2.

### 2.2. Generators

In this subsection, we choose generators of the mod $p$ cohomology of $BG$. We set notation. Hereafter, let $p$ be a prime greater than 5. Recall that the integral homology of $G$ is $p$-torsion free for $p > 5$, and so the mod $p$ cohomology of $BG$ is given as (1). For a homomorphism $\rho : H \to K$ between Lie groups, we denote the induced map $BH \to BK$ ambiguously by $\rho$.

We first choose generators of the mod $p$ cohomology of $BE_8$. Let $T$ be a maximal torus of $E_8$. Then as in [MT], since $p > 5$, the inclusion $T \to E_8$ induces an isomorphism

$$H^*(BE_8;\mathbb{Z}/p) \cong H^*(BT;\mathbb{Z}/p)^{W(E_8)},$$

where the right hand side is the ring of invariants of the Weyl group $W(E_8)$. We calculate invariants of $W(E_8)$ through a maximal rank subgroup of $E_8$. Let $\epsilon_1, \ldots, \epsilon_8$ be the standard basis of $\mathbb{R}^8$ which is regarded as the Lie algebra of $T$. As in [MT], we choose simple roots of $E_8$ as

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \quad \alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_3 = \epsilon_{i-1} - \epsilon_{i-2} \quad (3 \leq i \leq 8),$$

and
by which the extended Dynkin diagram of $E_8$ is described as

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & -\tilde{\alpha} \\
\end{array}
\]

where $\tilde{\alpha}$ is the dominant root. Removing $\alpha_1$ from the diagram, we get the maximal rank subgroup of $E_8$ which is of type $D_8$. Then there is a homomorphism $\rho_1 : \text{Spin}(16) \to E_8$ which induces a monomorphism

\[\rho_1^* : H^*(BE_8; \mathbb{Z}/p) \to H^*(B\text{Spin}(16); \mathbb{Z}/p).\]

By putting $t_1 = -\epsilon_1$, $t_8 = -\epsilon_8$ and $t_i = \epsilon_i$ ($2 \leq i \leq 7$), $H^*(BT; \mathbb{Z}/p)$ is identified with the polynomial ring $\mathbb{Z}/p[t_1, \ldots, t_8]$. Let $c_i$ and $p_i$ be the $i$-th elementary symmetric functions in $t_1, \ldots, t_8$ and in $t_1^2, \ldots, t_8^2$, respectively. As in (2), we have an isomorphism

\[H^*(B\text{Spin}(16); \mathbb{Z}/p) \cong \mathbb{Z}[t_1, \ldots, t_8]^{W(D_8)} = \mathbb{Z}/p[p_1, \ldots, p_7, c_8],\]

and then since $W(E_8)$ is generated by $W(D_8)$ and the reflection $\varphi$ corresponding to the simple root $\alpha_1$, it follows from (2) that

\[H^*(BE_8; \mathbb{Z}/p) \cong \mathbb{Z}/p[p_1, \ldots, p_7, c_8] \cap \mathbb{Z}/p[t_1, \ldots, t_8]^\varphi. \quad (3)\]

Hence generators of $H^*(BE_8; \mathbb{Z}/p)$ are chosen as elements of $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ which are invariant under $\varphi$. In [HK], the action of $\varphi$ on $p_1, \ldots, p_8, c_8 \in \mathbb{Z}/p[t_1, \ldots, t_8]$ is described as

\[\varphi(p_1) = p_1, \quad \varphi(p_i) \equiv p_i + h_i c_1, \quad \varphi(c_8) \equiv c_8 - \frac{1}{4}c_7 c_1 \mod (c_1^2)\]

for $2 \leq i \leq 8$, where

\[
\begin{align*}
    h_2 &= \frac{3}{2}c_3, &
    h_3 &= -\frac{1}{2}(5c_5 + c_3c_2), &
    h_4 &= \frac{1}{2}(7c_7 + 3c_5c_2 - c_4c_3), \\
    h_5 &= -\frac{1}{2}(5c_7c_2 - 3c_6c_3 + c_5c_4), &
    h_6 &= -\frac{1}{2}(5c_8c_3 - 3c_7c_4 + c_6c_5), &
    h_7 &= \frac{1}{2}(3c_8c_5 - c_7c_6).
\end{align*}
\]
We put

\[ \hat{x}_4 = p_1, \]

\[ \hat{x}_{16} = 12p_4 - \frac{18}{5}p_5p_1 + p_2^2 + \frac{1}{10}p_2p_1^2 + 168c_8, \]

\[ \hat{x}_{24} = 60p_6 - 5p_5p_1 - 5p_4p_2 + 3p_2^2 - p_3p_2p_1 + \frac{5}{36}p_2^3 + 110c_8p_2, \]

\[ \hat{x}_{28} = 480p_7 + 40p_5p_2 - 12p_4p_3 - p_3p_2^2 - 3p_4p_2p_1 + \frac{24}{5}p_2^2p_1 + \frac{11}{36}p_2^3p_1 + 312c_8p_3 - 82c_8p_2p_1, \]

\[ \hat{x}_{36} = 480p_7p_2 + 72p_6p_3 - 30p_5p_4 - \frac{25}{2}p_5p_2^2 + 9p_4p_3p_2 - \frac{18}{5}p_3^3 - \frac{1}{4}p_3p_2^3 + 1020c_8p_5 + 102c_8p_3p_2 \]

\[ - 42p_6p_2p_1 + 9p_5p_3p_1 - \frac{3}{2}p_4p_2p_1 + \frac{9}{5}p_3p_2p_1 + \frac{1}{24}p_2^2p_1 - 330c_8p_4p_1 - \frac{89}{2}c_8p_2p_1 - 300c_8p_1 \]

\[ + \frac{89}{4}p_5p_2p_1^2 - \frac{15}{2}p_4p_3p_1^2 - \frac{11}{20}p_3p_2p_1^2 + 156c_8p_3p_1^2 + \frac{5}{16}p_4p_2p_1^2 + \frac{9}{8}p_5p_1^3 + \frac{27}{320}p_3p_1^3 \]

\[ - \frac{383}{8}c_8p_2p_1^3 - \frac{195}{32}p_5p_1^4 - \frac{13}{64}p_3p_2p_1^4 - \frac{7}{192}p_2^2p_1^5 + \frac{195}{32}c_8p_1^5 + \frac{3}{32}p_3p_1^6 - \frac{1}{1024}p_2p_1^7, \]

\[ \hat{x}_{40} = 480p_7p_3 + 50p_6p_2^2 + 50p_5^2 - 10p_5p_3p_2 - \frac{25}{2}p_2^2p_2 + 9p_4p_2^2 - \frac{25}{36}p_4p_2^3 + \frac{3}{4}p_5p_2^2 + \frac{25}{864}p_5^2 \]

\[ + 2400c_8p_6 + 250c_8p_4p_2 + 3550c_8^2p_2 + 6c_8p_3^2 - \frac{175}{18}c_8p_2^3, \]

\[ \hat{x}_{48} = -200p_7p_5 - 60p_7p_3p_2 + 3p_6p_3^2 + \frac{25}{9}p_6p_2^2 + \frac{25}{3}p_5p_2^2 - \frac{5}{2}p_5p_3p_2 - \frac{25}{24}p_5p_3p_2^2 - \frac{25}{48}p_3p_2^2 \]

\[ + p_4p_5p_2 + \frac{25}{864}p_4p_2^2 - \frac{3}{10}p_3^3 - \frac{1}{36}p_2^3p_2 - \frac{25}{62208}p_2^6 - 400c_8p_6p_2 - 115c_8p_5p_3 - \frac{25}{12}c_8p_4p_2^2 \]

\[ + 3c_8p_3p_2 + \frac{25}{27}c_8p_2^2 + 75c_8p_1^2 - 300c_8^2p_4 - \frac{1525}{12}c_8^2p_2^2 + 300c_8^3. \]

We shall prove that the elements \( \hat{x}_i \) are invariant under \( \varphi \) and algebraically independent, implying that they are generators of \( H^4(B\mathcal{E}_8; \mathbb{Z}/p) \) through the isomorphism (3). Hamanaka and Kono \([HK]\) calculate \( \varphi \)-invariants in dimension 4, 16 and 24 as follows.

**Proposition 2.3** (Hamanaka and Kono \([HK]\)). Let \( \bar{x}_i \in \mathbb{Z}/p[p_1, \ldots, p_7, c_8] \) with \( |\bar{x}_i| = i \).

1. If \( \varphi(\bar{x}_i) \equiv \bar{x}_i \mod (c_7^2) \) in \( \mathbb{Z}/p[t_1, \ldots, t_8] \) for \( i = 4, 16 \), then

\[ \bar{x}_4 = \alpha \hat{x}_4 \quad \text{and} \quad \bar{x}_{16} = \beta \hat{x}_{16} + \gamma \hat{x}_4^4 \quad (\alpha, \beta, \gamma \in \mathbb{Z}/p). \]
2. If $\varphi(\bar{x}_{24}) \equiv \bar{x}_{24} \mod (c_1^2, c_2^2)$ in $\mathbb{Z}/p[t_1, \ldots, t_8]$, then

$$\bar{x}_{24} \equiv \alpha \hat{x}_{24} \ (\alpha \in \mathbb{Z}/p).$$

We further calculate $\varphi$-invariants in dimension 28, 36, 40, 48, where a partial calculation in dimension 28 is given in [KK].

**Proposition 2.4** (cf. [KK]). Let $\bar{x}_i \in \mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ with $|\bar{x}_i| = i$.

1. If $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \mod (c_1^2, c_2^2)$ in $\mathbb{Z}/p[t_1, \ldots, t_8]$, then

$$\bar{x}_{28} \equiv \alpha \hat{x}_{28} + \beta \hat{x}_4 \hat{x}_{24} \mod (p_1^2) \ (\alpha, \beta \in \mathbb{Z}/p).$$

2. If $\varphi(\bar{x}_{36}) \equiv \bar{x}_{36} \mod (c_1^2) \mod (c_1^2)$ in $\mathbb{Z}/p[t_1, \ldots, t_8]$, then

$$\bar{x}_{36} = \alpha_1 \hat{x}_{36} + \alpha_2 \hat{x}_4 \hat{x}_{16}^2 + \alpha_3 \hat{x}_4^2 \hat{x}_{28} + \alpha_4 \hat{x}_4^3 \hat{x}_{24} + \alpha_5 \hat{x}_4^5 \hat{x}_{16} + \alpha_6 \hat{x}_{16}^9 \ (\alpha_i \in \mathbb{Z}/p).$$

3. If $\varphi(\bar{x}_i) \equiv \bar{x}_i \mod (c_1^2, c_2^2)$ in $\mathbb{Z}/p[t_1, \ldots, t_8]$ for $i = 40, 48$, then

$$\hat{x}_{40} \equiv \alpha_1 \hat{x}_{40} + \alpha_2 \hat{x}_{24} \hat{x}_{16}, \quad \hat{x}_{48} \equiv \beta_1 \hat{x}_{48} + \beta_2 \hat{x}_{24}^2 + \beta_3 \hat{x}_{16}^3 \mod (p_1) \ (\alpha_i, \beta_i \in \mathbb{Z}/p).$$

**Proof.** The proof is the same as Proposition 2.3 given in [HK], and we only consider $\bar{x}_{28}$ since other cases are analogous. Excluding the indeterminacy $\hat{x}_4 \hat{x}_{24}$, we may suppose that $\bar{x}_{28}$ is a linear combination

$$\lambda_1 p_7 + \lambda_2 p_5 p_2 + \lambda_3 p_4 p_3 + \lambda_4 p_4 p_2 p_1 + \lambda_5 p_3 p_2 p_1 + \lambda_6 p_3 p_2 + \lambda_7 p_2 p_1 + \lambda_8 c_8 p_3 + \lambda_9 c_8 p_2 p_1$$

for $\lambda_i \in \mathbb{Z}/p$. By the congruence $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \mod (c_1^2, c_2^2)$ and the equality $p_i = \sum_{j+k=2i} (-1)^{j+k} c_j c_k$, we get linear equations in $\lambda_1, \ldots, \lambda_9$. Solving these equations, we see that $\bar{x}_{28} \equiv \alpha \hat{x}_{28} \mod (c_1^2, c_2^2)$, thus the proof is completed since the intersection of the ideal $(c_1^2, c_2^2)$ and the subring $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ of $\mathbb{Z}/p[t_1, \ldots, t_8]$ is the ideal $(p_1^2)$ in $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$. \qed

As an immediate consequence of Proposition 2.3 and 2.4, we obtain:

**Corollary 2.5.** We can choose a generator $x_i$ of $H^*(BE_8; \mathbb{Z}/p)$ for $i \neq 60$ in such a way that

$$\rho_i^*(x_i) = \hat{x}_i \quad (i = 4, 16, 36), \quad \rho_i^*(x_i) \equiv \hat{x}_i \mod (p_1^2) \quad (i = 24, 28)$$

$$\rho_i^*(x_i) \equiv \hat{x}_i \mod (p_1) \quad (i = 40, 48).$$
Hereafter, we choose generators of $H^*(BE_8; \mathbb{Z}/p)$ as in Corollary 2.5. From these generators, we next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G = F_4, E_6, E_7$. Recall that there is a commutative diagram of canonical homomorphisms

$$
\begin{align*}
F_4 & \xrightarrow{\alpha_3} E_6 \xrightarrow{\alpha_2} E_7 \xrightarrow{\alpha_1} E_8 \\
\rho_4 & \quad \rho_3 \quad \rho_2 \quad \rho_1 \\
\text{Spin}(9) & \xrightarrow{\theta_4} \text{Spin}(10) \xrightarrow{\theta_2} \text{Spin}(12) \xrightarrow{\theta_1} \text{Spin}(16).
\end{align*}
$$

Let us consider the induced map of arrows in the mod $p$ cohomology of the classifying spaces. Obviously, we have

$$
\begin{alignat}{3}
\theta_1^*(p_i) &= p_i & (i = 1, 2, 3, 4, 5), & \quad & \theta_1^*(p_6) &= c_6^2, & \quad & \theta_1^*(p_7) &= 0, & \quad & \theta_1^*(c_8) &= 0, & \quad & (5) \\
\theta_2^*(p_i) &= p_i & (i = 1, 2, 3, 4), & \quad & \theta_2^*(p_5) &= c_5^2, & \quad & \theta_2^*(c_6) &= 0, & \quad & (6) \\
\theta_3^*(p_i) &= p_i & (i = 1, 2, 3, 4), & \quad & \theta_3^*(c_5) &= 0. & \quad & (7)
\end{alignat}
$$

To determine the induced map of $\alpha_i$, we recall the results of [A, C, N, TW, W].

**Proposition 2.6.**

1. $H^*(E_6/\text{Spin}(10); \mathbb{Z}/p) = \mathbb{Z}/p[y_8]/(y_8^3) \otimes \Lambda(y_{17}), |y_i| = i$.
2. $H^*(E_6/F_4; \mathbb{Z}/p) = \Lambda(z_9, z_{17}), |z_i| = i$.
3. $H^*(E_7/E_6; \mathbb{Z}/p) = \mathbb{Z}/p[z_{10}, z_{18}], |z_1| = i$ for $* < 37$.
4. $H^*(E_8/E_7; \mathbb{Z}/p) = \mathbb{Z}/p[z_{12}, z_{20}], |z_i| = i$ for $* < 40$.

We next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G \neq E_8$. Let

$$
\hat{x}_{10} = c_5, \quad \hat{x}_{12} = -6p_3 + p_2p_1 - 60c_6, \quad \hat{x}_{18} = p_2c_5 \quad \text{and} \quad \hat{x}_{20} = p_5 + p_2c_6.
$$

We abbreviate $\theta_i(\hat{x}_j)$ by $\hat{x}_j$.

**Corollary 2.7.** We can choose a generator $x_i$ of $H^*(BE_7; \mathbb{Z}/p)$ so that

$$
\rho_2^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16, 36) \quad \text{and} \quad \rho_2^*(x_i) \equiv \hat{x}_i \mod (p_1^2) \quad (i = 20, 24, 28).
$$

**Proof.** Consider the Serre spectral sequence of the homotopy fiber sequence $E_8/E_7 \to BE_7 \to BE_8$. Then by Proposition 2.6, we get $\alpha_i^*(x_i) = x_i$ for $i = 4, 16, 24, 28, 36$, hence the desired result for $\rho_2^*(x_i)$ by Corollary 2.5. As in [BH], we can choose a generator $x_{12}$ of $H^*(BF_4; \mathbb{Z}/p)$ so that $\rho_4^*(x_{12}) = -6p_3 + p_2p_1$. On the other hand, it is calculated in [N] that $\rho_2^*(x_{12}) \equiv -6p_3 - 60c_6$.
modulo decomposables. Then we get $\rho_5^*(x_{12}) = \hat{x}_{12}$ by (6) and (7). By the Serre spectral sequence of the homotopy fiber sequence $E_6/\text{Spin}(10) \to B\text{Spin}(10) \to BE_6$ and Proposition 2.6, we have $\rho_3^*(x_{10}) \neq 0$. Then for a degree reason, we may choose $x_{10} \in H^*(BE_6; \mathbb{Z}/p)$ so that $\rho_3^*(x_{10}) = c_5$. Consider next the Serre spectral sequence of the homotopy fiber sequence $E_7/E_6 \to BE_6 \to BE_7$. Then it follows from Proposition 2.6 that we may choose $x_{20} \in H^*(BE_7; \mathbb{Z}/p)$ so that $\alpha_2^*(x_{20}) = x_{10}$, hence $\rho_5^*(x_{20}) \equiv p_5 + \alpha p_2 c_6 \mod (p_1^2)$ by (6), where $\alpha \in \mathbb{Z}/p$. For a degree reason, we have $\alpha_1^*(x_{30}) \equiv \lambda x_{20}^2 \mod (x_4, x_{12}, x_{16})$, hence

$$\theta_5^*(\dot{x}_{40}) = \lambda(p_5 + \alpha p_2 c_6)^2 \mod (\dot{x}_4, \dot{x}_{12}, \dot{x}_{16}).$$

Since $\theta_5^*(\dot{x}_{40}) \equiv 50p_5^2 - 10p_5p_3p_2 + \frac{1}{3}p_5^2p_2^2$ and $\dot{x}_{20}^2 \equiv p_5^2 - \frac{2}{5}p_5p_3p_2 + \frac{2}{15}p_5^2p_2^2 \mod (\dot{x}_4, \dot{x}_{12}, \dot{x}_{16})$, we get $\alpha = 1$ and $\lambda = 50$.

**Corollary 2.8.** We can choose a generator $x_i$ of $H^*(BE_6; \mathbb{Z}/p)$ so that

$$\rho_5^*(x_i) = \hat{x}_i \quad (i = 4, 10, 12, 16, 18) \quad \text{and} \quad \rho_3^*(x_{24}) = \hat{x}_{24} \mod (p_1^2).$$

**Proof.** By the Serre spectral sequence of the homotopy fiber sequence $E_7/E_6 \to BE_6 \to BE_7$ together with Proposition 2.6 and Corollary 2.7, we get $\alpha_2^*(x_i) = x_i$ for $i = 4, 12, 16, 24$. Then we obtain the desired result for $x_i$ ($i = 4, 12, 16, 24$) by Corollary 2.7. By Proposition 2.6, we have $\rho_3^*(x_{10}) \neq 0$, so we may put $\rho_3^*(x_{10}) = c_5$ for a degree reason. By Proposition 2.4, Corollary 2.7 and $\alpha_2 \circ \rho_3 = \rho_2 \circ \theta_2$, we see that $\rho_5^* \circ \alpha_2^*(x_{28})$ includes the term $p_2c_5^2$ which does not belong to $\rho_3^*(\mathbb{Z}/p[x_4, \ldots, \hat{x}_{18}, \ldots, x_{24}])$. Then we get $\rho_5^*(x_{18}) \neq 0$, implying that we may put $\rho_3^*(x_{18}) = p_2c_5$ for a degree reason.

**Corollary 2.9.** We can choose a generator $x_i$ of $H^*(BF_4; \mathbb{Z}/p)$ so that

$$\rho_4^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16) \quad \text{and} \quad \rho_4^*(x_{24}) = \hat{x}_{24} \mod (p_1^2).$$

**Proof.** The result follows from the Serre spectral sequence of the homotopy fiber sequence $E_6/F_4 \to BF_4 \to BE_6$ together with Proposition 2.6 and Corollary 2.8.

Recall that $G_2$ is a subgroup of $\text{Spin}(7)$. We denote the inclusion $G_2 \to \text{Spin}(7)$ by $\rho$.

**Proposition 2.10.** The induced map of $\rho : BG_2 \to B\text{Spin}(7)$ in mod $p$ cohomology satisfies

$$\rho^*(p_1) = x_4, \quad \rho^*(p_2) = 0 \quad \text{and} \quad \rho^*(p_3) = x_{12}.$$
Proof. It is well known that Spin(7)/G_2 = S^7. Then by considering the Serre spectral sequence of the homotopy fiber sequence Spin(7)/G_2 \to BG_2 \to BSpin(7), we obtain the desired result. \qed

For the rest of this paper, we choose generators of \( H^*(BG; \mathbb{Z}/p) \) as in Corollary 2.7, 2.8, 2.9, 2.10.

2.3. Calculation of \( P^1 \rho_i^*(x_j) \)

We first calculate the action of \( P^1 \) on \( H^*(BSpin(2m); \mathbb{Z}/p) \). Recall that \( H^*(BSpin(2m); \mathbb{Z}/p) = \mathbb{Z}/p[p_1, \ldots, p_{m-1}, c_m] \) as above.

Lemma 2.11. In \( H^*(BSpin(2m); \mathbb{Z}/p) \), we have

\[
P^1 p_i = \sum_{i_1 + 2i_2 + \cdots + mi_m = i + \frac{n+1}{2}} (-1)^{i_1 + \cdots + i_m + i + \frac{n+1}{2}} \frac{(i_1 + \cdots + i_m - 1)!}{i_1! \cdots i_m!} \times \left( 2i - 1 - \frac{\sum_{j=1}^{i-1} (2i + p - 1 - 2j)i_j}{i_1 + \cdots + i_m - 1} \right) p_1^{i_1} \cdots p_m^{i_m}
\]

and \( P^1 c_m = sp_{m-1}c_m \), where \( p_m = c_m^2 \) and \( s_k = t_1^k + \cdots + t_m^k \).

Proof. By [S], we have the mod \( p \) Wu formula

\[
P^1 c_i = \sum_{i_1 + 2i_2 + \cdots + 2mi_m = i + p - 1} (-1)^{i_1 + \cdots + i_m + 1} \frac{(i_1 + \cdots + i_m - 1)!}{i_1! \cdots i_m!} \times \left( i - 1 - \frac{\sum_{j=2}^{i-1} (i + p - 1 - j)i_j}{i_1 + \cdots + i_m - 1} \right) c_1^{i_1} \cdots c_m^{i_m}
\]

in \( H^*(BU(2m); \mathbb{Z}/p) \). Since the natural map \( c: BSpin(2m) \to BU(2m) \) satisfies \( c^*(c_2) = (-1)^i p_i \) and \( c^*(c_{2i+1}) = 0 \), we obtain the first equation. The second equation is obvious. \qed

We now calculate \( P^1 \rho_i^*(x_j) \).

Proposition 2.12. Define ideals \( I_j \) of \( \mathbb{Z}/p[p_1, \ldots, p_7, c_8] \) for \( j = 0, \ldots, 8 \) as

\[
I_0 = (p_1, p_2^2, p_3, p_4, p_6, c_8), \quad I_1 = I_0 + (p_3, p_6), \quad I_2 = I_0 + (p_2, p_3, p_4, p_7), \\
I_3 = I_0 + (p_2, p_3, p_6), \quad I_4 = I_0 + (p_2, p_3, p_4), \quad I_5 = I_0 + (p_2, p_3, p_4, p_6, p_7), \\
I_6 = I_0 + (p_2, p_3, p_4, p_6), \quad I_7 = I_0 + (p_2, p_3, p_4, p_6, p_7), \quad I_8 = I_0 + (p_2, p_4, p_7, \hat{x} 24).
\]
Then for a generator \(x_k \in H^*(BE_8; \mathbb{Z}/p)\), we have the following table.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(k)</th>
<th>(P^1 \rho_1^*(x_k) \mod I)</th>
<th>(I)</th>
<th>(p)</th>
<th>(k)</th>
<th>(P^1 \rho_1^*(x_k) \mod I)</th>
<th>(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>16</td>
<td>(9p_7^2p_5 + 24p_7p_5^2p_2 + 22p_5^3p_4)</td>
<td>(I_1)</td>
<td>37</td>
<td>4</td>
<td>(p_7^2p_5 + 34p_7p_5^2p_2 + 36p_5^3p_4)</td>
<td>(I_1)</td>
</tr>
<tr>
<td>24</td>
<td>(28p_7p_6p_5p_3 + 16p_5p_3^2)</td>
<td>(I_2)</td>
<td>16</td>
<td>8p_7^2p_5p_3 + 27p_7p_5^2 + 2p_5^3p_4p_3</td>
<td>(I_3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>(27p_7^2p_5p_3 + 30p_7p_5^2 + 30p_5^3p_4p_3)</td>
<td>(I_3)</td>
<td>24</td>
<td>(5p_7^3p_3 + 27p_7^2p_5^2 + 36p_6p_5^3p_3)</td>
<td>(I_4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>(p_7^3p_3 + 10p_5^2p_2 + 6p_0p_5^3p_3)</td>
<td>(I_4)</td>
<td>28</td>
<td>(7p_5^3)</td>
<td>(I_5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>(8p_5^5)</td>
<td>(I_5)</td>
<td>36</td>
<td>(20p_7^2p_5^2p_3 + 35p_7p_4)</td>
<td>(I_6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>(4p_7^2p_2p_3 + 5p_7p_3^4)</td>
<td>(I_6)</td>
<td>48</td>
<td>(36p_7p_4p_3 + 3p_5^3)</td>
<td>(I_7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>4</td>
<td>(35p_7p_6p_5p_3 + 40p_6p_5^3)</td>
<td>(I_7)</td>
<td>43</td>
<td>4</td>
<td>(3p_7^2p_5p_3 + 3p_7p_5^2 + 39p_6p_5p_4p_3)</td>
<td>(I_3)</td>
</tr>
<tr>
<td>16</td>
<td>(9p_7^3 + 38p_7p_5^2 + 16p_6p_5^3p_3)</td>
<td>(I_4)</td>
<td>16</td>
<td>(9p_5^5)</td>
<td>(I_5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>(7p_7^2p_2p_3 + 6p_7p_4^4)</td>
<td>(I_6)</td>
<td>24</td>
<td>(11p_7p_5^2p_3 + 40p_7p_4^4)</td>
<td>(I_6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>(34p_7p_5^3p_3 + 16p_6^5)</td>
<td>(I_7)</td>
<td>36</td>
<td>(35p_7p_5^3p_3 + 42p_6^5)</td>
<td>(I_7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>4</td>
<td>(p_7^3p_3 + 25p_7p_5^2 + 43p_6p_5^3p_3)</td>
<td>(I_4)</td>
<td>53</td>
<td>4</td>
<td>(6p_7^2p_5p_3 + p_7p_4^4)</td>
<td>(I_6)</td>
</tr>
<tr>
<td>16</td>
<td>(35p_7^2p_2p_3 + 10p_7p_4^3)</td>
<td>(I_6)</td>
<td>16</td>
<td>(23p_7p_4p_3 + 39p_6^5)</td>
<td>(I_7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>(17p_7p_5^3p_3 + 23p_6^5)</td>
<td>(I_7)</td>
<td>59</td>
<td>(5p_7p_5p_3 + 10p_5^5)</td>
<td>(I_7)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For \(p = 31\), we also have

\[
P^1 \rho_1^*(x_{48}) \equiv 17p_7p_3^2 + 4p_7p_5^2p_3 + 5p_7p_4^4 \mod I_8.
\]

Proof. For \(i = 4, 16, 24, 28, 36\), we have \(\rho^*_i(x_i) \equiv \hat{x}_i \mod (p_1)\). Since \(P^1(p_1^2) \subset (p_1)\) by the Cartan formula, we have \(P^1 \rho_1^*(x_i) \equiv P^1 \hat{x}_i \mod (p_1)\). For \(i = 40, 48\), we analogously have \(P^1 \rho_1^*(x_i) = P^1 \hat{x}_i + (P^1 p_1) q\) for some polynomial \(q\) in \(p_2, \ldots, p_7, c_8\). For a degree reason, we have \(q \equiv 0 \mod (p_1, p_2, p_3^2, p_4, p_6, c_8)\), implying that \(P^1 \rho_1^*(x_i) \equiv P^1 \hat{x}_i \mod I\) for the prescribed ideal \(I\). Thus in order to fill the table, we only need to calculate \(P^1 \hat{x}_i\) by Lemma 2.11.

For \(p = 31\), we have \(P^1 \rho_1^*(x_{48}) \equiv P^1 \hat{x}_{48} + (P^1 p_1) q \mod (p_1)\) for some polynomial \(q\) in \(p_2, \ldots, p_7, c_8\) as above. Since \(\hat{x}_i \in I_8\) for \(i = 4, 16, 24, 36\), we have \(P^1 p_1 \equiv 0 \mod I_8\) for a degree reason, hence \(P^1 \rho_1^*(x_{48}) \equiv P^1 \hat{x}_{48} \mod I_8\). Then we can calculate \(P^1 \rho_1^*(x_{48}) \mod I_8\) by Lemma 2.11. Since \(P^2 p_1 = p_1^5\) and \(\rho_1^*(x_{48}) \equiv \hat{x}_{48} \mod (p_1)\), we have \(P^2 \rho_1^*(x_{48}) \equiv P^2 \hat{x}_{48} \mod (p_1)\). Now \(P^2 \rho_1^*(x_{48})\) for \(p = 31\) can be calculated from Lemma 2.11 and the Adem relation \(P^1 P^1 = 2P^2\). \(\square\)

Quite similarly to Proposition 2.12, we can calculate \(P^1 \rho_1^*(x_j)\) for \(G = E_7, E_6\).
Proposition 2.13. For a generator $x_k \in H^*(BE_7; \mathbb{Z}/p)$, we have the following table.

| $p$ | $k$ | $|\mathcal{P}_{\nu}^k(x_k) \mod I|$ | $I$ |
|-----|-----|-----------------------------------|-----|
| 19  | 12  | $18p_5^2p_2 + 3p_5p_4p_3 + 15p_5p_3p_2^2 + 10p_4^3 + 17p_4^2p_2 + 6p_4p_2^2 + 15p_2^4$ | $(p_1, p_5^2, c_6)$ |
|     | 16  | $11p_5^2p_2 + 16p_5p_4p_2^2 + 15p_5p_2^4$ | $(p_1, p_3, c_6)$ |
|     | 20  | $p_5^2p_4 + 18p_5^2p_2^2 + 17p_5p_4p_3 + 8p_5p_3p_2^2 + 4c_6p_5p_4p_2 + 12c_6p_5p_3^2$ | $(p_1, p_3^2, c_6)$ |
|     |     | $+ 16c_6p_5p_3 + 8p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$ | |
|     | 24  | $13p_5^2p_4p_2 + 7p_5p_4p_3 + 8p_5p_2^5$ | $(p_1, p_3, p_5^2, c_6)$ |
|     | 28  | $14p_5^2p_4p_2 + p_5^2p_3^3 + 8p_5p_4p_3 + 10p_5p_4p_3p_2^2 + 17c_6p_5p_3p_2^4 + p_4^3 + 9p_3^2p_2^2$ | $(p_1, p_5^2, c_6)$ |
|     |     | $+ 6p_4p_2^4 + p_4p_2^6 + 3p_2^8$ | |
|     | 36  | $9p_5^2p_4p_2 + 4p_5^2p_3 + 6p_5^2p_1 + 17p_5p_4p_3p_2 + 15p_5p_3p_2^5 + 4p_4^3p_2 + 5p_3^4p_2^2$ | $(p_1, p_3^2, c_6)$ |
|     |     | $+ 2p_5^2p_5 + 11p_4p_2^7 + 3p_2^9$ | |
| 23  | 4   | $22p_5^2p_2 + 21p_5p_4p_3 + 3p_5p_3p_2^2 + 15p_4^3 + 13p_4^2p_2 + 22p_4p_2^4 + 4p_6^6$ | $(p_1, p_5^2, c_6)$ |
|     | 12  | $7p_5^2p_4 + 6p_5^2p_2^2 + 14p_5p_4p_3 + 13p_5p_3p_2^2 + 10p_4^3 + (p_4p_3 + 8p_4p_2^3 + 21p_4p_5^3$ | $(p_1, p_5^2, c_6)$ |
|     |     | $+ 4p_7^7 + 14c_6p_5p_4p_2 + 16c_6p_5p_3^3 + 7c_6p_5^2p_3 + 2p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$ | |
|     | 16  | $3p_5^2p_4p_2 + 20p_5p_4p_3^3 + 19p_5p_2^5$ | $(p_1, p_3, p_5^2, c_6)$ |
|     | 28  | $9p_5^2p_4p_2 + 3p_5^2p_4p_3^3 + 2p_5^2p_4p_3 + 10p_5p_4p_3p_2 + 10p_5p_4p_3p_2^2 + 8p_5p_3p_2^5$ | $(p_1, p_5^2, c_6)$ |
|     |     | $+ 14p_4^3p_2 + 15p_4p_2^3 + 14p_2^3 + 9p_2^3p_5 + 15p_4p_2^7$ | |
| 29  | 4   | $26p_5p_2p_2 + 4p_5p_4p_2 + 28p_5p_2^7$ | $(p_1, p_3, p_5^2, c_6)$ |
|     | 16  | $19p_5^2p_4 + p_5^2p_4p_3 + 19p_5p_4p_3 + 10p_5^2p_3p_2 + 6p_5p_4p_3p_2 + 13p_5p_3p_2^5$ | $(p_1, p_5^2, c_6)$ |
|     |     | $+ p_4^3p_2 + 7p_4^3p_2 + 2p_2^3p_2^5 + 16p_2^6p_2^7$ | |
| 31  | 12  | $p_5^2p_4p_2 + 17p_5p_4p_3^3 + 28p_5p_2p_2 + 18p_5p_4p_3p_2$ | $(p_1, p_5^2, c_6)$ |
|     |     | $+ 21p_2^4 + 3p_4^3p_2^3 + 6p_4^3 + 4p_5^2p_3 + 10c_6p_4^3 + 3c_6p_5p_3p_2 + 3c_6p_5p_2^3p_2 + 27c_6p_5p_4p_3^3 + c_6p_5p_2^5 + c_6p_5p_2^3 + 25c_6p_5p_3p_2^2 + 5c_6p_4p_3p_2^4 + 30c_6p_3p_2^6$ | |

Proposition 2.14. For a generator $x_k \in H^*(BE_6; \mathbb{Z}/p)$, we have the following table.
We finally calculate $P^1 x_k$ for a generator $x_k \in H^*(BG_2; \mathbb{Z}/p)$.

**Proposition 2.15.** For a generator $x_k \in H^*(BG_2; \mathbb{Z}/p)$, we have

\[
P^1 x_k = \begin{cases} 
  x_4 x_{12} + 2x_4^4 & (k, p) = (4, 7) \\
  6x_4^2 x_{12} + 2x_4^3 x_{12} & (k, p) = (12, 7) \\
  6x_4^2 + x_4^3 x_{12} + 2x_4^6 & (k, p) = (4, 11).
\end{cases}
\]

*Proof.* By Proposition 2.10 and the naturality of $P^1$, we have $P^1 x_{4k} = P^1 \rho^*(p_k) = \rho^*(P^1 p_k)$, hence the proof is completed by Lemma 2.11. \qed

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by using results in the previous section.

3.1. The case of $E_8$

Suppose that $E_8$ is $p$-regular, that is, $p > 30$. By an easy degree consideration, we see that if $P^1 x_k \mod (x_{2i} | i \in \mathfrak{t}(E_8))^3$ is nontrivial for a generator $x_k$ of $H^*(BE_8; \mathbb{Z}/p)$, it is as in the following table.
Let $I_k$ for $k = 1, \ldots, 8$ be the ideals of $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ as in Proposition 2.12.

(1) It is proved in [HK] that $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$.

(2) Since $\hat{x}_i \in I_1$ for $i = 4, 16, 24$, for a degree reason, we have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_2 \hat{x}_{28} \hat{x}_{48} + \lambda_3 \hat{x}_{36} \hat{x}_{40} \equiv -4000(24\lambda_2 p_2^2 p_5 + (\lambda_2 - 6\lambda_3)p_7 p_5^2 p_2) \mod I_1 + (p_4).$$

On the other hand, by the naturality of $\mathcal{P}^1$ and Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1(x_k) \equiv \begin{cases} 24p_7 p_7^2 p_2 + 9p_7^2 p_5 & (p = 31) \\ 34p_7 p_7^2 p_2 + p_7^2 p_5 & (p = 37) \end{cases} \mod I_1 + (p_4),$$

implying that $(\lambda_2, \lambda_3) = (19, 2), (5, 30)$ according as $p = 31, 37$. Since $\hat{x}_1, \hat{x}_{16}, \hat{x}_{24}, \hat{x}_{36} \in I_1 + (p_2, p_7)$, we also have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{16} \rho_1^*(x_{60}) + \lambda_3 \hat{x}_{36} \hat{x}_{40} \equiv \lambda_1 \hat{x}_{16} \rho_1^*(x_{60}) - 1500\lambda_3 p_6^3 p_4 \mod I_1 + (p_2, p_7),$$

and by Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1^*(x_k) \equiv \begin{cases} 22p_2^3 p_4 & (p = 31) \\ 36p_2^3 p_4 & (p = 37) \end{cases} \mod I_1 + (p_2, p_7).$$

Then we see that $\lambda_1 \hat{x}_{16} \rho_1^*(x_{60}) \equiv (1500\lambda_3 + \delta)p_5^2 p_4 \neq 0 \mod I_1 + (p_2, p_7)$ for $\delta = 22, 36$ according as $p = 31, 37$, implying $\lambda_1 \neq 0$. 

| $\mathcal{P}^1 x_k \mod (x_{2i} | i \in \mathfrak{t}(E_8))^3$ | $(k, p)$ |
|-----------------------------|---------------------|
| $\lambda_1 x_4 x_{60} + \lambda_2 x_{16} x_{48} + \lambda_3 x_{24} x_{40} + \lambda_4 x_{28} x_{36}$ | $(4, 31)$ |
| $\lambda_1 x_{16} x_{60} + \lambda_2 x_{28} x_{48} + \lambda_3 x_{36} x_{40}$ | $(16, 31), (4, 37)$ |
| $\lambda_1 x_{24} x_{60} + \lambda_2 x_{36} x_{48}$ | $(24, 31), (4, 41)$ |
| $\lambda_1 x_{28} x_{60} + \lambda_2 x_{40} x_{48}$ | $(28, 31), (16, 37), (4, 43)$ |
| $\lambda_1 x_{36} x_{60} + \lambda_2 x_{46}^2$ | $(36, 31), (24, 37), (16, 41), (4, 47)$ |
| $\lambda x_{40} x_{60}$ | $(40, 31), (28, 37), (16, 43)$ |
| $\lambda x_{48} x_{60}$ | $(48, 31), (36, 37), (28, 41), (24, 43), (16, 47), (4, 53)$ |
| $\lambda x_{60}^2$ | $(60, 31), (48, 37), (40, 41), (36, 43), (28, 47), (16, 53), (4, 59)$ |
(3) Since \( \hat{x}_i, \hat{x}_{2j} \in I_2 \) for \( i = 4, 16 \) and \( j = 24, 28, 36 \), we have
\[
\rho_i^*(P^1x) \equiv \lambda_1 \hat{x}_{24} \rho_i^*(x_{60}) + \lambda_2 \hat{x}_{36} \hat{x}_{48} \equiv \lambda_1 \hat{x}_{24} \rho_i^*(x_{60}) - 14400 \lambda_2 p_7 p_6 p_5 p_3 \mod I_2.
\]
By the naturality of \( P^1 \) and Proposition 2.12, we also have
\[
\rho_i^*(P^1x) = P^1 \rho_i^*(x) \equiv \begin{cases} 
28 p_7 p_6 p_5 p_3 + 16 p_6 p_5^3 & (p = 31) \\
35 p_7 p_6 p_5 p_3 + 40 p_6 p_5^3 & (p = 41)
\end{cases} \mod I_2,
\]
impacting that \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \) for both \( p = 31, 41 \).

(4) Since \( \hat{x}_i, \hat{x}_{28} \in I_3 + (p_3, p_4, p_5^2, \hat{x}_{40}) \) for \( i = 4, 16, 24, 36, 40 \), it follows from Proposition 2.12 that
\[
\lambda_1 \hat{x}_{28} \rho_i^*(x_{60}) \equiv \rho_i^*(P^1x) \equiv P^1 \rho_i^*(x) \equiv 0 \mod I_3 + (p_3, p_4, p_5^2, \hat{x}_{40})
\]
so \( \lambda_1 \neq 0 \). We can similarly get \( \lambda_2 \neq 0 \) by considering \( \rho_i^*(P^1x) \mod I_3 + (p_5^2, \hat{x}_{28}) \) since \( \hat{x}_i \in I_3 + (p_5^2, \hat{x}_{28}) \) for \( i = 4, 16, 24, 28 \).

(5), (6) and (7) We get \( \lambda \neq 0 \) similarly to (4) by considering \( \rho_i^*(P^1x) \mod I_4 + (p_7), I_5, I_6 + (\hat{x}_{40}) \) respectively for (5), (6) and (7) since \( \hat{x}_4, \hat{x}_{16}, \hat{x}_{24}, \hat{x}_{36} \in I_4 + (p_7), \hat{x}_i \in I_5 \) for \( i = 4, 16, 24, 18, 36 \) and \( \hat{x}_i \in I_6 + (\hat{x}_{40}) \) for \( i = 4, 16, 24, 36, 40 \).

(8) Suppose \( (k, p) \neq (60, 31) \). Since \( \hat{x}_i, \hat{x}_{28}^2, \hat{x}_{40}^3 \in I_7 + (\hat{x}_{40}^3) \) for \( i = 4, 16, 24, 36 \), we get \( \lambda \neq 0 \) by considering \( \rho_i^*(P^1x) \mod I_7 + (\hat{x}_{40}) \) as above.

Suppose next that \( (k, p) = (60, 31) \). By a degree reason, we have
\[
\rho_i^*(x_{60}) \equiv \alpha p_5^3 + \beta p_7 p_5 p_3 \mod I_8 + (\hat{x}_{40}^2)
\]
for \( \alpha, \beta \in \mathbb{Z}/p \). Since \( \hat{x}_i, \hat{x}_{40}^2 \in I_8 + (\hat{x}_{40}^2) \) for \( i = 4, 16, 24, 36 \) and \( \rho_i^*(x_{48}) \equiv -200 p_7 p_5 \mod I_8 \), we have
\[
\rho_i^*(P^1 x_{48}) \equiv \mu \hat{x}_{48} \rho_i^*(x_{60}) \equiv -200 \mu (\alpha p_7 p_5^4 + \beta p_7^2 p_5^2 p_3) \mod I_8 + (\hat{x}_{40}^2)
\]
for some \( \mu \in \mathbb{Z}/p \). By Proposition 2.12, we also have
\[
\rho_i^*(P^1 x_{48}) = P^1 \rho_i^*(x_{48}) \equiv 10 p_7 p_5^4 + 11 p_7^2 p_5^2 p_3 \mod I_8 + (\hat{x}_{40}^2)
\]
Then we may put \( (\alpha, \beta) = (17, 28) \) and \( \mu = 1 \). In the case (7), we have seen that \( P^1 x_{48} \equiv \mu x_{148} x_{60} \mod (x_{2i} | i \in \mathfrak{t}(E_8))^3 \), implying that \( P^1 P^1 x_{48} \equiv (\lambda + 1) x_{48} x_{60} \mod (x_{2i} | i \in \mathfrak{t}(E_8))^4 \), where \( P^1 x_{60} \equiv \lambda x_{60} \mod (x_{2i} | i \in \mathfrak{t}(E_8))^3 \). Then for a degree reason, we get
\[
\rho_i^*(P^1 P^1 x_{48}) \equiv (\lambda + 1) \hat{x}_{48} \rho_i^*(x_{60})^2 \equiv 21 (\lambda + 1) p_7^3 p_5^3 p_3 \mod I_8 + (\hat{x}_{40}^2).
\]
On the other hand, by the Adem relation $\mathcal{P}^1 \mathcal{P}^1 = 2\mathcal{P}^2$ and Proposition 2.12, we have

$$\rho^*_1(\mathcal{P}^1 \mathcal{P}^1 x_{48}) = \rho^*_1(2\mathcal{P}^2 x_{48}) = 2\mathcal{P}^2 \rho^*_1(x_{48}) \equiv 7p_7^3 p_5^3 p_3^2 \mod I_8 + (\hat{x}_{40}^2),$$

hence $\lambda \neq 0$.

3.2. The case of $E_7$

Suppose that $E_7$ is $p$-regular, that is, $p > 18$. Then if $\mathcal{P}^1 x_k \mod (x_{2i} \mid i \in \mathfrak{t}(E_7))^3$ is non-trivial, it is as in the following table.

<table>
<thead>
<tr>
<th>$\mathcal{P}^1 x_k \mod (x_{2i} \mid i \in \mathfrak{t}(E_7))^3$</th>
<th>$(k, p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 x_4 x_{36} + \lambda_2 x_{12} x_{28} + \lambda_3 x_{16} x_{24} + \lambda_4 x_{20}^2$</td>
<td>(4, 19)</td>
</tr>
<tr>
<td>$\lambda_1 x_{12} x_{36} + \lambda_2 x_{20} x_{28} + \lambda_3 x_{21}^2$</td>
<td>(12, 19), (4, 23)</td>
</tr>
<tr>
<td>$\lambda_1 x_{16} x_{36} + \lambda_2 x_{24} x_{28}$</td>
<td>(16, 19)</td>
</tr>
<tr>
<td>$\lambda_1 x_{20} x_{36} + \lambda_2 x_{28}^2$</td>
<td>(20, 19), (12, 23)</td>
</tr>
<tr>
<td>$\lambda x_{12} x_{36}$</td>
<td>(24, 19), (16, 23), (4, 29)</td>
</tr>
<tr>
<td>$\lambda x_{24} x_{36}$</td>
<td>(28, 19), (4, 31)</td>
</tr>
<tr>
<td>$\lambda x_{28} x_{36}$</td>
<td>(36, 19), (28, 23), (16, 29), (12, 31)</td>
</tr>
</tbody>
</table>

1. It is proved in [HK] that $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$.
2. Put $I = (p_1, p_3^2, c_6, \hat{x}_{16})$. Since $\hat{x}_4, \hat{x}_{12}, \hat{x}_{16} \in I$, by Corollary 2.7, we have

$$\rho^*_2(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_4 \hat{x}_{36} + \lambda_2 \hat{x}_{12} \hat{x}_{28} + \lambda_3 \hat{x}_{21}^2 \equiv 60\lambda_1 p_5 p_3 p_2^2 + 40\lambda_2 p_3^2 p_2 + \frac{25}{81} \lambda_3 p_2^6 \mod I.$$

On the other hand, it follows from Proposition 2.13 that

$$\rho^*_2(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho^*_2(x_k) \equiv \begin{cases} 18p_5^2 p_2 + 10p_5 p_3 p_2^2 + p_2^6 & (p = 19) \mod I, \\ 22p_5^2 p_2 + 7p_5 p_3 p_2^2 + 7p_2^6 & (p = 23) \end{cases}$$

hence $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$.

3. In this case, we have $(k, p) = (16, 19)$. Put $I = (p_1, p_3, c_6, \hat{x}_{16}^2)$. Since $\hat{x}_4, \hat{x}_{12}, \hat{x}_{16}^2 \in I$, it follows from Proposition 2.7 that

$$\rho^*_2(\mathcal{P}^1 x_{16}) \equiv \lambda_1 \hat{x}_{16} \hat{x}_{36} + \lambda_2 \hat{x}_{24} \hat{x}_{28} \equiv (13\lambda_1 + 9\lambda_2) p_5 p_4 p_2^2 + (9\lambda_1 + 14\lambda_2) p_5 p_2^4 \mod I.$$

By Proposition 2.13, we also have $\rho^*_2(\mathcal{P}^1 x_{16}) = \mathcal{P}^1 \rho^*_2(x_{16}) \equiv 11p_5 p_4 p_2^2 + 14p_5 p_2^4 \mod I$, implying $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.  

16
(4) Put \( I = (p_1, p_2^2, c_6, \hat{x}_{12}, \hat{x}_{16}, \hat{x}_{24}, \hat{x}_{16}\hat{x}_{20}\hat{x}_{24}) \). Since \( \hat{x}_i, \hat{x}_{16}, \hat{x}_{16}\hat{x}_{20}\hat{x}_{24} \in I \) for \( i = 4, 12, 24 \), we have

\[
\rho_2^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{20}\hat{x}_{36} + \lambda_2^2 \hat{x}_{28}^2 \equiv (-10\lambda_1 + 1600\lambda_2)p_5^2p_2^2 + (\frac{2}{3}\lambda_1 - \frac{320}{3}\lambda_2)p_5p_3p_2^3 \mod I.
\]

By Proposition 2.13, we also have

\[
\rho_2^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_2^*(x_k) \equiv \begin{cases} 
10p_5^2p_2^2 + 12p_5p_3p_2^3 & (p = 19) \\
15p_5^2p_2^2 + 22p_5p_3p_2^3 & (p = 23)
\end{cases} \mod I,
\]

hence \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \).

(5) and (7) Put \( I = (p_1, p_3, p_5, c_6, \hat{x}_{16}) \) and \( J = (p_1, p_3^2, c_6^2, \hat{x}_{12}, \hat{x}_{16}, \hat{x}_{20}, \hat{x}_{24}, \hat{x}_{20}\hat{x}_{24}\hat{x}_{28}) \). Then since \( \hat{x}_i, \hat{x}_{20}^2 \in I \) for \( i = 4, 12, 16 \) and \( \hat{x}_i, \hat{x}_{20}, \hat{x}_{24}, \hat{x}_{20}\hat{x}_{24}\hat{x}_{28} \in J \) for \( i = 4, 12, 16 \), we have \( \lambda \neq 0 \) similarly to (4) of \( E_8 \) by considering \( \rho_2^*(\mathcal{P}^1 x_k) \) modulo \( I \) and \( J \) respectively for (5) and (7).

(6) The case \( p = 31 \) follows from the above case of \( E_8 \) together with Corollary 2.7. Then we consider the case \( p = 19 \). Put \( I = (p_1, p_3^2, c_6, \hat{x}_{12}, \hat{x}_{16}, \hat{x}_{20}, \hat{x}_{24}) \).

Since \( \hat{x}_i, \hat{x}_j^2 \in I \) for \( i = 4, 12, 16 \) and \( j = 20, 24 \), we get \( \lambda \neq 0 \) as above by considering \( \rho_2^*(\mathcal{P}^1 x_k) \mod I \).

### 3.3. The cases of \( E_6 \) and \( F_4 \)

We first consider the case of \( E_6 \). Suppose that \( E_6 \) is \( p \)-regular, that is, \( p \geq 13 \). By an easy dimensional consideration, we see that if \( \mathcal{P}^1 x_k \not\equiv 0 \mod (x_{2i} \mid i \in \mathfrak{t}(E_6))^3 \), it is as in the following table.

<table>
<thead>
<tr>
<th>( \mathcal{P}^1 x_k \mod (x_{2i} \mid i \in \mathfrak{t}(E_6))^3 )</th>
<th>( (k, p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) ( \lambda_1 x_{14} x_{24} + \lambda_2 x_{10} x_{18} + \lambda_3 x_{12} x_{16} )</td>
<td>(4, 13)</td>
</tr>
<tr>
<td>(2) ( \lambda_1 x_{10} x_{24} + \lambda_2 x_{16} x_{18} )</td>
<td>(10, 13)</td>
</tr>
<tr>
<td>(3) ( \lambda_1 x_{12} x_{24} + \lambda_2 x_{18}^2 )</td>
<td>(12, 13), (4, 17)</td>
</tr>
<tr>
<td>(4) ( \lambda x_{16} x_{24} )</td>
<td>(16, 13), (4, 19)</td>
</tr>
<tr>
<td>(5) ( \lambda x_{18} x_{24} )</td>
<td>(18, 13), (10, 17)</td>
</tr>
<tr>
<td>(6) ( \lambda x_{24}^2 )</td>
<td>(24, 13), (16, 17), (12, 19), (4, 23)</td>
</tr>
</tbody>
</table>

When \( p = 19, 23 \), the result follows from the above case of \( E_7 \) and Corollary 2.8.

(1) It is proved in [HK] that \( \lambda_1 \neq 0, \lambda_2 \neq 0 \) and \( \lambda_3 \neq 0 \).
(2) Put \( I = (p_1, p_3^2, c_5^2) \). Since \( \hat{x}_4, \hat{x}_{10}^2, \hat{x}_{12}^2 \in I \), we have

\[
\rho_3^*(\mathcal{P}^1 x_{10}) \equiv \lambda_1 \hat{x}_{10} \hat{x}_{24} + \lambda_2 \hat{x}_{16} \hat{x}_{18} \equiv 5\lambda_1 (-p_4 p_2 c_5 + \frac{1}{36} p_2^3 c_5) + \lambda_2 (12 p_4 p_2 c_5 + p_2^3 c_5) \mod I,
\]

where \( \hat{x}_{10} = c_5 \) and \( \hat{x}_{18} = p_2 c_5 \). On the other hand, by Proposition 2.14, we have

\[
\rho_3^*(\mathcal{P}^1 x_{10}) = \mathcal{P}^1 \rho_3^*(x_{10}) \equiv 6p_4 p_2 c_5 + 7p_2^3 c_5 \mod I \text{ for } p = 13, \text{ hence } \lambda_1 \neq 0 \text{ and } \lambda_2 \neq 0.
\]

(3) Put \( I = (p_1, p_3^2, \hat{x}_{16}) \). It is sufficient to consider the case \( p = 13, 17 \). Since \( \hat{x}_i, \hat{x}_{12}^2 \in I \) for \( i = 4, 16 \),

\[
\rho_3^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{12} \hat{x}_{24} + \lambda_2 \hat{x}_{18}^2 \equiv -\frac{10}{3} \lambda_1 p_3 c_5^2 + \lambda_2 p_2^2 c_5^2 \mod I.
\]

By Proposition 2.14, we have

\[
\rho_3^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_3^*(x_k) \equiv \begin{cases} 
9p_3 p_2^3 + 5c_5^2 p_2^2 & (p = 13) \\
13p_3 p_2^3 - 11c_5^2 p_2^2 & (p = 17) \end{cases} \mod I,
\]

implying \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \).

(4), (5) and (6) Put \( I = (p_1, p_3, p_4, c_5), J = (p_1, p_3, c_5^2, \hat{x}_{16}) \) and \( K = (p_1, p_3, c_5, \hat{x}_{16}) \). Then since \( \hat{x}_i \in I \) for \( i = 4, 10, 12, \hat{x}_i, \hat{x}_{10}^2 \in J \) for \( i = 4, 12, 16 \) and \( \hat{x}_i \in K \) for \( i = 4, 12, 10, 16 \), we get \( \lambda \neq 0 \) similarly to (4) of \( E_8 \) by considering \( \rho_3^*(\mathcal{P}^1 x_k) \) modulo \( I, J, K \) respectively for (4), (5) and (6).

We next consider the case of \( F_4 \). Notice that \( F_4 \) is \( p \)-regular if and only if so is \( E_6 \), and that as in the proof of Corollary 2.9, the map \( \alpha_3^* : H^*(BE_6; \mathbb{Z}/p) \to H^*(BF_4; \mathbb{Z}/p) \) is surjective. Then the result for \( F_4 \) follows from that for \( E_6 \) above.

3.4. The case of \( G_2 \)

For a degree reason, if \( G_2 \) is \( p \)-regular and \( \mathcal{P}^1 x_k \neq 0 \mod (x_{2i} | i \in \mathfrak{t}(G_2))^3 \), then \( (k, p) = (4, 7), (12, 7), (4, 11) \). Hence Theorem 2.2 for \( G_2 \) readily follows from Proposition 2.15.

References


