KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

Discussion Paper No.909

"Dynamic Voluntary Advertising under Partial Market Coverage"

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December 2014

KYOTO UNIVERSITY

KYOTO, JAPAN

Dynamic Voluntary Advertising under Partial Market Coverage*

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November 19, 2014

Abstract

We consider a dynamic voluntary advertising model with a duopoly. Firms can use advertising and price as competitive tools where product quality is a given and the market is not fully covered by consumers. Advertising also plays a role as a public good. In this situation, we investigate how advertising, profits, and welfare respond to changes in consumer preference and product quality. We mainly find that a higher maximum preference value leads to increases in advertising, profits, and consumer surplus but a decrease in incumbent consumers' utility. We further find that a technology improvement by a low-quality firm increase its profit and consumer surplus if the technology gap is relatively large but if this is not the case, then the innovation could have different effects on firms' profits and consumer surplus.

Keywords Advertising, product quality, differential games, duopoly.
JEL Classification Numbers C72, C73, L13, M37.

*This research is financially supported by Grant-in-Aid for JSPS Fellows (No. 25 · 5882 and No. 26 · 3190).
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1 Introduction

It is generally thought that a firm advertises to persuade consumers to purchase its goods, which in turn increases the firm's market share. A substantial body of literature analyzing this situation exists in dynamic advertising models (Jørgensen (1982), Feichtinger et. al. (1994), Dockner et. al. (2000), and Huang, Leng, and Liang (2012)). Advertising can also play a role as a public good (Friedman (1983), Roberts and Samuelson (1988), Martin (1993), and Piga (1998)). These studies consider that even though firms compete with one another in an industry, they voluntarily advertise to persuade customers to buy their products over those of other firms. This occurs due to the knowledge that a positive externality exists in terms of voluntary advertising; that is, advertising benefits all other firms within an industry that produce the same industrial products. As the number of customers increases, all firms within an industry increase their profits. Voluntary advertising is frequently used in emerging industries and those that produce luxury goods. In emerging industries, competition over formats, such as that which occurred between Blu-ray and HD DVD manufacturers, is common. Firms that produce products with a unique proprietary format use advertising to increase their market size. In industries that produce luxury goods, such as cigarettes, jewels, and brand-name goods, firms advertise to persuade new customers to buy their products.

The first purpose of the present study is to investigate, using a dynamic voluntary advertising model, how firm's advertising behavior responds to changes in product quality and consumers' preference. These changes are analyzed in terms of how they affect a firm's profit. In the model, product goods are vertically differentiated and produced by duopolistic firms. The market is assumed not to be fully covered by consumers; that is, a fraction of consumers chooses not to purchase anything. Given product quality and consumers' preferences, firms use advertising and price as competitive tools. Piga (1998) investigates the relationship between advertising and product quality by using the voluntary advertising model, but he used the Hotelling (1929) location model where the goods are horizontally differentiated. Tenryu and Kamei (2013) extend Piga's (1998) model to a vertical product differentiation model. However, they focus on the interior solution case of a full market coverage model.

The second purpose of this study is to use the utility-based approach to analyze the relationships between welfare and (1) product quality, and (2) heterogeneity of preferences. There is little research investigating consumer behavior in a differential game. Huang, Leng, and Liang (2012) reports the following:

> we find that most publications have assumed general or specific differential functions to address a variety of dynamic advertising problems for an aggregate market, whereas very few publications have investigated consumer behavior using the utility-based approach. In the future, it would be interesting to apply the utility theory to construct and analyze dynamic advertising models.

Several studies (Colombo and Lambertini (2003), Lambertini (2005), Lambertini and Palestini (2009), Bertuzzi and Lambertini (2010)) use the utility-based approach to investigate

the role of a firm's advertising investment. Lambertini (2005) analyzes the advertising behavior of a monopoly firm based on the Hotelling (1929) model. Lambertini and Palestini (2009) and Bertuzzi and Lambertini (2010) follow Lambertini (2005)'s utility-based approach. Lambertini (2005) and subsequent research assume that generic consumers have homogenous preferences with regards to product quality. In contrast, the present model, a vertical product differentiation model, assumes that consumers have heterogeneous preferences with regards to product quality and, hence, explains the evaluation method of a consumer surplus in the presence of heterogeneous preferences. They also assume that advertising directly affects each consumer's reservation demand; that is, an increase in advertising investment increases demand. However, we assume that while advertising affects market size and number of customers, it does not affect the reservation demands of existing customers. Hence, in our model, the consumer surplus responds to the change in the number of consumers through advertising effects. Colombo and Lambertini (2003) use a vertical product differentiation model to investigate the effect of advertising and welfare, but do not deal with voluntary advertising strategies, unlike the present research.

We obtain the following results. First, when the technology gap between two firms is relatively large, a technological improvement by the low-quality firm enables it to increase its profits. Second, the effect of changes in the heterogeneity of preferences and technology on incumbent consumers can be different from the effect of changes on consumer surplus. Third, a more heterogeneous preference has a positive effect on firms and consumer surplus but has a negative effect on incumbent consumers. Fourth, if the technology gap is relatively large, innovations by either firm lead to increase in both firms' profits and consumer surplus, but if this is not the case, i.e., the technological gap so large, the innovations have different effects on firms' profits and consumer surplus. Finally, the low-quality firm has no incentive to improve its technology level above a certain degree of the technological gap.

The structure of the remainder of the study is as follows. Section 2 contains the basic setup. Section 3 derives equilibrium strategies, a steady state, and market shares. Section 4 analyzes how changes in preference and product quality affect the price, total advertising volume, firm profits, and total number of customers in the market. Consumer surplus and welfare are analyzed in section 5. Section 6 provides the conclusion.

2 The Model

We consider a partial market coverage economy; that is, where a fraction of consumers chooses not to purchase anything. The economy contains a high-quality firm, H, and a low-quality firm, L. The high-quality firm produces high-quality goods, and the low-quality firm produces low-quality ones. Each firm's technology level is exogenously given by s_i for $i \in \{L, H\}$ and is assumed to satisfy the relation, $s_H > s_L$.[1]

[1] It is well known that the high-quality firm invests more in R&D than the low-quality firm, and thus the high-quality firm has a higher inventory of intellectual property rights or patents. Many patented technologies are embedded in the high-quality good relative to the low-quality one.

Consumers are uniformly distributed along the line with density, N, and have several preferences for goods, θ, ($\underline{\theta} < \theta < \bar{\theta}$). The parameter, θ, represents each consumer's marginal willingness to pay and $\bar{\theta}$ ($\underline{\theta}$) is exogenously the maximum (minimum) value. Given consumer preferences and the partial market coverage, each consumer is assumed to decide whether to purchase one unit of a good and, if purchasing, is assumed to buy the good from either a high-quality firm or a low-quality firm. We define the indirect utility function at time t as $u^i_\theta(t) = \theta s_i - p_i(t)$, where $u^i_\theta(t)$ is the instantaneous utility generated from consumption of a good i at time t and $p_i(t)$ is good i's price at time t. If consumers do not buy anything, they obtain zero utility. The lifetime discounted present value of utility for the consumer with θ, $\bar{u}^i_\theta$, is represented as

$$\bar{u}^i_\theta = \int_0^\infty (\theta s_i - p_i(t))e^{-\rho t}dt, \;\; i \in \{L, H\},$$

where ρ is the discount rate.

In the partial market coverage economy, two thresholds divide the market. One characterizes a consumer who is indifferent between buying the high-quality good or the low-quality good:

$$\tilde{\theta}(t) = \frac{p_H(t) - p_L(t)}{s_H - s_L}. \tag{1}$$

The other threshold determines consumers' decision regarding whether to buy the low-quality goods:

$$\hat{\theta}(t) = \frac{p_L(t)}{s_L}, \tag{2}$$

where $\hat{\theta}(t) > \underline{\theta}$. In other words, for any time, consumers with the preference, $\theta \in [\hat{\theta}, \tilde{\theta}]$ buy low-quality goods, consumers with the preference, $\theta \in [\tilde{\theta}, \bar{\theta}]$ buy high-quality goods, and consumers with the preference, $\theta \in [\underline{\theta}, \hat{\theta})$ buy nothing and exit the market. Accordingly, $N(t)(\bar{\theta} - \hat{\theta}(t))$ represents the number of consumers in the industry at time t. Figure1 depicts these consumer preference distributions.

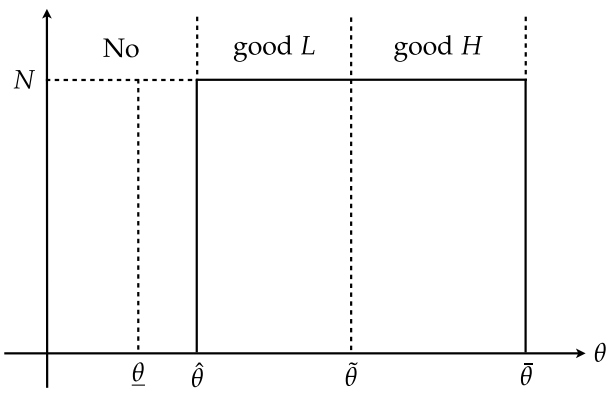


Figure 1: Market structure

Using the indirect utility function, we can derive the following demand function:

$$N(t)y_L(t) = N(\tilde{\theta}(t) - \hat{\theta}(t)) = N(t)\left(\frac{p_H(t) - p_L(t)}{s_H - s_L} - \frac{p_L(t)}{s_L}\right), \tag{3}$$

$$N(t)y_H(t) = N(\bar{\theta} - \tilde{\theta}(t)) = N(t)\left(\bar{\theta} - \frac{p_H(t) - p_L(t)}{s_H - s_L}\right), \tag{4}$$

where $y_L(t)$ and $y_H(t)$ represent the respective shares of consumers who buy low-quality goods and high-quality goods at time t. Since both firms face a common density of consumers, we can interpret y_i as firm i's market share.

The sum of the discounted present value of firm i's profit, V_i, is

$$V_i = \int_0^\infty \pi_i(t)e^{-\rho t}dt = \int_0^\infty \left[N(t)y_i(t)(p_i(t) - c(s_i)) - \mu A_i(t)^2\right] e^{-\rho t}dt, \tag{5}$$

where $\pi_i(t)$ is firm i's profit at time t, $c(s_i)$ is the exogenous unit cost at time t, $A_i(t)$ is the investment in advertising at time t, $\mu A_i(t)^2$ is the investment cost at time t, and μ is the exogenous positive parameter. Suppose that the unit cost is a function of a firm's technology level. Since investing in R&D costs quite a bit of money, the unit cost-technology level relationship can be interpreted as follows: the cost to produce goods embedded by patented technology rises as the quantity of patents increases. For analytical simplicity, the following discussion assumes that production costs are a linear function of product quality; that is, $c(s_i) = s_i$, and that a firm's discount rate equals the consumer's discount rate, ρ. Furthermore, for each firm's market share to be non-negative, we assume the maximum preference value is equal and larger than one, $\bar{\theta} > 1$.

The state variable evolves according to the following state equation,

$$\dot{N}(t) = \alpha(A_H(t) + A_L(t)) - \lambda N(t), \tag{6}$$

where $\alpha(> 0)$ is the advertising efficiency parameter and $N(0)(> 0)$ is the initial stock. The parameter $\lambda(> 0)$ is the depreciation rate, which implies that consumers will lose their interest in the industry's goods if firms do not advertise. This law of motion implies that advertising is cooperative behavior in the sense that advertising activity by one firm benefits not only itself but also other firms. Namely, advertising is interpreted as a public good.

It is noteworthy that the state equation and firms' profit functions are linear with respect to the state variable. In addition, as will be discussed below, the control variables are independent of the state variable, and open-loop strategies do not depend on the initial state. This is called a linear state game, for which has the open-loop Nash equilibrium is Markov perfect.[2] Therefore, in the analysis below, we use the Hamiltonian function to solve this duopolistic game.

3 Characterization of the Nash equilibrium

The Hamiltonian equation for firm i is represented as follows:

$$\mathcal{H}_i = N(t)y_i(t)(p_i(t) - s_i) - \mu A_i(t)^2$$

[2]See Dockner, et al. (2000), section 7.3.

$$+ \phi_i(t)[\alpha(A_H(t) + A_L(t)) - \lambda N(t)]. \tag{7}$$

This leads to the following optimality conditions for both firms. The low quality firm's optimal conditions are as follows:

$$p_L(t) = \frac{s_L p_H(t) + s_H s_L}{2s_H}, \tag{8}$$

$$\phi_L(t) = \frac{2\mu}{\alpha} A_L(t), \tag{9}$$

$$\dot{\phi}_L(t) = (\lambda + \rho)\phi_L(t) - y_L(t)(p_L(t) - s_L), \tag{10}$$

$$0 = \lim_{t\to\infty} \phi_L(t)N(t)e^{-\rho t}. \tag{11}$$

Similarly, the optimality conditions of the high quality firm are as follows:

$$p_H(t) = \frac{\bar{\theta}(s_H - s_L) + s_H + p_L(t)}{2}, \tag{12}$$

$$\phi_H(t) = \frac{2\mu}{\alpha} A_H(t), \tag{13}$$

$$\dot{\phi}_H(t) = (\lambda + \rho)\phi_H(t) - y_H(t)(p_H(t) - s_H), \tag{14}$$

$$0 = \lim_{t\to\infty} \phi_H(t)N(t)e^{-\rho t}. \tag{15}$$

3.1 Optimal and steady state values

From (8) and (12), we get the optimal prices

$$p_L^* = \frac{s_L\bar{\theta}(s_H - s_L) + 3s_H s_L}{4s_H - s_L} \quad \text{and} \quad p_H^* = \frac{2s_H\bar{\theta}(s_H - s_L) + 2s_H^2 + s_H s_L}{4s_H - s_L}. \tag{16}$$

Remark that both prices are independent of the state variable and constant over time and that firm H's price is always higher than firm L's price under the assumption, $s_H > s_L$. From (10), (14) and (16), in the steady state, the condition $\dot{\phi}_i = 0$ leads to the following equations.

$$\phi_L^* = \frac{s_H s_L(\bar{\theta} - 1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda + \rho)} \quad \text{and} \quad \phi_H^* = \frac{4s_H^2(\bar{\theta} - 1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda + \rho)}. \tag{17}$$

These equations, (9), and (13) immediately lead to equilibrium advertising strategies:

$$A_L^* = \frac{\alpha s_H s_L(\bar{\theta} - 1)^2(s_H - s_L)}{2\mu(4s_H - s_L)^2(\lambda + \rho)} \quad \text{and} \quad A_H^* = \frac{2\alpha s_H^2(\bar{\theta} - 1)^2(s_H - s_L)}{\mu(4s_H - s_L)^2(\lambda + \rho)}. \tag{18}$$

From (6), the condition $\dot{N} = 0$ leads to

$$N^* = \frac{\alpha}{\lambda}(A_H^* + A_L^*) = \frac{\alpha^2 s_H(4s_H + s_L)(s_H - s_L)(\bar{\theta} - 1)^2}{2\lambda\mu(4s_H - s_L)^2(\lambda + \rho)}, \tag{19}$$

which is the steady state consumer density.[3] Finally, let us consider the trajectory of consumer density. We obtain this by solving (6). Substitute (18) and (19) into it yields the following equation:

$$N(t) = \frac{\alpha^2 s_H(4s_H + s_L)(s_H - s_L)(\bar{\theta} - 1)^2}{2\lambda\mu(4s_H - s_L)^2(\lambda + \rho)} + \left[N(0) - \frac{\alpha^2 s_H(4s_H + s_L)(s_H - s_L)(\bar{\theta} - 1)^2}{2\lambda\mu(4s_H - s_L)^2(\lambda + \rho)}\right] e^{-\lambda t}. \tag{20}$$

3.2 Market shares and thresholds

The two thresholds are defined in (1) and (2). Using the equilibrium prices derived above, we obtain the optimal threshold values as follows.

$$\tilde{\theta}^* = \frac{p_H^* - p_L^*}{s_H - s_L} = \frac{2s_H(\bar{\theta} + 1) - \bar{\theta}s_L}{4s_H - s_L}, \quad \hat{\theta}^* = \frac{p_L^*}{s_L} = \frac{\bar{\theta}(s_H - s_L) + 3s_H}{4s_H - s_L}. \tag{21}$$

Since prices are constant over time, the two thresholds are also constant. Responses to changes in exogenous parameters, $\bar{\theta}$, s_H, and s_L are as follows.

$$\frac{\partial\tilde{\theta}^*}{\partial\bar{\theta}} = \frac{2s_H - s_L}{4s_H - s_L} > 0, \quad \frac{\partial\tilde{\theta}^*}{\partial s_H} = \frac{2s_L(\bar{\theta} - 1)}{(4s_H - s_L)^2} > 0, \quad \frac{\partial\tilde{\theta}^*}{\partial s_L} = -\frac{2s_H(\bar{\theta} - 1)}{(4s_H - s_L)^2} < 0.$$
$$\frac{\partial\hat{\theta}^*}{\partial\bar{\theta}} = \frac{s_H - s_L}{4s_H - s_L} > 0, \quad \frac{\partial\hat{\theta}^*}{\partial s_H} = \frac{3s_L(\bar{\theta} - 1)}{(4s_H - s_L)^2} > 0, \quad \frac{\partial\hat{\theta}^*}{\partial s_L} = -\frac{3s_H(\bar{\theta} - 1)}{(4s_H - s_L)^2} < 0.$$

An increase in s_H (s_L) leads to an increase (decrease) in the thresholds $\tilde{\theta}^*$ and $\hat{\theta}^*$, with the results that the differential coefficients of $\hat{\theta}^*$ have larger absolute values than those of $\tilde{\theta}^*$. This implies that a technological improvement has larger influence on the lower threshold $\hat{\theta}^*$ than on the upper threshold $\tilde{\theta}^*$.

Constant thresholds cause both firms' market shares to be invariant to time.

$$y_H^* = \bar{\theta} - \tilde{\theta}^* = \frac{2s_H(\bar{\theta} - 1)}{4s_H - s_L}, \quad y_L^* = \tilde{\theta}^* - \hat{\theta}^* = \frac{s_H(\bar{\theta} - 1)}{4s_H - s_L}. \tag{22}$$

The market share of firm H is always higher than and twice as that of firm L, $y_H^* = 2y_L^*$. For both firms to have positive markets shares, the maximum value of consumer preference must be larger than one.

Comparative statics of market shares leads to the equivalent results due to $y_H^* = 2y_L^*$ so that it suffices to deal with firm H's market share.

$$\frac{\partial y_H^*}{\partial\bar{\theta}} = \frac{2s_H}{4s_H - s_L} > 0, \quad \frac{\partial y_H^*}{\partial s_H} = -\frac{2s_L(\bar{\theta} - 1)}{(4s_H - s_L)^2} < 0, \quad \frac{\partial y_H^*}{\partial s_L} = \frac{2s_H(\bar{\theta} - 1)}{4s_H - s_L} > 0. \tag{23}$$

The degree of change in firm H's share is larger than that in firm L's share.

[3] If $\bar{\theta} = 1$, the steady state value N^* is zero and thus both firms do not advertise, $A_i^* = \phi_i^* = 0$, for $i \in \{H, L\}$. Firms' prices are, therefore, $p_L^* = s_L$ and $p_H^* = s_H$. This means that both firms' profits are always zero and that threshold values are $\bar{\theta} = \tilde{\theta} = \hat{\theta} = 1$, so that the industry has no market share and disappears.

3.3 Transversality conditions and stability

In this subsection, we derive the conditions needed for the transversality conditions and the stability of our model to be satisfied. We use equations (3), (4), (16), and (22) to solve differential equations for the co-state variables, (10) and (14). The following equations are obtained.

$$\phi_L(t) = \frac{s_H s_L(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)} + \left[\phi_L(0) - \frac{s_H s_L(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)}\right] e^{(\lambda+\rho)t}, \tag{24}$$

$$\phi_H(t) = \frac{4s_H^2(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)} + \left[\phi_H(0) - \frac{4s_H^2(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)}\right] e^{(\lambda+\rho)t}. \tag{25}$$

Using these equations, we confirm that the transversality conditions of both firms are satisfied. The result is summarized in the following lemma.

Lemma 1. *For the transversality conditions of firm L and firm H to be satisfied the following conditions are required.*

$$\phi_L(0) = \frac{s_H s_L(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)} \text{ and } \phi_H(0) = \frac{4s_H^2(\bar{\theta}-1)^2(s_H - s_L)}{(4s_H - s_L)^2(\lambda+\rho)}. \tag{26}$$

Proof. See Appendix A. □

The lemma implies that each firm makes an advertising decision as it maintains a constant advertising investment over time. The reason is as follows. Both co-state variables are constant for all times t under condition (26) and the relationship between the co-state variable and advertising is given as $A_i(t) = \alpha\phi_i(t)/(2\mu)$. These imply that advertising investments must be invariant over time. In other words, each firm chooses its advertising investment level to satisfy the condition of the lemma.

The complete dynamical system in the model is, therefore, represented with only the law of motion for consumer density (6). The trajectory of consumer density obtained in (20) converges to the steady state value, N^*, as $t \to \infty$ due to $\lambda > 0$. As a result, the dynamical system in the model is globally stable. We summarize these as the following lemma.

Lemma 2. *In open-loop Nash equilibrium, prices and the two thresholds are also invariant over time. Advertising investments and co-state variables are also constant over time and their values are equal to the steady state values.*

4 Producer surplus

4.1 Firms' profits

In this subsection, we calculate firms' profits. For notational simplicity, we define a firm's instantaneous profit per consumer density, $y_i^*(p_i^* - s_i)$, as z_i^*.[4] Using (5), equilibrium val-

[4]These are defined in Appendix A.

ues, and the density function, we can calculate the sum of the discounted present value of firm i's profit.

$$V_i = \int_0^\infty \left[N(t) z_i^* - \mu (A_i^*)^2 \right] e^{-\rho t} dt = \frac{N(0) z_i^*}{\lambda + \rho} + \frac{\lambda N^* z_i^* - (\lambda + \rho) \mu (A_i^*)^2}{\rho (\lambda + \rho)}. \tag{27}$$

According to Lemma 2, co-state variables, market shares, and prices are constant on the equilibrium path. Using Lemma 1, (9), (10), (13), and (14), we can derive the equation that relates a firm's instantaneous profit per density to its advertising investment.

$$z_i^* = \frac{2\mu(\lambda + \rho)}{\alpha} A_i^*, \quad i \in \{L, H\}. \tag{28}$$

We substitute this into the above profit function, (27), and rewrite it as follows:

$$V_i^* = \frac{2\mu}{\rho} \left[\frac{\rho N(0)}{\alpha} A_i^* + \frac{(A_i^*)^2}{2} + A_i^* A_j^* \right], \ j \neq i. \tag{29}$$

This equation implies that each firm's profits benefit not only from its own advertising investment but also from the opponent's advertising investment. This is a positive externality represented in the third term within the square bracket. However, both firms have a definite incentive to advertise to entice new customers into the market; that is, they never choose not to advertise and simply enjoy a free ride on the opponent's advertisement behavior.

Next, we compare both firms' profits.

$$V_H^* - V_L^* = \mu (A_H^* - A_L^*) \left[\frac{2N(0)}{\alpha} + \frac{A_H^* + A_L^*}{\rho} \right].$$

The left hand side sign depends on which firm invests more in advertising. According to (18), we easily confirm that firm H's advertising investment is always larger than firm L's.

$$A_H^* - A_L^* = \frac{\alpha s_H (\bar{\theta} - 1)^2 (s_H - s_L)}{2\mu (4 s_H - s_L)(\lambda + \rho)} > 0.$$

We summarize the result as the following proposition.

Proposition 1. *Advertising investment and lifetime profit of firm H are always larger than those of firm L.*

The difference in lifetime profits is proportional to the difference in advertising investment levels. This is obtained from the fact that firm H's market share is always higher than firm L's.

4.2 Comparative statics for prices, advertising, consumer density, and profits

In this subsection, we investigate how each firm responds to a change in maximum preference and technology. First, we calculate comparative statics for both firms' prices. It is easy to confirm that both firms increase their prices as consumer preferences become increasingly heterogeneous. The reasons are as follows. When maximum willingness to pay increases from $\bar{\theta}$ to $\bar{\theta}'$, two market thresholds increase; $\tilde{\theta}^*$ and $\hat{\theta}^*$ rise to $\tilde{\theta}'$ and $\hat{\theta}'$, respectively.[5] In other words, consumers with a high preference $\theta \in [\tilde{\theta}^*, \tilde{\theta}']$ change their decision and purchase good H and not good L, and consumers with low preference $\theta \in [\hat{\theta}^*, \hat{\theta}']$ become disinclined to buy good L and exit the market.[6] This implies that, since consumers who have higher appreciation for good H enter the market and undertake a purchase, firm H has an incentive to drive its price higher and thus firm L faces the same pressure.

Similarly, we can conclude that the both firms raise prices when firm H improves its technology by R&D and acquire new patents and that good H is reduced in price when firm L develops its technology. However, the pricing response by firm L to a rise in s_L is complicated. According to section 3.2, the two threshold values become smaller, which means that firm L is likely to cut good L's price. In contrast, the first term in the utility functions of consumers who purchase a good L, increases as s_L rises to s'_L; i.e., from θs_L to $\theta s'_L$.[7] Therefore, it may be possible that firm L can derive its price even if the threshold $\hat{\theta}^*$ becomes smaller.

Let us consider this problem directly by differentiating firm L's equilibrium price (16). The derivative of p_L^* with respect to s_L is

$$\frac{\partial p_L^*}{\partial s_L} = \frac{4(\bar{\theta}+3)s_H^2 - 8\bar{\theta}s_H s_L + \bar{\theta}s_L^2}{(4s_H - s_L)^2}.$$

The derivative's sign is dependent on that of the numerator, thus it suffices to consider the numerator. We represent the numerator as $\xi(\bar{\theta}, s_H, s_L)$. First rearranging the function ξ we investigate the determination of the sign:

$$\xi(\bar{\theta}, s_H, s_L) = \bar{\theta}\left(4s_H^2 - 8s_H s_L^2 + s_L^2\right) + 12s_H^2.$$

Since for $s_H > \frac{2+\sqrt{3}}{2}s_L$, the terms in parentheses is positive,[8] it is guaranteed that, for $s_H \geq \frac{2+\sqrt{3}}{2}s_L$, $\xi(\bar{\theta}, s_H, s_L)$ is positive. Next we consider the range $s_L < s_H < \frac{2+\sqrt{3}}{2}s_L$. If s_L is quite close to s_H, the numerator $\xi(\bar{\theta}, s_H, s_H) = -3s_H^2(\bar{\theta} - 4)$. Therefore,

$$\xi(\bar{\theta}, s_H, s_H) \begin{cases} < 0 & \text{if } \bar{\theta} > 4 \\ \geq 0 & \text{if } 1 < \bar{\theta} \leq 4. \end{cases}$$

[5] See Section 3.2.

[6] See Figure 4 below.

[7] The utility function is defined as $u_\theta^i(t) = \theta s_i - p_i(t)$.

[8] We can find s_H which satisfies $4s_H^2 - 8s_H s_L^2 + s_L^2 = 0$, which are $s_H = \frac{2-\sqrt{3}}{2}s_L$ and $s_H = \frac{2+\sqrt{3}}{2}s_L$.

If $s_H = \frac{7}{4}s_L$, the numerator $\xi(\bar{\theta}, s_H, \frac{4}{7}s_H) = -\frac{12}{49}s_H^2(\bar{\theta} - 49)$ and thus

$$\xi\left(\bar{\theta}, s_H, \frac{4}{7}s_H\right) \begin{cases} < 0 & \text{if } \bar{\theta} > 49 \\ \geq 0 & \text{if } 1 < \bar{\theta} \leq 49. \end{cases}$$

Hence, in this case, it is observed that for relatively high $\bar{\theta}$ and/or for s_L very close to s_H, the derivative is negative and otherwise positive, respectively.

Second, we investigate how firms change their advertising strategies in response to changes in exogenous parameters. As can be seen from equations (19) and (29), in the present study, advertising strategies play the most important role in both steady state consumer density and each firms' profits. By differentiating equilibrium advertising (18) with respect to $\bar{\theta}$, s_H, and s_L, we obtain the following proposition:

Proposition 2. *When the maximum preference value and technology level of firm H increase, each firm increases the levels of investment in advertising. When the technology level of firm L increases, firm H decreases its investment in advertising, and firm L decreases its investment if $s_L < s_H < \frac{7}{4}s_L$, increases it if $\frac{7}{4}s_L < s_H$, and keeps it unchanged if $s_H = \frac{7}{4}s_L$.*

Proof. It is straightforward to derive the first part of the proposition. First, we differentiate (18) with respect to $\bar{\theta}$,

$$\frac{\partial A_L^*}{\partial \bar{\theta}} = \frac{\alpha s_H s_L (s_H - s_L)(\bar{\theta} - 1)}{\mu(4s_H - s_L)^2(\lambda + \rho)} > 0 \quad \text{and} \quad \frac{\partial A_H^*}{\partial \bar{\theta}} = \frac{4\alpha s_H^2 (s_H - s_L)(\bar{\theta} - 1)}{\mu(4s_H - s_L)^2(\lambda + \rho)} > 0,$$

and differentiate (18) with respect to s_H,

$$\frac{\partial A_L^*}{\partial s_H} = \frac{\alpha s_L^2 (2s_H + s_L)(\bar{\theta} - 1)^2}{2\mu(4s_H - s_L)^3(\lambda + \rho)} > 0 \quad \text{and} \quad \frac{\partial A_H^*}{\partial s_H} = \frac{2\alpha s_H (4s_H^2 - 3s_H s_L + 2s_L^2)(\bar{\theta} - 1)^2}{\mu(4s_H - s_L)^3(\lambda + \rho)}.$$

The term $4s_H^2 - 3s_H s_L + 2s_L^2$ in the latter equation is positive for region $s_H > s_L$ because

$$4s_H^2 - 3s_H s_L + 2s_L^2 = 4\left(s_H - \frac{3}{8}s_L\right)^2 + \frac{23}{16}s_L^2,$$

and thus the sign is positive for $s_H > s_L$. To prove the second part, we differentiate (18) with respect to s_L;

$$\frac{\partial A_L^*}{\partial s_L} = \frac{\alpha s_H^2 (4s_H - 7s_L)(\bar{\theta} - 1)^2}{2\mu(4s_H - s_L)^3(\lambda + \rho)} \quad \text{and} \quad \frac{\partial A_H^*}{\partial s_L} = -\frac{2\alpha s_H^2 (2s_H + s_L)(\bar{\theta} - 1)^2}{\mu(4s_H - s_L)^3(\lambda + \rho)} < 0.$$

The sign of the derivative A_L^* depends on the sign of the term $4s_H - 7s_L$ and thus

$$\frac{\partial A_L^*}{\partial s_L} \begin{cases} > 0 & \text{if} \quad s_H > \frac{7}{4}s_L \\ = 0 & \text{if} \quad s_H = \frac{7}{4}s_L \\ < 0 & \text{if} \quad s_L < s_H < \frac{7}{4}s_L. \end{cases}$$

□

An increase in s_L has a different effect on each firm; the advertising investment of firm H always decreases but the sign of the advertising investment made by firm L is determined by the technology difference between the two firms. If the difference is relatively small (large), firm L decreases (increases) its advertising activities. In contrast, in the internal case of covered market, firm L always decreases its advertising activity.[9]

Third, we examine comparative statics for consumer density. When an exogenous parameter changes, consumer density is affected through changes in firms' advertising behavior. We call this the advertising effect. According to (19) and (20), the variation in consumer density at time t hinges on density in the steady state value.[10] It suffices to investigate the derivative of consumer density in the steady state. Differentiating (19) with respect to exogenous parameters, we obtain the following result.

Proposition 3. *The instantaneous consumer density, $N(t)$, and the steady state consumer density, N^*, are increasing functions of the maximum preference value and the technology level of firm H and decreasing functions of firm L's technology level.*

Proof. It suffices to prove the case of the steady state consumer density. Using (19) and Proposition 2, we confirm that the derivatives of N^* with respect to $\bar{\theta}$ and s_H are increasing functions,

$$\frac{\partial N^*}{\partial \bar{\theta}} = \frac{\alpha}{\lambda}\left(\frac{\partial A_L^*}{\partial \bar{\theta}} + \frac{\partial A_H^*}{\partial \bar{\theta}}\right) > 0 \quad \text{and} \quad \frac{\partial N^*}{\partial s_H} = \frac{\alpha}{\lambda}\left(\frac{\partial A_L^*}{\partial s_H} + \frac{\partial A_H^*}{\partial s_H}\right) > 0,$$

and a decreasing function,

$$\frac{\partial N^*}{\partial s_L} = \frac{\alpha}{\lambda}\left(\frac{\partial A_L^*}{\partial s_L} + \frac{\partial A_H^*}{\partial s_L}\right) = -\frac{\alpha^2 s_H^2 (4s_H + 11s_L)(\bar{\theta} - 1)^2}{\lambda \mu (4s_H - s_L)^3 (\lambda + \rho)} < 0.$$

□

The consumer density in the steady state is dependent on the direction of changes in both firms' advertising strategies. According to Proposition 2, changes in maximum preference and technology level of firm H have the same effect on firms' advertising and hence their effect on the consumer density is unique. When firm L succeeds in R&D, then the question of whether firm L will increase its advertising is determined by the technology difference between the two firms. Even if, however, firm L increases its advertising investment level, the effect is relatively small and is dominated by the effect of firm H and thus the aggregate effect is unique and negative.

[9] See Tenryu and Kamei (2013, 2014).

[10] Consumer density at time t is represented as $N(t) = N^* + [N(0) - N^*]e^{-\lambda t}$. We defferentiate it with respect to x; thus

$$\frac{\partial N(t)}{\partial x} = \left(1 - \frac{1}{e^{\lambda t}}\right)\frac{\partial N^*}{\partial x},$$

where x is an exogenous parameter; $\bar{\theta}$ or s_i. Since, for any time $t > 0$, λt is positive, the value in parentheses is positive whereas the sign of $\partial N(t)/\partial x$ is determined by a change in the steady state value.

Finally, we calculate comparative statics for both firms' lifetime profits. The sum of the discounted present value of a firm's profit is given by (29), which is represented as a function of both firms' advertising investment amounts. The effects of exogenous parameters on advertising strategies are discussed in Propositions 2 and 3. Using this and differentiating it with respect to exogenous parameters, we obtain the following two propositions.

Proposition 4. *Lifetime profits of both firms are increasing functions of the maximum preference level and firm H's technology level. The lifetime profit of firm H is a decreasing function of firm L's technology level. If $s_L < s_H \leq \frac{7}{4}s_L$, firm L's lifetime profit is a decreasing function and otherwise a threshold value σ exists such that*

$$\sigma \equiv \frac{\alpha A_L^*(16s_H^2 - 32s_H s_L - 11s_L^2)}{\rho s_L(7s_L - 4s_H)}.$$

Further, if the initial consumer density exceeds the threshold value, firm L's lifetime profit is an increasing function of s_L. In contrast, if initial consumer density is smaller than the threshold value, its profit is a decreasing function of s_L.

$$\frac{\partial V_L^*}{\partial s_L} \begin{cases} > 0 & \text{if } N(0) > \sigma \\ = 0 & \text{if } N(0) = \sigma \\ < 0 & \text{if } N(0) < \sigma. \end{cases}$$

Proof. We differentiate (29) with respect to $\bar{\theta}$: for $i(\neq j)$,

$$\frac{\partial V_i^*}{\partial \bar{\theta}} = \frac{2\mu}{\rho}\left[\frac{\rho N(0)}{\alpha}\frac{\partial A_i^*}{\partial \bar{\theta}} + (A_i^* + A_j^*)\frac{\partial A_i^*}{\partial \bar{\theta}} + A_i^*\frac{\partial A_j^*}{\partial \bar{\theta}}\right], \quad i \in \{L, H\}.$$

According to Proposition 2, this is positive, and thus V_i^* is an increasing function of $\bar{\theta}$.

Similarly, we calculate the derivatives of lifetime profits with respect to s_H: for $i(\neq j)$,

$$\frac{\partial V_i^*}{\partial s_H} = \frac{2\mu}{\rho}\left[\frac{\rho N(0)}{\alpha}\frac{\partial A_i^*}{\partial s_H} + (A_i^* + A_j^*)\frac{\partial A_i^*}{\partial s_H} + A_i^*\frac{\partial A_j^*}{\partial s_H}\right], \; i \in \{L, H\}.$$

This is positive because both derivatives of advertising are positive according to Proposition 2. Differentiating firm H's lifetime profit with respect to s_L we obtain

$$\frac{\partial V_H^*}{\partial s_L} = \frac{2\mu}{\rho}\left[\left(\frac{\rho N(0)}{\alpha} + A_L^*\right)\frac{\partial A_H^*}{\partial s_H} + A_H^*\left(\frac{\partial A_H^*}{\partial s_H} + \frac{\partial A_L^*}{\partial s_H}\right)\right].$$

As discussed in Propositions 2 and 3, the derivative of A_H^* is negative and

$$\frac{\partial A_L^*}{\partial s_L} + \frac{\partial A_H^*}{\partial s_L} = -\frac{\alpha s_H^2(4s_H + 11s_L)(\bar{\theta} - 1)^2}{\mu(4s_H - s_L)^3(\lambda + \rho)} < 0.$$

Therefore, firm H's lifetime profit is a decreasing function of firm L's technology level.

Finally, we differentiate (29) with respect to s_L and obtain

$$\frac{\partial V_L^*}{\partial s_L} = \frac{2\mu}{\rho}\left[\left(\frac{\rho N(0)}{\alpha} + A_H^*\right)\frac{\partial A_L^*}{\partial s_L} + A_L^*\left(\frac{\partial A_H^*}{\partial s_L} + \frac{\partial A_L^*}{\partial s_L}\right)\right]. \tag{30}$$

According to Proposition 2, firm L increases its advertising investment level if $s_H > \frac{7}{4}s_L$ and decreases it if $s_L < s_H < \frac{7}{4}s_L$. When firm L improves its technology level, firm H always decreases its advertising investment level. Therefore, we consider the sign of (30) by dividing the proof into two cases.

Case 1: $s_L < s_H \leq \frac{7}{4}s_L$

In this case, both advertising functions are decreasing functions of s_L for the region $s_L < s_H < \frac{7}{4}s_L$. Meanwhile, for $s_H = \frac{7}{4}s_L$, firm L does not change its advertising strategy but firm H decreases its advertising investment level. Therefore, the sign of (30) is negative.

Case 2: $\frac{7}{4}s_L < s_H$

In this region, the derivatives of A_H^* and A_L^* have opposite signs, $\frac{\partial A_H^*}{\partial s_L} < 0$ and $\frac{\partial A_H^*}{\partial s_L} > 0$. Thus, the sign of (30) is ambiguous:

$$\frac{\partial V_L^*}{\partial s_L} = \frac{2\mu}{\rho}\left[\left(\frac{\rho N(0)}{\alpha} + A_H^*\right)\underbrace{\frac{\partial A_L^*}{\partial s_L}}_{(+)} + A_L^*\underbrace{\left(\frac{\partial A_H^*}{\partial s_L} + \frac{\partial A_L^*}{\partial s_L}\right)}_{(-)}\right].$$

To investigate the sign, we arrange this as follows:

$$\frac{\partial V_L^*}{\partial s_L} = \frac{\rho s_H^2(\bar{\theta}-1)^2 s_L(4s_H - 7s_L)}{\alpha s_L(4s_H - s_L)^3(\lambda+\rho)}\left[N(0) + \frac{\alpha A_L^*(16s_H^2 - 32s_H s_L - 11s_L^2)}{\rho s_L(4s_H - 7s_L)}\right].$$

We can confirm that, for region $s_H \geq \frac{4+3\sqrt{3}}{4}s_L$, the equation $16s_H^2 - 32s_H s_L - 11s_L^2$ is non-negative and hence $\frac{\partial V_L^*}{\partial s_L}$ is positive.

Next, we consider the remaining region, namely $\frac{7}{4}s_L < s_H < \frac{4+3\sqrt{3}}{4}s_L$. Within the region, the second term in the square brackets is , making the sign ambiguous. Let us define σ as

$$\sigma \equiv \frac{\alpha A_L^*(16s_H^2 - 32s_H s_L - 11s_L^2)}{\rho s_L(7s_L - 4s_H)}.$$

We can easily confirm that the sign in the square brackets is positive (zero or negative) if $N(0) > \sigma$ ($N(0) = \sigma$ or $N(0) < \sigma$). □

Proposition 4 states that both firms enjoy a more heterogeneous consumer preference and firm H enjoys a wider technology gap between firms: an increase in s_H or a decrease

in s_L. Although firm L also benefits from firm H's technological improvement, whether firm L receives benefits from an increase in its technology level depends on the situation. The results for firm L are summarized in Figure 2. According to the discussion on price and Proposition 2, even if firm s_L succeeds in R&D, the question of whether it invests more in advertisement is not uniquely determined. In the case that technological gap between two firms is relatively large, say $s_H > \frac{4+3\sqrt{3}}{4}s_L$, firm L obtains additional profit. In the case that the gap is relatively moderate, the marginal advertising investment is positive but whether firm L benefits from its R&D depends upon the initial consumer density. If initial consumer density is relatively large (small), say $N(0) > (<)\sigma$, firm L has a positive (negative) marginal profit. The reason is as follows. According to (28), the marginal profit per density is proportional to the marginal advertising investment level and firm L obtains the positive marginal profit per density. At the same time, the marginal density is negative because the effect from firm H always dominates firm L's effect. Therefore, in the case where the technology gap is high, $s_H > \frac{4+3\sqrt{3}}{4}s_L$, the former dominates the latter. In the moderate case, $\frac{7}{4}s_L < s_H < \frac{4+3\sqrt{3}}{4}s_L$, since the latter becomes stronger, the second term of the right-hand side in (27) is negative. Thus if initial consumer density is large the first term of the right-hand side in (27) exceeds the second term and firm L can earn an additional profit. In emerging industries, the initial consumer density is almost zero and thus the marginal profit becomes negative in the intermediate region.

In a partial market, when the technology gap between firm H and L is relatively large, firm L has an opportunity to obtain positive benefits by improving its technology. For firm L, an increase in s_L increases both its market share and unit profits, and thus increases z_L^* and its advertising investment level. The increase in the advertising investment level makes the consumer density losses smaller. As a result, these effects exceed the advertising effect. In contrast, in the internal case of a covered market (Tenryu and Kamei (2013, 2014)), firm L cannot obtain an additional profit from improving its technology. Technology improvement has no effect on the threshold value, $\tilde{\theta}$, and decreases firm L's unit profit. Thus the profit per density of firm L, z_L^*, always decreases and firm L cannot enjoy an additional benefit.

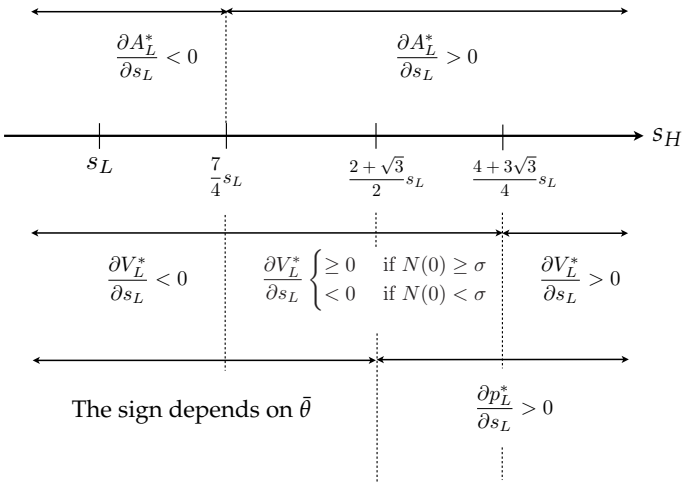


Figure 2: Comparative statics for firm L

5 Consumer surplus

5.1 Instantaneous utility and consumer surplus

The instantaneous utility of a consumer with preference θ is defined as $u_\theta^i(t) = \theta s_i - p_i(t)$. In equilibrium, the price is invariant so that instantaneous utility remains constant over time. Before proceeding to an analysis of consumers' lifetime utility, this subsection and the next subsection will investigate utility and consumer surplus at time t.

The instantaneous utility that consumers with preference $\theta \in [\tilde{\theta}, \bar{\theta}]$ derives from buying one unit of good H is represented as u_θ^H,

$$u_\theta^H = \theta s_H - p_H^* = \frac{s_H^2(4\theta - 2\bar{\theta} - 2) + s_H s_L(2\bar{\theta} - \theta - 1)}{4s_H - s_L}, \tag{31}$$

The instantaneous utility that consumers with preference $\theta \in [\hat{\theta}, \tilde{\theta}]$ derives from purchasing one unit of good L is represented as u_θ^L,

$$u_\theta^L = \theta s_L - p_L^* = \frac{s_L^2(\bar{\theta} - \theta) + s_H s_L(-\bar{\theta} + 4\theta - 3)}{4s_H - s_L}. \tag{32}$$

Consumers with preference $\tilde{\theta}^*$ are indifferent between purchasing either good L or good H so that the utility derived from the respective good corresponds, $u_{\tilde{\theta}^*}^H = u_{\tilde{\theta}^*}^L$, and utility function kinks at this point because the slope of obtained utility changes from s_L to s_H at $\tilde{\theta}$. In addition, consumers with preference $\hat{\theta}^*$ are indifferent between purchasing good L and buying nothing and hence they derive zero utility, $u_{\hat{\theta}^*}^H = u_{\hat{\theta}^*}^L = 0$. To sum up, we can illustrate consumers' instantaneous utility in Figure 3.

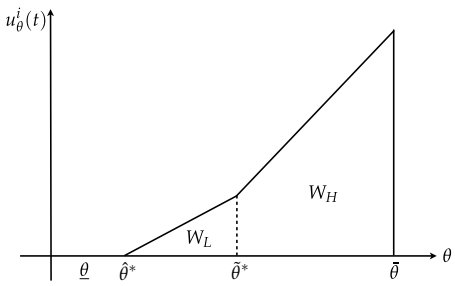


Figure 3: The instantaneous utility function

In Figure 3, W_i, $i \in \{L, H\}$, represents the instantaneous sum of utility derived by consumers who purchase good i as consumer surplus. We define this as follows:

$$W_L \equiv \int_{\hat{\theta}^*}^{\tilde{\theta}^*} (s_L\theta - p_L^*)d\theta, \quad \text{and} \quad W_H \equiv \int_{\tilde{\theta}^*}^{\bar{\theta}} (s_H\theta - p_H^*)d\theta. \tag{33}$$

In the next subsection, we focus on the utility of incumbent consumers and calculate comparative statics for their individual utility and consumer surplus at time t.

5.2 Comparative statics for instantaneous utility and surplus

As discussed in section 3.2, when an exogenous parameter increases, thresholds change and thus new consumers may enter and some incumbent consumers may exit the market. The remaining incumbent consumers remain in the market and continue to buy one unit of good but their utility may or may not change. In this subsection, we assume that the consumer density is given to investigate how the incumbent consumers' utility and consumer surplus respond to changes in the maximum preference value, firm H's technology, or firm L's technology. We consider the lifetime case, consumer density varies over time, in the subsequent subsection.

First, we calculate comparative statics for incumbent consumers' instantaneous utility. Differentiating it with respect to $\bar{\theta}$ and s_i, we obtain the following results.

Lemma 3. *Incumbent consumers become worse off for an increase in the maximum preference value and better off for an increase in firm L's technology. When an increases in firm H's technology level, incumbent consumers with preference $\theta \in [\hat{\theta}, \theta_t)$ become worse off, consumers with θ_t keep their utility, and consumers with $\theta \in (\theta_t, \bar{\theta}]$ become better off.*

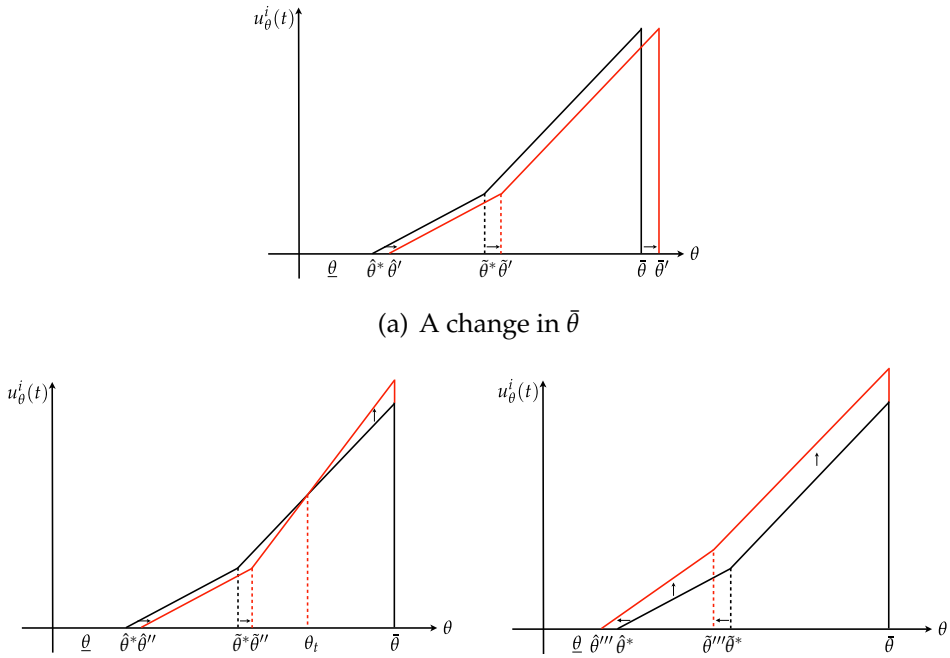
Proof. See Appendix B. □

It is easy to understand this proposition by considering Figure 4. Whether incumbent consumers are better off is determined by how firms raise their goods' prices in response to parameter changes. An increase in the maximum preference value has no effect on firms' technology, but increases the price of a respective firm's goods, which is undesirable for incumbent consumers (see Figure 4(a)).

Since the utility function is divided into two terms, θs_i and p_i, then the case where the technology improvement occurs has two effects on utility. Consumers buying good i appreciate firm i's technological improvement. Thus, we call the effect of the first term the appreciation effect and the effect of the second term the price effect. Henceforth, we collectively call these two effects the utility effect.

When the technology of firm H is improved, both firms raise the price of their goods. This improvement has no appreciation effect on consumers purchasing behavior toward good L because θs_L is independent of a change in s_H. However, the price increase through s_H is not desirable and thus their utility drops. This leads that a segment of consumers buying good L to exit the market. In contrast, consumers purchasing good H benefit from the appreciation effect but they simultaneously receive good H's price increase. Due to these opposite effects on utility, the overall effect is ambiguous. Lemma 3 claims that the overall effect is determined by their preference. That is, consumers with over θ_t is better off because the former dominates the latter and conversely consumers with under θ_t is worse off because the latter dominates the former (see Figure 4(b)).

As firm L's technology increases, good H's price is decreasing, but good L's price increases in some situations but decreases in other situations. Those who buy good H can benefit from the drop in the price while their appreciations are not changed, and are thus better off. In contrast, consumers buying good L enjoy the technological improvement of firm L, and the effect of the improvement is larger than that of the change in price even if the price increases. Therefore, they are always better off and new customers who want to buy good L enter the market (see Figure 4(c)).



(a) A change in $\bar{\theta}$

(b) A change in s_H

(c) A change in s_L

Figure 4: Comparative statics for instantaneous utility

Next, we calculate comparative statics for incumbent consumer surplus given by (33). Differentiating these functions with respect to $\bar{\theta}$ and s_i yields the following result.

Lemma 4. *Given the current consumer density, both incumbent consumer surplus functions are increasing functions of the maximum preference value and firm L's technology. W_L is an increasing function of firm H's technology and W_H is*

$$\frac{\partial W_H}{\partial s_H} \begin{cases} > 0 & \text{if } s_H > \frac{3+\sqrt{41}}{8} s_L \\ \leq 0 & \text{if } s_L < s_H \leq \frac{3+\sqrt{41}}{8} s_L \end{cases}$$

The second equation is zero only if the equality in the condition is satisfied.

Proof. See Appendix C. □

We can intuitively interpret this finding by comparing values between the utility functions before and after changes in exogenous parameters (see Figure 4). To do so, we add the amount of change in incumbent consumer utility to the amount of utility that new consumers derive from purchasing one unit of good. For example, we consider the case that $\bar{\theta}$ rises. Consumers with preference $\theta \in [\hat{\theta}, \hat{\theta}']$ exit the market and consumers with preference $\theta \in [\hat{\theta}', \tilde{\theta}]$ continue to buy a good L but their utility declines. Consumers with preference $\theta \in [\tilde{\theta}, \tilde{\theta}']$ change their mind and get to purchase good L, which increases the consumer surplus in the good L market. The latter effect dominates the former so that W_L increases.

In the good H market, consumers with preference $\theta \in [\tilde{\theta}, \tilde{\theta}']$ desist buying good H and move into the good L market,[11] whereas the utility of consumers with preference $\theta \in [\tilde{\theta}', \bar{\theta}]$ decreases, and consumers with preference $\theta \in [\bar{\theta}, \bar{\theta}']$ enter the good H market. The third effect is positive for W_H and larger than the other negative effects. As a result, W_H increases. We can similarly interpret the remaining cases.

5.3 Lifetime utility and surplus

We now calculate changes in incumbent consumers' lifetime utility. In our model, prices constant over time mean that instantaneous utility is also constant, and thus we can separate instantaneous utility from the discount rate. Hence, we obtain the following:

$$\bar{u}_\theta^i = \int_0^\infty (\theta s_i - p_i^*) e^{-\rho t} dt = \frac{1}{\rho} u_\theta^i, \;\; i \in \{L, H\}$$

This implies that the comparative statics is the same result as that of Lemma 3.

To analyze welfare in the economy, we must consider the evolution of consumer density, $N(t)$. To do so, we define the lifetime consumer surplus as follows:

$$W = \int_{\hat{\theta}^*}^{\tilde{\theta}^*} \int_0^\infty N(t)(\theta s_L - p_L^*) e^{-\rho t} dt d\theta + \int_{\tilde{\theta}^*}^{\bar{\theta}} \int_0^\infty N(t)(\theta s_H - p_H^*) e^{-\rho t} dt d\theta.$$

In our model, since instantaneous utility is constant and the state variable is independent of consumer preferences, no interactions occur between θ and t. Therefore, the lifetime surplus function can be rearranged as follows:

$$W = \int_0^\infty N(t) e^{-\rho t} (W_L + W_H) dt = \frac{\lambda N^* + \rho N(0)}{\rho(\lambda + \rho)} \left[\frac{s_H^2 (4 s_H + 5 s_L)(\bar{\theta} - 1)^2}{2(4 s_H - s_L)^2} \right]. \tag{34}$$

Using this equation, we analyze the effects of the maximum preference value and a respective firm's technological improvement. The steady state consumer density and instantaneous consumer surplus play important roles in the analysis. Using this equation enables us dissect the overall effect in two parts: the utility effect and the advertising effect. Results obtained above predict the result that, in the case where $\bar{\theta}$ increases, the direction of change in W is unique, but in the case where s_i increases, it is not unique. In the following proposition, we prove the prediction is true.

Proposition 5. *Lifetime surplus increases as the maximum preference value increases. When a respective firm's technology improves, consumer lifetime surplus increases if the technological gap between both firms is relatively large. If the gap is relatively small, when firm H's (L's) technology increases, the lifetime surplus increases if the initial consumer density, discount rate, or depreciation rate is relatively small (large), or the advertising efficiency parameter is relatively large (small).*

[11] This is completely identical the latter effect in the good L market.

Proof. Differentiating (34) with respect to $\bar{\theta}$, we obtain

$$\frac{\partial W}{\partial \bar{\theta}} = \underbrace{\frac{\lambda(W_H + W_L)}{\rho(\lambda+\rho)}\frac{\partial N^*}{\partial \bar{\theta}}}_{\text{the advertising effect}} + \underbrace{\frac{\lambda N^* + \rho N(0)}{\rho(\lambda+\rho)}\left(\frac{\partial W_H}{\partial \bar{\theta}} + \frac{\partial W_L}{\partial \bar{\theta}}\right)}_{\text{the utility effect}},$$

which is positive because of Proposition 3 and Lemma 4.

Next, we derive the derivative with respect to s_H.

$$\frac{\partial W}{\partial s_H} = \frac{s_H(\bar{\theta}-1)^2}{\rho(\lambda+\rho)(4s_H - s_L)^3}\left[\frac{\alpha^2 s_H \psi(s_H, s_L)(\bar{\theta}-1)^2}{4\mu(4s_H - s_L)^2(\lambda+\rho)} + \rho N(0)(2s_H + s_L)(4s_H - 5s_L)\right],$$

where $\psi(s_H, s_L) = 128s_H^4 - 64s_H^3 s_L - 40s_H^2 s_L^2 + 96s_H s_L^3 + 15s_L^4$. The first term is positive because the function $\psi(s_H, s_L)$ is arranged as

$$\psi(s_H, s_L) = 8s_H^2(s_H - s_L)(13s_H + 5s_L) + 14s_H(s_H^3 + s_L^3) + s_L^3(82s_H + 15s_L),$$

and this is always positive for $s_H > s_L$. Let us consider the case where $s_H \geq \frac{5}{4}s_L$. As, in this region, the second term in the square brackets is non-negative, we confirm that lifetime consumer surplus increases as firm H's technology improves. In the case where $s_L < s_H < \frac{5}{4}s_L$, however, an increase in s_H has opposite impacts on the steady state consumer density and instantaneous consumer surplus. The sign of the derivative depends on the sign in the square brackets so that

$$\frac{\partial W}{\partial s_H} \gtreqless 0 \quad \Longleftrightarrow \quad N(0) \lesseqgtr \frac{\alpha^2 s_H \psi(s_H, s_L)(\bar{\theta}-1)^2}{4\mu\rho(2s_H + s_L)(5s_L - 4s_H)(4s_H - s_L)^2(\lambda+\rho)}.$$

This implies that the lifetime consumer surplus increases if the initial consumer density is relatively low, but it decreases if the density is relatively high.

Finally, we calculate the derivative of the lifetime consumer surplus with respect to firm L's technology level.

$$\frac{\partial W}{\partial s_L} = \frac{\lambda(W_H + W_L)}{\rho(\lambda+\rho)}\frac{\partial N^*}{\partial s_L} + \frac{\lambda N^* + \rho N(0)}{\rho(\lambda+\rho)}\left(\frac{\partial W_H}{\partial s_L} + \frac{\partial W_L}{\partial s_L}\right)$$

The first term is negative due to Proposition 3 and the second term is positive due to Lemma 4 and thus the derivative's sign is ambiguous. To investigate the sign, we rearrange the equation as follows:

$$\frac{\partial W}{\partial s_L} = \frac{s_H^2(\bar{\theta}-1)^2}{2\rho(\lambda+\rho)(4s_H - s_L)^3}\left[\frac{\alpha^2 s_H \pi(s_H, s_L)(\bar{\theta}-1)^2}{2\mu(4s_H - s_L)^2(\lambda+\rho)} + \rho N(0)(28s_H + 5s_L)\right],$$

where $\pi(s_H, s_L) = 80s_H^3 - 192s_H^2 s_L - 153s_H s_L^2 - 5s_L^3$. For a relatively large gap between s_H and s_L, say for about $s_H > 3.037s_L$, then π is positive. In this condition, the derivative's sign is always positive. In contrast, for a relatively small gap, the sign can be determined as followed:

$$\frac{\partial W}{\partial s_L} \gtreqless 0 \quad \Longleftrightarrow \quad N(0) \gtreqless -\frac{\alpha^2 s_H \pi(s_H, s_L)(\bar{\theta}-1)^2}{2\rho\mu(28s_H + 5s_L)(4s_H - s_L)^2(\lambda+\rho)}.$$

□

This proposition implies that, when the technology gap between two firms is relatively large, welfare in the economy is likely to be better off by improving both firms' technology levels. When the gap is relatively small, the effect on welfare is determined by the relative size of initial consumer density. In emerging industries, the initial consumer density is zero. In this situation, $\frac{\partial W}{\partial s_H}$ is always positive and $\frac{\partial W}{\partial s_L} \gtreqless 0$ only if approximately $s_H \gtreqless 3.037 s_L$.

6 Concluding remarks

We consider a dynamic voluntary advertising model with a duopoly following the research direction reported by Huang, Leng, and Liang (2012). For a given product quality, firms can use advertising and price as competitive tools where the market is not fully covered by consumers. In this situation, we investigate how firm's advertising behavior responds to changes in product quality and consumer preference as well as how this change in behavior affects firm's profit. We then analyze the welfare effect of any changes in product quality and preferences.

We have obtained the following results. First, when the technology gap is relatively large, an improvement of firm L's technology level enables it to obtain additional profits. Second, the effect of changes in exogenous parameters upon incumbent consumers can vary from the effect on consumer surplus. For example, when consumer preferences become heterogeneous, say $\bar{\theta}$ increases, the lifetime utility of each incumbent consumer decreases while the consumer surplus increases. In the situation where consumers are heterogeneous, we have to carefully consider not only individual consumers but also the aggregate consumer. Third, the effect of the maximum preference has a positive effect on the economy except for the case of incumbent consumers. Fourth, if the technology gap is relatively large, innovations by either firm leads to increases in both firms' profit and consumer surplus but, if this is not the case, the innovations have a different effect on both firms' profits and consumer surplus. In emerging industries, say $N(0)$ is almost zero, then when the gap is moderate,[12] the technological improvement of firm L increases its profits but decreases the consumer surplus. Finally, firm L has no incentive to improve its technology level above a certain degree of technological gap.

The model here is only a beginning to investigate consumer behavior using a utility-based approach. Since we use a linear state game, the control variables are independent of the state variable. This makes the analysis here easy and excludes the interesting dynamics; that is, the model's dynamics described only by the state equation, and the only variable varying over time is consumer density. Therefore, the model could be extended in several directions. One direction is to consider a non-linear state game and an interaction between prices and advertising investment levels. Advertising investment strategies would therefore be changed from the linear function to a Cobb-Douglas function or a quadratic function. In such a situation, control variables explicitly depend on state variables so that we can analyze more interesting dynamics of the advertising problem like the existing literature. Another direction is to introduce another advertising model into our model. For example,

[12] We consider the case where $s_H \in \left(\frac{4+3\sqrt{3}}{4} s_L, 3.037 s_L\right)$.

we could use the Lanchester model to extend the model to two industries model.

Appendix A. Proof of Lemma 1

To begin with, for notational simplicity we define $y_i^*(p_i^* - s_i)$ as z_i^*,

$$z_L^* = y_L^*(p_L^* - s_L) = \frac{s_H s_L (s_H - s_L)(\bar{\theta} - 1)^2}{(4s_H - s_L)^2},$$
$$z_H^* = y_H^*(p_H^* - s_H) = \frac{4s_H^2 (s_H - s_L)(\bar{\theta} - 1)^2}{(4s_H - s_L)^2}.$$

Using (6), (9), (13), (24), and (25), we obtain the following differential equation.

$$\dot{N} + \lambda N(t) = \frac{\alpha^2}{2\mu}\left(\frac{z_L^* + z_H^*}{\lambda + \rho} + \left[\phi_L(0) + \phi_H(0) - \left(\frac{z_L^* + z_H^*}{\lambda + \rho}\right)\right] e^{(\lambda+\rho)t}\right).$$

Solving this for N, we can derive the equilibrium law of motion for the state variable.

$$N(t) = \frac{z_N^*}{\lambda} + \left[N(0) - \frac{z_N^*}{\lambda} - \frac{\phi_N(0) - z_N^*}{2\lambda + \rho}\right] e^{-\lambda t} + \frac{\phi_N(0) - z_N^*}{2\lambda + \rho} e^{(\lambda+\rho)t}.$$

where

$$z_N^* = \frac{\alpha^2 (x_L + x_H)}{2\mu(\lambda + \rho)} \quad \text{and} \quad \phi_N(0) = \frac{\alpha^2(\phi_H(0) + \phi_L(0))}{2\mu}. \tag{35}$$

We now confirm that the transversality conditions hold. From (11) and (20),

$$\begin{aligned}
&\lim_{t\to\infty} \phi_L(t) N(t) e^{-\rho t} \\
&= \lim_{t\to\infty} \Bigg\{ M \frac{z_L^*}{\lambda + \rho} e^{-(\lambda+\rho)t} + M\left(\phi_L(0) - \frac{z_L^*}{\lambda + \rho}\right) + \frac{z_L^* z_N^*}{\lambda(\lambda + \rho)} e^{-\rho t} \\
&\qquad + \frac{z_N^*}{\lambda}\left(\phi_L(0) - \frac{z_L^*}{\lambda + \rho}\right) e^{\lambda t} + \frac{z_L^*}{\lambda + \rho}\left(\frac{\phi_N(0) - z_N^*}{2\lambda + \rho}\right) e^{\lambda t} \\
&\qquad + \left(\phi_N(0) - \frac{z_N^*}{2\lambda + \rho}\right)\left(\frac{\phi_N(0) - z_N^*}{2\lambda + \rho}\right) e^{(2\lambda+\rho)t} \Bigg\}
\end{aligned}$$

where

$$M = N(0) - \frac{z_N^*}{\lambda} - \frac{\phi_N(0) - z_N^*}{2\lambda + \rho}$$

For the transversality condition to be satisfied, it is necessary that

$$\phi_L(0) = \frac{z_L^*}{\lambda + \rho} \quad \text{and} \quad \phi_N(0) = z_N^*. \tag{36}$$

Similarly, from (15) and (20) the transversality condition of firm H is

$$\lim_{t\to\infty} \phi_H(t) N(t) e^{-\rho t}$$

$$= \lim_{t\to\infty}\left\{ M\frac{z_H^*}{\lambda+\rho}e^{-(\lambda+\rho)t} + M\left(\phi_H(0) - \frac{z_H^*}{\lambda+\rho}\right) + \frac{z_H^* z_N^*}{\lambda(\lambda+\rho)}e^{-\rho t} \right.$$
$$+ \frac{z_N^*}{\lambda}\left(\phi_L(0) - \frac{z_H^*}{\lambda+\rho}\right)e^{\lambda t} + \frac{z_H^*}{\lambda+\rho}\left(\frac{\phi_N(0) - z_N^*}{2\lambda+\rho}\right)e^{\lambda t}$$
$$\left. + \left(\phi_N(0) - \frac{z_N^*}{2\lambda+\rho}\right)\left(\frac{\phi_N(0) - z_N^*}{2\lambda+\rho}\right)e^{(2\lambda+\rho)t} \right\}.$$

For the transversality condition to be satisfied, it is necessary that

$$\phi_H(0) = \frac{z_H^*}{\lambda+\rho} \quad \text{and} \quad \phi_N(0) = z_N^*. \tag{37}$$

According to (35),

$$\phi_N(0) = z_N^* \quad \Longleftrightarrow \quad \phi_L(0) + \phi_H(0) = \frac{z_L^* + z_H^*}{\lambda+\rho},$$

so that, when the conditions $\phi_L(0) = z_L^*/(\lambda+\rho)$ and $\phi_H(0) = z_H^*/(\lambda+\rho)$ are simultaneously satisfied, the condition $\phi_N(0) = z_N^*$ immediately holds. As a result, satisfying the transversality conditions requires that

$$\phi_L(0) = \frac{z_L^*}{\lambda+\rho} \quad \text{and} \quad \phi_H(0) = \frac{z_H^*}{\lambda+\rho}. \tag{26}$$

Appendix B. Proof of Lemma 3

We differentiate (31) and (32) with respect to $\bar{\theta}$ and s_i and confirm their signs. First, we can easily confirm that both utility functions are decreasing with respect to $\bar{\theta}$:

$$\frac{\partial u_\theta^L}{\partial \bar{\theta}} = -\frac{s_L(s_H - s_L)}{4s_H - s_L} < 0 \quad \text{and} \quad \frac{\partial u_\theta^H}{\partial \bar{\theta}} = -\frac{2s_H(s_H - s_L)}{4s_H - s_L} < 0.$$

Second, we calculate the derivatives with respect to s_H.

$$\frac{\partial u_\theta^L}{\partial s_H} = -\frac{3s_L^2(\bar{\theta}-1)}{(4s_H - s_L)^2} < 0 \quad \text{and} \quad \frac{\partial u_\theta^H}{\partial s_H} = \frac{2s_H(2s_H - s_L)(4\theta - 2\bar{\theta} - 2) - s_L^2(2\bar{\theta} - \theta - 1)}{(4s_H - s_L)^2}$$

The sign of the term $(4\theta - 2\bar{\theta} - 2)$ is ambiguous so that the derivative of u_θ^H is not uniquely determined. To determine the sign, we investigate the characteristics of the derivative. The cross derivative of u_θ^H is positive:

$$\frac{\partial^2 u_\theta^H}{\partial\theta\partial s_H} = \frac{8s_H(2s_H - s_L)}{(4s_H - s_L)^2} > 0.$$

If θ is $\tilde{\theta}$,

$$4\tilde{\theta} - 2\bar{\theta} - 2 = -\frac{2s_L(\bar{\theta}-1)}{4s_H - s_L} < 0,$$

so that the derivative's sign is negative. If θ is $\bar{\theta}$, $4\bar{\theta} - 2\bar{\theta} - 2$ is positive and we can rearrange the derivative as follows.

$$\frac{\partial u_\theta^H}{\partial s_H} = \frac{(\bar{\theta}-1)(8s_H^2 - 4s_H s_L - s_L^2)}{(4s_H - s_L)^2}.$$

Given that, the term $(8s_H^2 - 4s_H s_L - s_L^2)$ is always positive for the region $s_H > s_L$, the sign is positive. Hence, there exists θ_t such that $\frac{\partial u_\theta^H}{\partial s_H} = 0$ is satisfied for $[\tilde{\theta}, \bar{\theta}]$:

$$\theta_t = \frac{4s_H(2s_H - s_L)(\bar{\theta}+1) + s_L^2(2\bar{\theta}-1)}{8s_H(2s_H - s_L) + s_L^2}.$$

To sum up, we obtain the following relationship,

$$\frac{\partial u_\theta^H}{\partial s_H} \begin{cases} > 0 & \text{if } \theta_t < \theta \leq \bar{\theta} \\ < 0 & \text{if } \tilde{\theta} \leq \theta < \theta_t \\ = 0 & \text{if } \theta = \theta_t. \end{cases}$$

Finally, we derive the derivatives of u_θ^L and u_θ^H with respect to s_L.

$$\frac{\partial u_\theta^L}{\partial s_L} = \frac{4s_H^2(4\theta - \bar{\theta} - 3) + s_L(8s_H - s_L)(\bar{\theta} - \theta)}{(4s_H - s_L)^2} \quad \text{and} \quad \frac{\partial u_\theta^H}{\partial s_L} = \frac{6s_H^2(\bar{\theta}-1)}{(4s_H - s_L)^2} > 0.$$

Similarly, we can confirm that the sign of the former derivative is positive. The cross derivative is positive and at $\theta = \hat{\theta}$ the derivative's sign is positive:

$$\frac{\partial u_{\hat{\theta}}^L}{\partial s_L} = \frac{3s_H s_L(4s_H - s_L)\bar{\theta} + s_H\{20s_H(s_H - s_L) + 3(4s_H^2 + s_L^2)\}}{(4s_H - s_L)^2} > 0.$$

Therefore, $\frac{\partial u_\theta^L}{\partial s_L}$ is positive for $[\hat{\theta}, \tilde{\theta}]$.

Appendix C. Proof of Lemma 4

We differentiate W_H and W_L with respect to $\bar{\theta}$, s_L, and s_H, respectively.

$$\frac{\partial W_H}{\partial \bar{\theta}} = -\int_{\tilde{\theta}}^{\bar{\theta}} \frac{\partial p_H^*}{\partial \bar{\theta}} d\theta + (s_H\bar{\theta} - p_H^*) - (s_H\tilde{\theta} - p_H^*)\frac{\partial \tilde{\theta}}{\partial \bar{\theta}} = \frac{4s_H^2(s_H + s_L)(\bar{\theta}-1)}{(4s_H - s_L)^2} > 0$$

$$\frac{\partial W_H}{\partial s_H} = \int_{\tilde{\theta}}^{\bar{\theta}} \left(\theta - \frac{\partial p_H^*}{\partial s_H}\right) d\theta - (s_H\tilde{\theta} - p_H^*)\frac{\partial \tilde{\theta}}{\partial s_H} = \frac{2s_H(4s_H^2 - 3s_H s_L - 2s_L^2)(\bar{\theta}-1)^2}{(4s_H - s_L)^3}$$

$$\frac{\partial W_H}{\partial s_L} = -\int_{\tilde{\theta}}^{\bar{\theta}} \frac{\partial p_H^*}{\partial s_L} d\theta - (s_L\tilde{\theta} - p_H^*)\frac{\partial \tilde{\theta}}{\partial s_L} = \frac{2s_H(6s_H + s_L)(\bar{\theta}-1)^2}{(4s_H - s_L)^3} > 0$$

$$\frac{\partial W_L}{\partial \bar{\theta}} = -\int_{\hat{\theta}}^{\tilde{\theta}} \frac{\partial p_L^*}{\partial \bar{\theta}} d\theta + (s_L\hat{\theta} - p_L^*)\frac{\partial \tilde{\theta}}{\partial \bar{\theta}} - (s_L\hat{\theta} - p_L^*)\frac{\partial \hat{\theta}}{\partial \bar{\theta}} = \frac{s_H^2 s_L(\bar{\theta}-1)}{(4s_H - s_L)} > 0$$

$$\frac{\partial W_L}{\partial s_H} = -\int_{\hat{\theta}}^{\tilde{\theta}} \frac{\partial p_L^*}{\partial s_H} d\theta + (s_L\tilde{\theta} - p_L^*)\frac{\partial \tilde{\theta}}{\partial s_H} - (s_L\hat{\theta} - p_L^*)\frac{\partial \hat{\theta}}{\partial s_H} = -\frac{s_H s_L^2(\bar{\theta}-1)^2}{(4s_H - s_L)^3} < 0$$

$$\frac{\partial W_L}{\partial s_L} = \int_{\hat{\theta}}^{\tilde{\theta}} \left(\theta - \frac{\partial p_L^*}{\partial s_L}\right) d\theta + (s_L\tilde{\theta} - p_L^*)\frac{\partial \tilde{\theta}}{\partial s_L} - (s_L\hat{\theta} - p_L^*)\frac{\partial \hat{\theta}}{\partial s_L} = \frac{s_H^2(4s_H + s_L)(\bar{\theta}-1)^2}{2(4s_H - s_L)^3} > 0$$

To determine the sign of the derivative of W_H with respect to s_H, we must investigate the characteristics of the quadratic function in the numerator. It is easy to confirm that the function is larger than zero for the region $s_H > \frac{3+\sqrt{41}}{8}s_L$.[13] In the region $\left(s_L, \frac{3+\sqrt{41}}{8}s_L\right]$, the function is non-positive and zero only if $s_H = \frac{3+\sqrt{41}}{8}s_L$.

[13]Solving $4s_H^2 - 3s_Hs_L - 2s_L^2 = 0$ for s_H, we obtain solutions $s_H = \frac{3\pm\sqrt{41}}{8}s_L$. Since we assume $s_H > s_L$, the region where the function is positive for s_H is over $\frac{3+\sqrt{41}}{8}s_L$.

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