RATIONALITY OF BERSHADSKY-POLYAKOV VERTEX ALGEBRAS

TOMOYUKI ARAKAWA

ABSTRACT. We prove the conjecture of Kac-Wakimoto on the rationality of exceptional W-algebras for the first non-trivial series, namely, for the Bershadsky-Polyakov vertex algebras $W_3^{(2)}$ at level $k=p/2-3$ with $p=3,5,7,9,\dots$. This gives new examples of rational conformal field theories.

1. INTRODUCTION

Recently, a remarkable family of W-algebras associated with simple Lie algebras and their *non-principal* nilpotent elements, called *exceptional W-algebras*, has been discovered by Kac and Wakimoto [10]. In [10] it was conjectured that with an exceptional W-algebra one can associate a rational conformal field theory.

As a first step to resolve the Kac-Wakimoto conjecture we have proved in the previous article [3] that exceptional W-algebras are *lisse*, or equivalently [2], C_2-cofinite. Therefore it remains [15, 6] to show that exceptional W-algebras are *rational*, i.e., that the representations are completely reducible, in order to prove the conjecture. In this article we prove the rationality of the first non-trivial series of exceptional W-algebras, that is, the *Bershadsky-Polyakov (vertex) algebras* $W_3^{(2)}$ [13, 4] at level $k=p/2-3$ with $p=3,5,7,9,\dots$. The vertex algebra $W_3^{(2)}$ is the W-algebra associated with $\mathfrak{g}=\mathfrak{sl}_3$ and it minimal nilpotent element.

Let us state our main result more precisely: Let $\mathcal{W}_k$ denote the unique simple quotient of $W_3^{(2)}$ at level $k\neq -3$.

Main Theorem (Conjectured by Kac and Wakimoto [10]). *Let p be an odd integer equal or greater than 3, $k=p/2-3$. Then the vertex algebra $\mathcal{W}_k$ is rational. The simple $\mathcal{W}_k$-modules are parameterized by the set of integral dominant weights of $\widehat{\mathfrak{sl}}_3$ of level $p-3$. These simple modules can be obtained by the quantum BRST reduction from irreducible admissible representations of $\widehat{\mathfrak{sl}}_3$ of level k.*

For $p=3$, $\mathcal{W}_{3/2-3}$ is one-dimensional. In the remaining cases $\mathcal{W}_{p/2-3}$ are conformal with negative central charges.

We note that Zhu's algebra of $W_3^{(2)}$ is closely related with Smith's algebra [14] which is a deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$, and that the rational quotient $\mathcal{W}_{p/2-3}$ has features in common with the $\widehat{\mathfrak{sl}}_2$-integrable affine vertex algebras in the sense that the following relations hold:

$$:G^+(z)^{p-2}:=:G^-(z)^{p-2}:=0,$$

where $G^+(z)$ and $G^-(z)$ are the standard generating fields of $\mathcal{W}_{p/2-3}$, see below.

This work is partially supported by the JSPS Grant-in-Aid for Scientific Research (B) No. 20340007.

2. Bershadsky-Polyakov algebras at exceptional levels.

Let $\mathcal{W}^k$ denote the Bershadsky-Polyakov (vertex) algebra $W_3^{(2)}$ at level $k \neq -3$, which is the vertex algebra freely generated by the fields $J(z), G^{\pm}(z), T(z)$ with the following OPE's:

$$\begin{aligned}
&J(z)J(w) \sim \frac{2k+3}{3(z-w)^2}, \quad G^{\pm}(z)G^{\pm}(w) \sim 0,\\
&J(z)G^{\pm}(w) \sim \pm\frac{1}{z-w}G^{\pm}(w),\\
&T(z)T(w) \sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w),\\
&T(z)G^{\pm}(w) \sim \frac{3}{2(z-w)^2}G^{\pm}(w) + \frac{1}{z-w}\partial G^{\pm}(w),\\
&T(z)J(w) \sim \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}\partial J(w),\\
&G^{+}(z)G^{-}(w) \sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)}{(z-w)^2}J(w)\\
&\qquad\qquad + \frac{1}{z-w}\left(3 : J(w)^2 : + \frac{3(k+1)}{2}\partial J(w) - (k+3)T(w)\right).
\end{aligned}$$

As in introduction we denote by $\mathcal{W}_k$ the unique simple quotient of $\mathcal{W}^k$.

Theorem 2.1 ([3]). *Let k, p be as in Main Theorem. Then $\mathcal{W}_k$ is lisse, or equivalently, C_2-cofinite.*

Set

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = T(z) + \frac{1}{2}\partial J(w).$$

This defines a conformal vector of $\mathcal{W}^k$ with central charge

$$c(k) = -\frac{4(k+1)(2k+3)}{k+3} = -\frac{4(p-4)(p-3)}{p},$$

which gives J, G^+, G^- conformal weights 1, 1, and 2, respectively. Hence $\mathcal{W}^k$ is $\mathbb{Z}_{\geq 0}$-graded with respect to the Hamiltonian L_0. We expand the corresponding fields accordingly:

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \quad G^+(z) = \sum_{n\in\mathbb{Z}} G^+_n z^{-n-1}, \qquad G^-(z) = \sum_{n\in\mathbb{Z}} G^-_n z^{-n-2}.$$

We have

$$\begin{aligned}
&[J_m, J_n] = \frac{2k+3}{3} m\delta_{m+n,0}, \quad [J_m, G_n] = G_{m+n}, \quad [J_m, F_n] = -F_{m+n},\\
&[L_m, J_n] = -nJ_{m+n} - \frac{(2k+3)(m+1)m}{6}\delta_{m+n,0},\\
&[L_m, G^+_n] = -nG^+_{m+n}, \quad [L_m, G^-_n] = (m-n)G^-_{m+n},\\
&[G^+_m, G^-_n] = 3(J^2)_{m+n} + (3(k+1)m - (2k+3)(m+n+1))\, J_{m+n} - (k+3)L_{m+n}\\
&\qquad\qquad + \frac{(k+1)(2k+3)m(m+1)}{2}\delta_{m+n,0},
\end{aligned}$$

where $\sum_{n\in\mathbb{Z}}(J^2)_n z^{-n-2} \overset{\text{def}}{=} :J(z)^2:$.

For $(\xi,\chi)\in\mathbb{C}^2$, let $L(\xi,\chi)$ be the irreducible representation of $\mathcal{W}^k$ generated by the vector $|\xi,\chi\rangle$ such that

$$\begin{aligned}
&J_0|\xi,\chi\rangle=\xi|\xi,\chi\rangle, \quad J_n|\xi,\chi\rangle=0 \quad \text{for } n>0,\\
&L_0|\xi,\chi\rangle=\chi|\xi,\lambda\rangle, \quad L_n|\xi,\chi\rangle=0 \quad \text{for } n>0,\\
&G_n^-|\xi,\chi\rangle=0 \quad \text{for } n\geq 0, \quad G_n^+|\xi,\chi\rangle=0 \quad \text{for } n\geq 1.
\end{aligned}$$

By Theorem 2.1, any simple $\mathcal{W}_k$-module is of the form $L(\xi,\lambda)$ with some ξ and χ. (It is important that the lisse condition is defined independent of the choice of a conformal vector.)

For a $\mathcal{W}^k$-module M set

$$M_{a,d}=\{m\in M; J_0m=am,\ L_0m=dm\}.$$

It is clear that $L(\xi,\chi)=\bigoplus_{\substack{(a,d)\in\mathbb{C}^2\\ d\in\chi+\mathbb{Z}_{\geq 0}}} L(\xi,\chi)_{a,d}$, $\dim L(\xi,\chi)_{\xi,\chi}=1$. Let

$$L(\xi,\chi)_{\text{top}}=\{v\in L(\xi,\chi); L_0v=\chi v\}=\bigoplus_a L(\xi,\chi)_{a,\chi}.$$

By definition $L(\xi,\chi)_{\text{top}}$ is spanned by the vectors $(G_0^+)^i|\xi,\chi\rangle$ with $i\geq 0$.

Following [14] set

$$g(\xi,\chi)=-(3\xi^2-(2k+3)\xi-(k+3)\chi),$$

so that $G_0^-G_0^+|\xi,\chi\rangle=g(\xi,\chi)|\xi,\chi\rangle$. We have

$$G_0^-(G_0^+)^i|\xi,\chi\rangle=ih_i(\xi,\chi)(G_0^+)^{i-1}|\xi,\chi\rangle,$$

where

$$\begin{aligned}
h_i(\xi,\chi)&=\frac{1}{i}(g(\xi,\chi)+g(\xi+1,\chi)+\cdots+g(\xi+i-1,\chi))\\
&=-i^2+ki-3\xi i+3i-3\xi^2-k+2k\xi+6\xi+k\chi+3\chi-2.
\end{aligned}$$

Hence we have the following assertion.

Proposition 2.2. *If the space $L(\xi,\chi)_{\text{top}}$ is n-dimensional, then $h_n(\xi,\chi)=0$.*

Define

$$\Delta(-J,z)=z^{-J_0}\exp\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{-J_k}{kz^k}\right),$$

and set

$$\sum_{n\in\mathbb{Z}}\psi(a_{(n)})z^{-n-1}=Y(\Delta(-J,z)a,z)$$

for $a\in\mathcal{W}^k$. For any $\mathcal{W}^k$-module M, we can define on M a new $\mathcal{W}^k$-module structure by twisting the action of $\mathcal{W}^k$ as $a_{(n)}\mapsto\psi(a_{(n)})$ ([11]). We denote by $\psi(M)$ thus obtained $\mathcal{W}^k$-module from M.

Proposition 2.3. *Suppose that* $\dim L(\xi,\chi)_{\text{top}}=i$. *Then*

$$\psi(L(\xi,\chi))\cong L(\xi+i-1-\frac{2k+3}{3},\chi-(\xi-i+1)+\frac{2k+3}{3}).$$

Proof. The assertion follows from the fact that

$$\psi(J_n) = J_n - \frac{2k+3}{3}\delta_{n,0}, \quad \psi(L_n) = L_n - J_n + \frac{2k+3}{3},$$
$$\psi(G_n^+) = G_{n-1}^+, \quad \psi(G_n^-) = G_{n+1}^-.$$

□

By solving the equation

$$h_i(\xi,\chi) = h_j(\xi + i - 1 - \frac{2k+3}{3}, \chi - (\xi - i + 1) + \frac{2k+3}{3})$$

we obtain the following assertion.

Proposition 2.4. *Suppose that* $\dim L(\xi,\chi)_{\mathrm{top}} = i$ *and* $\dim \psi(L(\xi,\chi))_{\mathrm{top}} = j$. *Then*

$$\xi = \xi_{i,j} \overset{\mathrm{def}}{=} \frac{1}{3}(-2i - j + 2k + 6),$$
$$\chi = \chi_{i,j} \overset{\mathrm{def}}{=} \frac{i^2 + ji - ki - 3i + j^2 - 6j - 2jk + 3k + 6}{3(k+3)}.$$

Proposition 2.5. *Let k, p be as in Main Theorem. Then $(G_{-1}^+)^{p-2}\mathbf{1}$ belongs to the maximal ideal of $\mathcal{W}^k$.*

Proof. Since $\xi_{1,p-2} = \chi_{1,p-2} = 0$, the correspondence $\mathbf{1} \mapsto |\xi_{1,p-2}, \chi_{1,p-2}\rangle$ gives an isomorphism $\mathcal{W}_k \cong L(\xi_{1,p-2}, \chi_{1,p-2})$. Because

$$h_{p-2}(\xi_{1,p-2} - (2k+3)/2, \chi_{1,p-2} + (2k+3)/3) = 0,$$

from Proposition 2.3 it follows that $\psi(\mathcal{W}_k)_{\mathrm{top}}$ is at most $p-2$-dimensional. Hence $(G_{-1}^+)^{p-2}\mathbf{1} = 0$. □

Remark 2.6. One can show that in fact $(G_{-1}^+)^{p-2}$ generates the maximal ideal of $\mathcal{W}^k$. However we do not need this fact.

Proposition 2.7. *Let k, p be as in Main Theorem. Then any simple $\mathcal{W}_k$-module is isomorphic to $L(\xi_{i,j}, \chi_{i,j})$ for some (i,j) such that $1 \le i \le p-2$, $1 \le j \le p-i-1$.*

Proof. Let $L(\xi,\chi)$ be a simple $\mathcal{W}_k$-module. As $: G^+(z)^{p-2} := 0$ on $L(\xi,\chi)$ by Proposition 2.5, $L(\xi,\chi)_{\mathrm{top}}$ is at most $(p-2)$-dimensional. Since $\psi(L(\xi,\chi))$ is also a $\mathcal{W}_k$-module we have $(\xi,\chi) = (\xi_{i,j}, \chi_{i,j})$ for some $1 \le i,j \le p-2$. Because $\psi(\psi(L(\xi_{i.j}, \chi_{i,j})))$ is also a $\mathcal{W}_k$-module it follows that $\xi_{i,j} + i - 1 - \frac{2k+3}{3} = \frac{i-j}{3} \le \frac{-2j-1+2k+6}{3} = \frac{p-2j-1}{3}$. Hence $j \le p-i-1$. □

The simple $\mathcal{W}^k$-modules $L(\xi_{i,j}, \chi_{i,j})$ with $1 \le i \le p-2$, $1 \le j \le p-i-1$, are mutually non-isomorphic since their highest weights are distinct.

3. Proof of Main Theorem

Let k, p be as in Main Theorem.

Let $\mathfrak{g} = \mathfrak{sl}_3$ as in introduction, $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrixes. Set $h_i = E_{i,i} - E_{i+1,i+1}$, $h_\theta = h_1 + h_2$, $e_i = e_{\alpha_i} = E_{i,i+1}$, $f_i = f_{\alpha_i} = E_{i+1,i}$ for $i = 1,2$, $e_\theta = E_{1,3}$, $f_\theta = E_{3,1}$, where $E_{i,j}$ is the matrix element. We equip $\mathfrak{g}$ the invariant form $(x|y) = \mathrm{tr}(xy)$. Set $\bar{\Lambda}_1 = (2h_1 + h_2)/3$, $\bar{\Lambda}_1 = (h_1 + 2h_2)/3$, so that $(\bar{\Lambda}_i|h_j) = \delta_{i,j}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the (non-twisted) affine Kac-Moody algebra associated with $\mathfrak{g}$, where K is the central element and D is the degree operator. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D \subset \widehat{\mathfrak{g}}$ the standard Cartan subalgebra, $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ the dual of $\widehat{\mathfrak{h}}$, where Λ_0 and δ are elements dual to K and D, respectively.

The vector f_θ is a the minimal nilpotent element of $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be the corresponding Dynkin grading: $\mathfrak{g}_j = \{u \in \mathfrak{g}; [h_\theta, u] = 2ju\}$. Denote by $H_{f_\theta}^{\frac{\infty}{2}+0}(?)$ the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f_\theta)$ and the Dynkin grading. We have [7, 9] the vertex algebra isomorphism

$$\mathcal{W}^k \xrightarrow{\sim} H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})),$$

which is given by the following assignment:

$$J(z) \mapsto J^{-\bar{\Lambda}_1+\bar{\Lambda}_2}(z) - :\Phi_1(z)\Phi_2(z):,$$
$$G^+(z) \mapsto J^{f_1}(z) - :J^{h_1}(z)\Phi_2(z): + :\Phi_1(z)\Phi_2(z)^2: - (k+1)\partial\Phi_2(z),$$
$$G^+(z) \mapsto -J^{f_2}(z) - :J^{h_2}(z)\Phi_1(z): - :\Phi_1(z)^2\Phi_2(z): - (k+1)\partial\Phi_1(z),$$

Here

$$J^u(z) = u(z) - \sum_{\beta,\gamma \in \{\alpha_1,\alpha_2,\theta\}} c_{u,f_\beta}^{f_\gamma} :\psi_\beta^*(z)\psi_\gamma(z):$$

for $u \in \mathfrak{g}$, $c_{u_1,u_2}^{u_3}$ is the structure constant, $\psi_\alpha(z)$, $\psi_\alpha^*(z)$ with $\alpha \in \{\alpha_1, \alpha_2, \theta\}$ are fermionic ghosts satisfying

$$\psi_\alpha(z)\psi_\beta^*(w) \sim \frac{\delta_{\alpha,\beta}}{z-w}, \quad \psi_\alpha(z)\psi_\beta(w) \sim \psi_\alpha^*(z)\psi_\beta(^*w) \sim 0, \tag{1}$$

$\Phi_1(z)$, $\Phi_2(z)$ are bosonic ghosts satisfying

$$\Phi_1(z)\Phi_2(w) \sim \frac{1}{z-w}, \quad \Phi_i(z)\Phi_i(w) \sim 0,$$

and the BRST differential is the zero mode of the field

$$Q(z) = \sum_{\alpha \in \{\alpha_1,\alpha_2,\theta\}} e_\alpha(z)\psi_\alpha^*(w) - :\psi_{\alpha_1}^*(z)\psi_{\alpha_2}^*(z)\psi_\theta(z):$$
$$+\Phi_1(z)\psi_{\alpha_1}^*(z) + \Phi_2(z)\psi_{\alpha_2}(z) + \psi_\theta(z).$$

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k, $\mathbf{L}_\lambda$ the irreducible representation of $\widehat{\mathfrak{g}}$ with highest weight λ. Denote by $\mathcal{W}^k$-Mod the category of $\mathcal{W}^k$-modules.

Theorem 3.1 ([1]).

(i) *The functor* $H_{f_\theta}^{\frac{\infty}{2}+0}(?) : \mathcal{O}_k \to \mathcal{W}^k$-Mod, $M \mapsto H_{f_\theta}^{\frac{\infty}{2}+0}(M)$, *is exact.*

(ii) *For* $\lambda \in \widehat{\mathfrak{h}}^*$ *we have* $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda) = 0$ *if and only if* $\lambda(\alpha_0^\vee) \in \{0, 1, 2, 3, \dots\}$. *Otherwise* $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda)$ *is irreducible.*

Let Adm^k be the set of admissible weights [8] of $\widehat{\mathfrak{g}}$ of level k, and put

$$Adm_+^k = \{\lambda \in Adm^k; \bar{\lambda} \text{ is an integral dominant weight of } \mathfrak{g}\},$$

where $\widehat{\mathfrak{h}}^* \ni \lambda \mapsto \bar{\lambda} \in \mathfrak{h}^*$ is the restriction. Then

$$Adm_+^k = \{\bar{\mu} + k\Lambda_0; \mu \in \widehat{P}_{++}^{p-3}\},$$

where $\widehat{P}_{++}^{p-3}$ is the set of integral dominant weights of $\widehat{\mathfrak{g}}$ of level $p-3$. Explicitly, we have

$$Adm_+^k = \{\lambda_{i,j}; 1 \leq i \leq p-2,\ 1 \leq j \leq p-i-1\},$$

where

$$\lambda_{i,j} = (i-1)\bar{\Lambda}_1 + (p-i-j-1)\bar{\Lambda}_2 + k\Lambda_0.$$

Note that

$$\xi_{i,j} = (\lambda_{i,j}| - \bar{\Lambda}_1 + \bar{\Lambda}_2), \quad \chi_{i,j} = \frac{(\lambda_{i,j}|\lambda_{i,j} + 2\bar{\rho})}{2(k+3)} - (\lambda_{i,j}|\bar{\Lambda}_2), \tag{2}$$

where $\bar{\rho} = \bar{\Lambda}_1 + \bar{\Lambda}_2$.

Recall the following result of Malikov and Frenkel [12].

Theorem 3.2 ([12, Corollary 5.2.2])**.** *For $\lambda \in Adm_+^k$, $\mathbf{L}_\lambda$ is a module over $\mathbf{L}_{k\Lambda_0}$.*

Proposition 3.3. *For $\lambda_{i,j} \in Adm_+^k$, $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is a simple $\mathcal{W}_k$-module isomorphic to $L(\xi_{i,j}, \chi_{i,j})$.*

Proof. By Theorem 3.1 we have $\mathcal{W}_k \cong H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{k\Lambda_0})$. Hence by the functoriality of $H_{f_\theta}^{\frac{\infty}{2}+0}(?)$, Theorem 3.2 immediately gives that $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is a module over $\mathcal{W}_k$. By Theorem 3.1, $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is (nonzero and) irreducible. Let v be the image of the highest weight vector of $\mathbf{L}_{\lambda_{i,j}}$ in $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$. By (2) and the fact [9] that the image of $L(z)$ in $\mathcal{W}^k$ is cohomologous to

$$L_{\mathfrak{g}}(z) + L_{\mathrm{ch}}(z) + L_\Phi(z) + \partial J^{\bar{\Lambda}_2}(z),$$

where $L_{\mathfrak{g}}(z)$ is the Sugawara operator of $\mathfrak{g}$, $L_{\mathrm{ch}}(z) = -\sum_{\alpha=\alpha_1,\alpha_2,\theta} : \phi_\alpha(z)\partial\phi_\alpha^*(z)$, $L_\Phi(z) = \frac{1}{2}(: \Phi_2(z)\partial\Phi_1(z) : - : \partial\Phi_1(z)\Phi_2(z))$, it is straightforward to check that the assignment $|\xi_{i,j}, \chi_{i,j}\rangle \mapsto v$ gives a $\mathcal{W}^k$-module homomorphism. By the irreducibility, this must be an isomorphism. □

By Propositions 2.7 and 3.3, the set $\{H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda); \lambda \in Adm_+^k\}$ gives the complete set of isomorphism classes of simple $\mathcal{W}_k$-modules. Therefore Main Theorem now follows immediately from the following important result of Gorelik and Kac [5].

Theorem 3.4 ([5, Corollary 8.8.9])**.** *For any $\lambda, \mu \in Adm^k$, we have*

$$\mathrm{Ext}^1_{\mathcal{W}^k\text{-Mod}}(H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda), H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\mu)) = 0.$$

References

[1] T. Arakawa. Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture. *Duke Math. J.*, Vol. 130, No. 3, pp. 435–478, 2005.

[2] T. Arakawa. A remark on the C_2 cofiniteness condition on vertex algebras. *Math. Z.*, Vol. 270, No. 1-2, pp. 559–575, 2012.

[3] T. Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras. *preprint*, arXiv:1004.1554v2.

[4] Michael Bershadsky. Conformal field theories via Hamiltonian reduction. *Comm. Math. Phys.*, Vol. 139, No. 1, pp. 71–82, 1991.

[5] Maria Gorelik and Victor Kac. On complete reducibility for infinite-dimensional Lie algebras. *Adv. Math.*, Vol. 226, No. 2, pp. 1911–1972, 2011.

[6] Yi-Zhi Huang. Vertex operator algebras and the Verlinde conjecture. *Commun. Contemp. Math.*, Vol. 10, No. 1, pp. 103–154, 2008.

[7] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, Vol. 241, No. 2-3, pp. 307–342, 2003.
[8] V. G. Kac and M. Wakimoto. Classification of modular invariant representations of affine algebras. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, Vol. 7 of *Adv. Ser. Math. Phys.*, pp. 138–177. World Sci. Publ., Teaneck, NJ, 1989.
[9] Victor G. Kac and Minoru Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, Vol. 185, No. 2, pp. 400–458, 2004.
[10] Victor G. Kac and Minoru Wakimoto. On rationality of W-algebras. *Transform. Groups*, Vol. 13, No. 3-4, pp. 671–713, 2008.
[11] Haisheng Li. The physics superselection principle in vertex operator algebra theory. *J. Algebra*, Vol. 196, No. 2, pp. 436–457, 1997.
[12] F. G. Malikov and I. B. Frenkel′. Annihilating ideals and tilting functors. *Funktsional. Anal. i Prilozhen.*, Vol. 33, No. 2, pp. 31–42, 95, 1999.
[13] A. M. Polyakov. Gauge transformations and diffeomorphisms. *Internat. J. Modern Phys. A*, Vol. 5, No. 5, pp. 833–842, 1990.
[14] S. P. Smith. A class of algebras similar to the enveloping algebra of sl(2). *Trans. Amer. Math. Soc.*, Vol. 322, No. 1, pp. 285–314, 1990.
[15] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, Vol. 9, No. 1, pp. 237–302, 1996.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

E-mail address: arakawa@kurims.kyoto-u.ac.jp