J. reine angew. Math. **677** (2013), 15—70
DOI 10.1515/crelle.2012.009

Journal für die reine und
angewandte Mathematik

$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space, II: A structure theorem for $r(M) > 10$

By *Ken-Ichi Yoshikawa* at Kyoto and Seoul

Abstract. We study the structure of the invariant of $K3$ surfaces with involution, which we obtained using equivariant analytic torsion. It was known before that the invariant is expressed as the Petersson norm of an automorphic form on the moduli space. When the rank of the invariant sublattice of the $K3$ lattice with respect to the involution is strictly bigger than 10, we prove that this automorphic form is expressed as the tensor product of an explicit Borcherds lift and Igusa's Siegel modular form.

Contents

The author is partially supported by JSPS Grants-in-Aid (B) 19340016, (S) 17104001, (S) 22224001.

Introduction

In this paper, we study the structure of the invariant of $K3$ surfaces with involution introduced in [62]. Let us recall briefly this invariant.

A $K3$ surface with holomorphic involution (X, ι) is called a 2-elementary $K3$ surface if ι acts non-trivially on the holomorphic 2-forms on X. Let $\mathbb{L}_{K3}$ be the $K3$ lattice, i.e., an even unimodular lattice of signature $(3, 19)$, which is isometric to $H^2(X, \mathbf{Z})$ endowed with the cup-product pairing. Let M be a sublattice of $\mathbb{L}_{K3}$ with rank $r(M)$. A 2-elementary $K3$ surface (X, ι) is of type M if the invariant sublattice of $H^2(X, \mathbf{Z})$ with respect to the ι-action is isometric to M. By [46], $M \subset \mathbb{L}_{K3}$ must be a primitive 2-elementary Lorentzian sublattice. The rank of the discriminant group of M is denoted by $l(M)$ and the parity of the 2-elementary lattice M is denoted by $\delta(M) \in \{0, 1\}$ (cf. [48] and Section 1.2).

Let $M^{\perp}$ be the orthogonal complement of M in $\mathbb{L}_{K3}$. Let $\Omega_{M^{\perp}}$ be the period domain for 2-elementary $K3$ surfaces of type M, which is an open subset of a quadric hypersurface of $\mathbf{P}(M^{\perp} \otimes \mathbf{C})$. We fix a connected component $\Omega^+_{M^{\perp}}$ of $\Omega_{M^{\perp}}$, which is isomorphic to a bounded symmetric domain of type IV of dimension $20 - r(M)$. Let $\mathscr{D}_{M^{\perp}}$ be the discriminant locus of $\Omega^+_{M^{\perp}}$, which is a reduced divisor on $\Omega^+_{M^{\perp}}$. Let $O(M^{\perp})$ be the group of isometries of $M^{\perp}$. Then $O(M^{\perp})$ acts properly discontinuously on $\Omega_{M^{\perp}}$ and $\mathscr{D}_{M^{\perp}}$. Let $O^+(M^{\perp})$ be the subgroup of $O(M^{\perp})$ with index 2 that preserves $\Omega^+_{M^{\perp}}$. The coarse moduli space of 2-elementary $K3$ surfaces of type M is isomorphic to the analytic space $\mathscr{M}^o_{M^{\perp}} = (\Omega^+_{M^{\perp}} \backslash \mathscr{D}_{M^{\perp}})/O^+(M^{\perp})$ via the period map by the global Torelli theorem [51], [15]. The period of a 2-elementary $K3$ surface (X, ι) of type M is denoted by $\overline{\varpi}_M(X, \iota) \in \mathscr{M}^o_{M^{\perp}}$.

Let (X, ι) be a 2-elementary $K3$ surface of type M. Let κ be a ι-invariant Kähler form on X. Let X^{ι} be the set of fixed points of ι and let $X^{\iota} = \sum_i C_i$ be the decomposition into the connected components. Let $\eta \in H^0(X, \Omega^2_X) \backslash \{0\}$. In [62], we introduced a real-valued invariant

$$\tau_M(X,\iota) = \mathrm{Vol}\big(X,(2\pi)^{-1}\kappa\big)^{\frac{14-r(M)}{4}}\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)\prod_i \mathrm{Vol}\big(C_i,(2\pi)^{-1}\kappa|_{C_i}\big)\tau(C_i,\kappa|_{C_i})$$
$$\times\exp\left[\frac{1}{8}\int_{X^\iota}\log\left(\frac{\eta\wedge\bar\eta}{\kappa^2/2!}\cdot\frac{\mathrm{Vol}\big(X,(2\pi)^{-1}\kappa\big)}{\|\eta\|_{L^2}^2}\right)\bigg|_{X^\iota} c_1(X^\iota,\kappa|_{X^\iota})\right],$$

where $\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)$ is the equivariant analytic torsion of (X,κ) with respect to the $\mathbf{Z}_2$-action induced by ι, $\tau(C_i,\kappa|_{C_i})$ is the analytic torsion of $(C_i,\kappa|_{C_i})$, and $c_1(X^\iota,\kappa|_{X^\iota})$ is the first Chern form of $(X^\iota,\kappa|_{X^\iota})$ (see [5], [6], [52] and Section 5). Since $\tau_M(X,\iota)$ depends only on the isomorphism class of (X,ι), we get the function

$$\tau_M : \mathscr{M}^o_{M^\perp} \ni \overline{\varpi}_M(X,\iota) \to \tau_M(X,\iota) \in \mathbf{R}_{>0}.$$

By [62], [66], there exists an automorphic form Φ_M on $\Omega^+_{M^\perp}$ with values in a certain $O^+(M^\perp)$-equivariant holomorphic line bundle on $\Omega^+_{M^\perp}$, such that

$$\tau_M = \|\Phi_M\|^{-\frac{1}{2\nu}},\quad \mathrm{div}\,\Phi_M = \nu\mathscr{D}_{M^\perp},\quad \nu\in\mathbf{Z}_{>0}.$$

Here $\|\cdot\|$ denotes the Petersson norm. By [62], Φ_M is given by the Borcherds Φ-function [8], [9] when M is one of the two exceptional lattices in Proposition 2.1. For an arithmetic counterpart of the invariant τ_M, we refer the reader to [39].

In this paper, we give an explicit formula for τ_M for a class of non-exceptional M. We use two kinds of automorphic forms to express τ_M, i.e., the Borcherds lift $\Psi_{M^\perp}(\cdot,F_{M^\perp})$ and Igusa's Siegel modular form χ_g, which we explain briefly.

In [8], [10], Borcherds developed the theory of automorphic forms with infinite product expansion over domains of type IV. For an even 2-elementary lattice Λ of signature $\big(2,r(\Lambda)-2\big)$, we define the Borcherds lift $\Psi_\Lambda(\cdot,F_\Lambda)$ as follows.

Let A_Λ be the discriminant group of Λ, which is a vector space over $\mathbf{Z}/2\mathbf{Z}$. Let $\mathbf{C}[A_\Lambda]$ be the group ring of A_Λ and let $\rho_\Lambda : \mathrm{Mp}_2(\mathbf{Z})\to\mathrm{GL}(\mathbf{C}[A_\Lambda])$ be the Weil representation, where $\mathrm{Mp}_2(\mathbf{Z})$ is the metaplectic double cover of $\mathrm{SL}_2(\mathbf{Z})$. Let $\{\boldsymbol{e}_\gamma\}_{\gamma\in A_\Lambda}$ be the standard basis of $\mathbf{C}[A_\Lambda]$. Let $\eta(\tau)$ be the Dedekind η-function and set $\eta_{1^{-8}2^84^{-8}}(\tau)=\eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$. Let $\theta_{\mathbb{A}_1^+}(\tau)$ be the theta function of the (positive-definite) A_1-lattice. Then $\eta_{1^{-8}2^84^{-8}}(\tau)$ and $\theta_{\mathbb{A}_1^+}(\tau)$ are modular forms for the subgroup $\mathrm{M}\Gamma_0(4)\subset\mathrm{Mp}_2(\mathbf{Z})$ corresponding to the congruence subgroup $\Gamma_0(4)\subset\mathrm{SL}_2(\mathbf{Z})$. Following [11] and [55], we define a $\mathbf{C}[A_\Lambda]$-valued holomorphic function $F_\Lambda(\tau)$ on the complex upper half-plane $\mathfrak{H}$ as

$$F_\Lambda(\tau) = \sum_{g\in\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})}\{\eta_{1^{-8}2^84^{-8}}\theta_{\mathbb{A}_1^+}^{12-r(\Lambda)}\}|_g(\tau)\rho_\Lambda(g^{-1})\boldsymbol{e}_0. \tag{0.1}$$

Here we used the notation $\phi|_g(\tau)=\phi\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-k}$ for a modular form $\phi(\tau)$ for $\mathrm{M}\Gamma_0(4)$ of weight k with certain character and $g=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\sqrt{c\tau+d}\right)\in\mathrm{Mp}_2(\mathbf{Z})$. By [11] and [55], $F_\Lambda(\tau)$ is an elliptic modular form for $\mathrm{Mp}_2(\mathbf{Z})$ of type ρ_Λ with weight $2-\frac{r(\Lambda)}{2}$. Then $\Psi_\Lambda(\cdot,F_\Lambda)$ is defined as the Borcherds lift of $F_\Lambda(\tau)$, which is an automorphic form on Ω_{Λ}^{+} for $O^{+}(\Lambda)$ by [10] (see (8.5) for an explicit infinite product expression of $\Psi_{\Lambda}(\cdot, F_{\Lambda})$). The Petersson norm $\|\Psi_{M^{\perp}}(\cdot, F_{M^{\perp}})\|^{2}$ is an $O^{+}(M^{\perp})$-invariant function on $\Omega_{M^{\perp}}^{+}$ and the value $\|\Psi_{M^{\perp}}(\varpi_{M}(X,\iota), F_{M^{\perp}})\|$ makes sense.

Recall that χ_g is the Siegel modular form on the Siegel upper half-space $\mathfrak{S}_g$ of degree g defined as the product of all even theta constants (cf. [31])

$$\chi_g(\Sigma) = \prod_{(a,b)\text{ even}} \theta_{a,b}(\Sigma), \quad \Sigma \in \mathfrak{S}_g, \quad \chi_0 = 1. \tag{0.2}$$

Then χ_g gives rise to another function on $\mathscr{M}_{M^{\perp}}^{o}$ as follows. For a 2-elementary $K3$ surface (X,ι) of type M, let X^{ι} denote the set of fixed points of ι. By [48], X^{ι} is the disjoint union of (possibly empty) compact Riemann surfaces, whose topological type is determined by M. Let $g(M) \in \mathbf{Z}_{\geqq 0}$ denote the total genus of X^{ι}. The period of X^{ι} is denoted by $\Omega(X^{\iota}) \in \mathfrak{S}_{g(M)}/\mathrm{Sp}_{2g(M)}(\mathbf{Z})$. By [62], there exist a proper $O^{+}(M^{\perp})$-invariant Zariski closed subset $Z \subset \mathscr{D}_{M^{\perp}}$ and an $O^{+}(M^{\perp})$-equivariant holomorphic map

$$J_M : \Omega_{M^{\perp}} \backslash Z \to \mathfrak{S}_{g(M)}/\mathrm{Sp}_{2g(M)}(\mathbf{Z})$$

that induces the map of moduli spaces

$$\mathscr{M}_{M^{\perp}}^{o} \ni \overline{\varpi}_{M}(X,\iota) \to \Omega(X^{\iota}) \in \mathfrak{S}_{g(M)}/\mathrm{Sp}_{2g(M)}(\mathbf{Z}).$$

Then $J_M^{*}\|\chi_{g(M)}\|^{2}$ is an $O^{+}(M^{\perp})$-invariant C^{∞} function on $\Omega_{M^{\perp}}^{o}$.

The following structure theorem for τ_M is the main result of this paper:

Theorem 0.1 (cf. Theorem 9.1). *Let M be a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$. If $r(M) > 10$ or $(r(M), \delta(M)) = (10,1)$, then there exists a constant C_M depending only on the lattice M such that the following identity holds for every 2-elementary $K3$ surface (X,ι) of type M:*

$$\tau_M(X,\iota)^{-2^{g(M)+1}(2^{g(M)}+1)} = C_M \|\Psi_{M^{\perp}}(\overline{\varpi}_M(X,\iota), F_{M^{\perp}})\|^{2^{g(M)}} \|\chi_{g(M)}(\Omega(X^{\iota}))\|^{16}.$$

It may be worth emphasizing that the structure of τ_M becomes transparent by considering elliptic modular forms for $M\Gamma_0(4)$ rather than $\mathrm{Mp}_2(\mathbf{Z})$. After Bruinier [14], Theorem 0.1 may not be surprising. Indeed, if $M^{\perp}$ contains an even unimodular lattice of signature $(2,2)$ as a direct summand and if there is a Siegel modular form S such that $\mathrm{div}(J_M^{*}S)$ is a Heegner divisor on $\Omega_{M^{\perp}}^{+}$, then Φ_M must be the product of a Borcherds lift and $J_M^{*}S$ by [14], Theorem 0.8. When $g(M) = 2$, this explains the existence of a factorization of τ_M like Theorem 0.1.[1)] It seems to be an interesting problem to understand the geometric origin of these common structures of the modular forms F_{Λ} and χ_g.

There are 43 isometry classes of primitive 2-elementary Lorentzian sublattices $M \subset \mathbb{L}_{K3}$ such that $r(M) > 10$ or $(r(M), \delta(M)) = (10,1)$ (cf. Table 1 in Section 9). In fact, Theorem 0.1 remains valid for a certain primitive 2-elementary Lorentzian sublattice

[1)] However, this does not seem to explain the common structures (0.1), (0.2) of the elliptic modular forms F_{Λ} and the Siegel modular forms χ_g appearing in the expression of τ_M.

$M \subset \mathbb{L}_{K3}$ with $r(M) = 9$ (see Theorem 9.4). By Theorems 0.1 and 9.4 and [62], Theorems 8.2 and 8.7, τ_M and Φ_M are determined for 46 isometry classes of M. Since the total number of the isometry classes of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ is 75 by Nikulin [48], the structures of τ_M and Φ_M are still open for the remaining 29 lattices.

Following [62], Theorem 8.7, we shall prove Theorem 0.1 by comparing the $O^+(M^\perp)$-invariant currents $dd^c \log \tau_M$, $dd^c \log \|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|^2$ and $dd^c \log J_M^* \|\chi_{g(M)}^8\|^2$ (see Section 9). The current $dd^c \log \tau_M$ was computed in [62]. In Section 8, the weight and the zero divisor of $\Psi_{M^\perp}(\cdot, F_{M^\perp})$ shall be determined, from which a formula for $dd^c \log \|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|^2$ follows. In Section 4, the current $dd^c \log J_M^* \|\chi_{g(M)}^8\|^2$ shall be computed, where the irreducibility of certain component of the divisor $\mathscr{D}_{M^\perp}/O^+(M^\perp)$ on $\Omega_{M^\perp}^+/O^+(M^\perp)$ plays a crucial role (see Appendix 11.3).

In Proposition 9.3, we shall prove that $\chi_{g(M)}$ vanishes identically on the locus $J_M(\Omega_{M^\perp}^+ \setminus \mathscr{D}_{M^\perp})$ when $\big(r(M), \delta(M)\big) = (10, 0)$ and M is not exceptional. Hence Theorem 0.1 does *not* hold in these four cases. This is similar to the exceptional case $\big(r(M), l(M), \delta(M)\big) = (10, 8, 0)$, where χ_g should be replaced by the product of two Jacobi Δ-functions [62], Theorem 8.7.

There is an application of the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ to the moduli space $\mathscr{M}_{M^\perp}^o$.

Theorem 0.2. *If $r(M) \geqq 9$ and $\big(r(M), l(M)\big) \neq (9, 9)$, then $\mathscr{M}_{M^\perp}^o$ is quasi-affine.*

When $\mathscr{M}_{M^\perp}^o$ is the coarse moduli space of Enriques surfaces, this was proved by Borcherds [9]. Since the coarse moduli space of ample M-polarized $K3$ surfaces (cf. [1], [18], [47]) is a finite covering of $\mathscr{M}_{M^\perp}^o$, its quasi-affinity follows from that of $\mathscr{M}_{M^\perp}^o$. The quasi-affinity of $\mathscr{M}_{M^\perp}^o$ is a consequence of the fact that $\Psi_{M^\perp}(\cdot, F_{M^\perp})$ vanishes only on the discriminant locus $\mathscr{D}_{M^\perp}$ when $r(M^\perp) \leqq 13$ and $\big(r(M^\perp), l(M^\perp)\big) \neq (13, 9)$. By [48], there are 53 isometry classes of primitive 2-elementary Lorentzian sublattices $M \subset \mathbb{L}_{K3}$ with $r(M) \geqq 9$ and $\big(r(M), l(M)\big) \neq (9, 9)$. In general, it is not easy to find a primitive sublattice $\Lambda \subset \mathbb{L}_{K3}$ of signature $\big(2, r(\Lambda) - 2\big)$ admitting an automorphic form on Ω_Λ^+ vanishing only on $\mathscr{D}_\Lambda$. For example, there is *no* automorphic form on the coarse moduli space of polarized $K3$ surfaces of degree $2d$ vanishing only on the discriminant locus, if the discriminant locus is irreducible, [36], Section 3.3, [49] (see, e.g., [8], [9], [11], [12], [28], II, [34], [55], [26] for affirmative examples). For another application of $\Psi_\Lambda(\cdot, F_\Lambda)$ to the negativity of the Kodaira dimension of $\mathscr{M}_{M^\perp} = \Omega_{M^\perp}^+/O^+(M^\perp)$, see [37]. In fact, $\mathscr{M}_{M^\perp}$ is always unirational and hence $\kappa(\mathscr{M}_{M^\perp}) = -\infty$ by S. Ma [37].

This paper is organized as follows. In Section 1, we recall lattices and orthogonal modular varieties. In Section 2, we recall 2-elementary $K3$ surfaces and their moduli spaces, and we study the singular fiber of an ordinary singular family of 2-elementary $K3$ surfaces. In Section 3, we recall log del Pezzo surfaces of index $\leqq 2$ and their relation with 2-elementary $K3$ surfaces. In Section 4, we study the current $dd^c J_M^* \|\chi_{g(M)}^8\|^2$ and we recall the notion of automorphic forms on $\Omega_{M^\perp}^+$. In Section 5, we recall the invariant τ_M. In Section 6, we recall Borcherds products. In Section 7, we construct the elliptic modular form $F_\Lambda(\tau)$. In Section 8, we study the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$. In Section 9, we prove Theorem 0.1. In Section 10, we interpret Theorem 0.1 as a statement about the equivariant determinant of the Laplacian on real $K3$ surfaces. In the Appendix, we prove some technical results about lattices.

Nota bene: In [62], we used the notation Ω_M, $\mathscr{M}_M$, $\mathscr{D}_M$ etc. in stead of $\Omega_{M^\perp}$, $\mathscr{M}_{M^\perp}$, $\mathscr{D}_{M^\perp}$ etc.

Acknowledgement. The author thanks Professors Jean-Michel Bismut, Jean-Benoît Bost, Kai Köhler, Vincent Maillot and Damian Roessler for helpful discussions about the invariant τ_M and the automorphic form Φ_M, in particular their arithmetic nature; and Professor Siye Wu for helpful discussions about the elliptic modular form $F_\Lambda(\tau)$ and S-duality. The main result of this paper is a refinement of [60], Theorem 8.2. The author thanks Professor Richard Borcherds, from whom he learned how to construct modular forms of type ρ_Λ from scalar-valued modular forms for $\Gamma_0(4)$ (cf. Proposition 7.1, [60], Proposition 8.1); the construction of $F_\Lambda(\tau)$ is an obvious extension of that of $F_k(\tau)$ in [60], Section 8.2. The author thanks Professor Shigeyuki Kondō, who suggested the proofs of some cases of Propositions 3.1 and 9.3 in the earlier version. The author thanks the referee for the suggestion on Question 8.6, which inspired improvements in Theorems 0.2 and 8.1 in the case $r(M) = 9$.

1. Lattices

A free $\mathbf{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. The rank of a lattice L is denoted by $r(L)$. The signature of L is denoted by $\mathrm{sign}(L) = (b^+(L), b^-(L))$. We define $\sigma(L) := b^+(L) - b^-(L)$. A lattice L is *Lorentzian* if $\mathrm{sign}(L) = (1, r(L) - 1)$. For a lattice $L = (\mathbf{Z}^r, \langle\cdot,\cdot\rangle)$, we define $L(k) := (\mathbf{Z}^r, k\langle\cdot,\cdot\rangle)$. The dual lattice of L is denoted by $L^\vee \subset L \otimes \mathbf{Q}$. The finite abelian group $A_L := L^\vee/L$ is called the *discriminant group* of L. For $\lambda \in L^\vee$, we write $\bar{\lambda} := \lambda + L \in A_L$. A lattice L is *even* if $\langle x, x\rangle \in 2\mathbf{Z}$ for all $x \in L$. A sublattice $M \subset L$ is *primitive* if L/M has no torsion elements. The *level* of an even lattice L is the smallest positive integer l such that $l\lambda^2/2 \in \mathbf{Z}$ for all $\lambda \in L^\vee$. The group of isometries of L is denoted by $O(L)$. We set $\Delta_L := \{d \in L; \langle d, d\rangle = -2\}$ and define

$$\Delta'_L := \{d \in \Delta_L; d/2 \notin L^\vee\}, \quad \Delta''_L := \{d \in \Delta_L; d/2 \in L^\vee\}.$$

Then Δ_L, Δ'_L, Δ''_L are preserved by $O(L)$. For $d \in \Delta_L$, the corresponding reflection $s_d \in O(L)$ is defined as $s_d(x) := x + \langle x, d\rangle d$ for $x \in L$.

1.1. Discriminant forms. For an even lattice L, the *discriminant form* q_L of A_L is the quadratic form on A_L with values in $\mathbf{Q}/2\mathbf{Z}$ defined as $q_L(\bar{l}) := l^2 + 2\mathbf{Z}$ for $\bar{l} \in A_L$. The corresponding bilinear form on A_L with values in $\mathbf{Q}/\mathbf{Z}$ is denoted by b_L. Then $b_L(\bar{l}, \bar{l'}) = \langle l, l'\rangle + \mathbf{Z}$ for $\bar{l}, \bar{l'} \in A_L$. Since $\lambda \in L^\vee$ lies in L if and only if $\langle \lambda, l\rangle \in \mathbf{Z}$ for all $l \in L^\vee$, the bilinear form b_L is non-degenerate, i.e., if $b_L(\gamma, x) \equiv 0 \bmod \mathbf{Z}$ for all $x \in A_L$, then $\gamma = 0$ in A_L. We often write γ^2 (resp. $\langle\gamma, \delta\rangle$) for $q_L(\gamma)$ (resp. $b_L(\gamma, \delta)$). The automorphism group of (A_L, q_L) is denoted by $O(q_L)$. See [46] for more details.

1.2. 2-elementary lattices. Set $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. An even lattice L is 2-*elementary* if there is an integer $l \in \mathbf{Z}_{\geqq 0}$ with $A_L \cong \mathbf{Z}_2^l$. For a 2-elementary lattice L, we set $l(L) := \dim_{\mathbf{Z}_2} A_L$. Then $r(L) \geqq l(L)$ and $r(L) \equiv l(L) \bmod 2$ by [46], Theorem 3.6.2 (2). We define

$$\delta(L) := \begin{cases} 0 & \text{if } x^2 \in \mathbf{Z} \text{ for all } x \in L^\vee, \\ 1 & \text{if } x^2 \notin \mathbf{Z} \text{ for some } x \in L^\vee. \end{cases}$$

The triplet $(\mathrm{sign}(L), l(L), \delta(L))$ determines the isometry class of an *indefinite* even 2-elementary lattice L by [46], Theorem 3.6.2.

Since the mapping $A_L \ni \gamma \to \gamma^2 \in \frac{1}{2}\mathbf{Z}/\mathbf{Z} \cong \mathbf{Z}_2$ is $\mathbf{Z}_2$-linear and since b_L is non-degenerate, there exists a unique element $\mathbf{1}_L \in A_L$ such that $\langle \gamma, \mathbf{1}_L \rangle \equiv \gamma^2 \bmod \mathbf{Z}$ for all $\gamma \in A_L$. By the uniqueness of $\mathbf{1}_L$, we have $g(\mathbf{1}_L) = \mathbf{1}_L$ for all $g \in O(q_L)$. By definition, $\mathbf{1}_L = 0$ if and only if $\delta(L) = 0$. If $L = L' \oplus L''$, then $\mathbf{1}_L = \mathbf{1}_{L'} \oplus \mathbf{1}_{L''}$.

Let $\mathbb{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\mathbb{A}_1$, $\mathbb{D}_{2k}$, $\mathbb{E}_7$, $\mathbb{E}_8$ be the *negative-definite* Cartan matrix of type A_1, D_{2k}, E_7, E_8 respectively, which are identified with the corresponding even lattices. Then $\mathbb{U}$ and $\mathbb{E}_8$ are unimodular, and $\mathbb{A}_1$, $\mathbb{D}_{2k}$ and $\mathbb{E}_7$ are 2-elementary. Set

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8.$$

For a sublattice $\Lambda \subset \mathbb{L}_{K3}$, we define $\Lambda^\perp := \{l \in \mathbb{L}_{K3}; \langle l, \Lambda \rangle = 0\}$. When $\Lambda \subset \mathbb{L}_{K3}$ is primitive, then $(A_\Lambda, -q_\Lambda) \cong (A_{\Lambda^\perp}, q_{\Lambda^\perp})$ by [46], Corollary 1.6.2. In particular, one has $l(\Lambda) \leqq \min\{r(\Lambda), 22 - r(\Lambda)\}$ for a primitive 2-elementary sublattice $\Lambda \subset \mathbb{L}_{K3}$.

1.3. Lorentzian lattices. Let L be a Lorentzian lattice. The set

$$\mathscr{C}_L := \{v \in L \otimes \mathbf{R}; v^2 > 0\}$$

is called the positive cone of L. Since L is Lorentzian, $\mathscr{C}_L$ consists of two connected components, which are written as $\mathscr{C}_L^+$, $\mathscr{C}_L^-$. For $l \in L \otimes \mathbf{R}$, we set $h_l := \{v \in \mathscr{C}_L^+; \langle v, l \rangle = 0\}$. Then $h_l \neq \emptyset$ if and only if $l^2 < 0$. We define $(\mathscr{C}_L^+)^o := \mathscr{C}_L^+ \setminus \bigcup_{d \in \Delta_L} h_d$. Any connected component of $(\mathscr{C}_L^+)^o$ is called a *Weyl chamber* of L.

Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let I_M be the involution on $M \oplus M^\perp$ defined as

$$I_M(x, y) = (x, -y), \quad (x, y) \in M \oplus M^\perp.$$

Then I_M extends uniquely to an involution on $\mathbb{L}_{K3}$ by [46], Corollary 1.5.2. We define

$$g(M) := \{22 - r(M) - l(M)\}/2, \quad k(M) := \{r(M) - l(M)\}/2.$$

For $d \in \Delta_{M^\perp}$, the smallest sublattice of $\mathbb{L}_{K3}$ containing M and $\mathbf{Z}d$ is given by

$$[M \perp d] := (M^\perp \cap d^\perp)^\perp.$$

By Lemma 11.3 below, $[M \perp d]$ is again a 2-elementary Lorentzian lattice such that

$$(1.1) \qquad I_{[M \perp d]} = s_d \circ I_M = I_M \circ s_d, \quad [M \perp d]^\perp = M^\perp \cap d^\perp.$$

By, e.g., [21], Appendix, Tables 1–3, M and $M^\perp$ are expressed as direct sums of the 2-elementary lattices $\mathbb{A}_1^+ := \mathbb{A}_1(-1)$, $\mathbb{A}_1$, $\mathbb{U}$, $\mathbb{U}(2)$, $\mathbb{D}_{2k}$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{E}_8(2)$.

1.4. Lattices of signature (2, *n*) and orthogonal modular varieties. Let Λ be a lattice with $\mathrm{sign}(\Lambda) = (2, n)$. Define

$$\Omega_\Lambda := \{[x] \in \mathbf{P}(\Lambda \otimes \mathbf{C}); \langle x, x\rangle = 0, \langle x, \bar{x}\rangle > 0\},$$

which has two connected components Ω_Λ^+ and Ω_Λ^-. Each of $\Omega_\Lambda^\pm$ is isomorphic to a bounded symmetric domain of type IV of dimension n. $O(\Lambda)$ acts projectively on Ω_Λ. Set

$$O^+(\Lambda) := \{g \in O(\Lambda); g(\Omega_\Lambda^\pm) = \Omega_\Lambda^\pm\},$$

which is a subgroup of $O(\Lambda)$ of index 2 such that $\Omega_\Lambda/O(\Lambda) = \Omega_\Lambda^+/O^+(\Lambda)$. Since $O^+(\Lambda)$ is an arithmetic subgroup of $\mathrm{Aut}(\Omega_\Lambda^+)$, $O^+(\Lambda)$ acts properly discontinuously on Ω_Λ^+. In particular, the stabilizer $O^+(\Lambda)_{[\eta]} := \{g \in O^+(\Lambda); g \cdot [\eta] = [\eta]\}$ is finite for all $[\eta] \in \Omega_\Lambda^+$, and the quotient

$$\mathscr{M}_\Lambda := \Omega_\Lambda/O(\Lambda) = \Omega_\Lambda^+/O^+(\Lambda)$$

is an analytic space. There exists a compactification $\mathscr{M}_\Lambda^*$ of $\mathscr{M}_\Lambda$, called the Baily–Borel–Satake compactification [3], such that $\mathscr{M}_\Lambda^*$ is an irreducible normal projective variety of dimension n with $\dim(\mathscr{M}_\Lambda^* \backslash \mathscr{M}_\Lambda) \leqq 1$.

For $\lambda \in \Lambda \otimes \mathbf{R}$, set $H_\lambda := \{[x] \in \Omega_\Lambda; \langle x, \lambda\rangle = 0\}$. Then $H_\lambda \neq \emptyset$ if and only if $\lambda^2 < 0$. We define the discriminant locus of Ω_Λ by

$$\mathscr{D}_\Lambda := \sum_{d \in \Delta_\Lambda/\pm 1} H_d,$$

which is a reduced divisor on Ω_Λ. We define the reduced divisors $\mathscr{D}'_\Lambda$ and $\mathscr{D}''_\Lambda$ by

$$\mathscr{D}'_\Lambda = \sum_{d \in \Delta'_\Lambda/\pm 1} H_d, \quad \mathscr{D}''_\Lambda = \sum_{d \in \Delta''_\Lambda/\pm 1} H_d.$$

Since $\Delta_\Lambda = \Delta'_\Lambda \amalg \Delta''_\Lambda$, we have $\mathscr{D}_\Lambda = \mathscr{D}'_\Lambda + \mathscr{D}''_\Lambda$.

Assume that Λ is a primitive 2-elementary sublattice of $\mathbb{L}_{K3}$. We set

$$\Omega_\Lambda^o := \Omega_\Lambda \backslash \mathscr{D}_\Lambda, \quad \mathscr{M}_\Lambda^o := \Omega_\Lambda^o/O(\Lambda).$$

For $d \in \Delta_\Lambda$, we have the relation

$$H_d \cap \Omega_\Lambda = \Omega_{\Lambda \cap d^\perp} = \Omega_{[\Lambda^\perp \perp d]^\perp}.$$

We define the subsets $H_d^o \subset H_d$ $(d \in \Delta_\Lambda)$ and $\mathscr{D}_\Lambda^o \subset \mathscr{D}_\Lambda$ by

$$H_d^o := \{[\eta] \in \Omega_\Lambda^+; O^+(\Lambda)_{[\eta]} = \{\pm 1, \pm s_d\}\}, \quad \mathscr{D}_\Lambda^o := \sum_{d \in \Delta_\Lambda/\pm 1} H_d^o.$$

If $H_d \neq \emptyset$ (resp. $\mathscr{D}_\Lambda \neq \emptyset$), then H_d^o (resp. $\mathscr{D}_\Lambda^o$) is a non-empty Zariski open subset of $\Omega_{\Lambda \cap d^\perp}$ (resp. $\mathscr{D}_\Lambda$) unless $M^\perp = (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1$ (cf. [66], proof of Theorem 4.1 and Section 5). Since $O(\Lambda)$ preserves $\mathscr{D}_\Lambda$ and $\mathscr{D}_\Lambda^o$, we define

$$\overline{\mathscr{D}}_\Lambda := \mathscr{D}_\Lambda/O(\Lambda), \quad \overline{\mathscr{D}}_\Lambda^o := \mathscr{D}_\Lambda^o/O(\Lambda) \subset \overline{\mathscr{D}}_\Lambda.$$

Then $\overline{\mathscr{D}}^o_\Lambda \cap \operatorname{Sing} \mathscr{M}_\Lambda = \emptyset$ by [62], Proposition 1.9 (5). For the irreducibility of $\mathscr{D}'_\Lambda/O(\Lambda)$, see Proposition 11.6 (5) below.

When $\Lambda = \mathbb{U}(N) \oplus L$, a vector of $\Lambda \otimes \mathbf{C}$ is denoted by (m, n, v), where $m, n \in \mathbf{C}$ and $v \in L \otimes \mathbf{C}$. The tube domain $L \otimes \mathbf{R} + i\mathscr{C}_L$ is identified with Ω_Λ via the map

$$(1.2) \qquad L \otimes \mathbf{R} + i\mathscr{C}_L \ni z \to [(-z^2/2, 1/N, z)] \in \Omega_\Lambda \subset \mathbf{P}(\Lambda \otimes \mathbf{C}), \quad z \in L \otimes \mathbf{C}$$

by [10], p. 542. The component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\mathscr{C}_L^+$ via the isomorphism (1.2) is written as Ω_Λ^+.

2. $K3$ surfaces with involution

2.1. $K3$ surfaces with involution and their moduli space. A compact, connected, smooth complex surface X is called a $K3$ *surface* if it is simply connected and has a trivial canonical line bundle Ω_X^2. Let X be a $K3$ surface. Then $H^2(X, \mathbf{Z})$ endowed with the cup-product pairing is isometric to the $K3$ lattice $\mathbb{L}_{K3}$. An isometry of lattices $\alpha : H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a *marking* of X. The pair (X, α) is called a *marked* $K3$ *surface*, whose period is defined as

$$\pi(X, \alpha) := [\alpha(\eta)] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C}), \quad \eta \in H^0(X, \Omega_X^2) \backslash \{0\}.$$

Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. A $K3$ surface equipped with a holomorphic involution $\iota : X \to X$ is called a 2-*elementary* $K3$ *surface of type* M if there exists a marking α of X satisfying

$$\iota^*|_{H^0(X, \Omega_X^2)} = -1, \quad \iota^* = \alpha^{-1} \circ I_M \circ \alpha.$$

Then $\alpha\big(H_+^2(X, \mathbf{Z})\big) = M$, where $H_\pm^2(X, \mathbf{Z}) := \{l \in H^2(X, \mathbf{Z}); \iota^* l = \pm l\}$.

Let (X, ι) be a 2-elementary $K3$ surface of type M and let α be a marking with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. Since $H^{2,0}(X, \mathbf{C}) \subset H_-^2(X, \mathbf{Z}) \otimes \mathbf{C}$, we have $\pi(X, \alpha) \in \Omega_{M^\perp}^o$ by [45], Theorem 3.10. By [62], Theorem 1.8, and Proposition 11.2 below, the $O(M^\perp)$-orbit of $\pi(X, \iota)$ is independent of the choice of a marking α with $\iota^* = \alpha^{-1} I_M \alpha$. The Griffiths period of (X, ι) is defined as the $O(M^\perp)$-orbit

$$\varpi_M(X, \iota) := O(M^\perp) \cdot \pi(X, \alpha) \in \mathscr{M}_{M^\perp}^o.$$

By [51], [15], [48], [18], as well as [62], Theorem 1.8, and by Proposition 11.2 below, the coarse moduli space of 2-elementary $K3$ surfaces of type M is isomorphic to $\mathscr{M}_{M^\perp}^o$ via the map ϖ_M. In the rest of this paper, we identify the point $\varpi_M(X, \iota) \in \mathscr{M}_{M^\perp}^o$ with the isomorphism class of (X, ι).

For a 2-elementary $K3$ surface (X, ι), set $X^\iota := \{x \in X; \iota(x) = x\}$.

Proposition 2.1. *Let (X, ι) be a 2-elementary $K3$ surface of type M.*

(1) *If* $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, *then* $X^\iota = \emptyset$.

(2) *If* $M \cong \mathbb{U} \oplus \mathbb{E}_8(2)$, *then* X^ι *is the disjoint union of two elliptic curves.*

(3) *If* $M \ncong \mathbb{U}(2) \oplus \mathbb{E}_8(2), \mathbb{U} \oplus \mathbb{E}_8(2)$, *then there exist a smooth irreducible curve* C *of genus* $g(M)$ *and smooth rational curves* $E_1, \dots, E_{k(M)}$ *such that*

$$X^\iota = C \amalg E_1 \amalg \cdots \amalg E_{k(M)}.$$

Proof. See [48], Theorem 4.2.2. □

After Proposition 2.1, a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$ is said to be *non-exceptional* if $M \ncong \mathbb{U}(2) \oplus \mathbb{E}_8(2), \mathbb{U} \oplus \mathbb{E}_8(2)$. Let (X, ι) be a 2-elementary $K3$ surface of type M. When M is non-exceptional and when $g(M) > 0$, the component of X^ι with genus $g(M)$ is called *the main component of* X^ι.

For $g \geqq 0$, let $\mathfrak{S}_g$ be the Siegel upper half-space of degree g. When $g = 1$, $\mathfrak{S}_1$ is the complex upper half-plane. We write $\mathfrak{H}$ for $\mathfrak{S}_1$. Let $\mathrm{Sp}_{2g}(\mathbf{Z})$ be the symplectic group of degree $2g$ over $\mathbf{Z}$ and let $\mathscr{A}_g := \mathfrak{S}_g/\mathrm{Sp}_{2g}(\mathbf{Z})$ be the Siegel modular variety of degree g. Then $\mathscr{A}_g$ is a coarse moduli space of principally polarized Abelian varieties of dimension g. The Satake compactification of $\mathscr{A}_g$ is denoted by $\mathscr{A}_g^*$. Then $\mathscr{A}_g^*$ has the stratification $\mathscr{A}_g^* = \mathscr{A}_g \amalg \mathscr{A}_{g-1} \amalg \cdots \amalg \mathscr{A}_0$.

For a 2-elementary $K3$ surface (X, ι) of type M, the period of X^ι, i.e., the period of $\mathrm{Jac}(X^\iota) := H^1(X^\iota, \mathscr{O}_{X^\iota})/H^1(X^\iota, \mathbf{Z})$, is denoted by $\Omega(X^\iota) \in \mathscr{A}_{g(M)}$. For a 2-elementary $K3$ surface (X, ι) of type M, we define

$$\bar{J}_M^o(X, \iota) = \bar{J}_M^o(\varpi_M(X, \iota)) := \Omega(X^\iota) \in \mathscr{A}_{g(M)}.$$

Let $\Pi_{M^\perp} : \Omega_{M^\perp} \to \mathscr{M}_{M^\perp}$ be the projection and set $J_M^o := \bar{J}_M^o \circ \Pi_{M^\perp}|_{\Omega_{M^\perp}^o}$. Then J_M^o is an $O(M^\perp)$-equivariant holomorphic map from $\Omega_{M^\perp}^o$ to $\mathscr{A}_{g(M)}$ with respect to the trivial $O(M^\perp)$-action on $\mathscr{A}_{g(M)}$. By [62], Theorem 3.3, J_M^o extends to an $O(M^\perp)$-equivariant holomorphic map $J_M : \Omega_{M^\perp}^o \cup \mathscr{D}_{M^\perp}^o \to \mathscr{A}_{g(M)}^*$. The corresponding holomorphic extension of $\bar{J}_M^o$ is denoted by $\bar{J}_M : \mathscr{M}_{M^\perp}^o \cup \bar{\mathscr{D}}_{M^\perp}^o \to \mathscr{A}_{g(M)}^*$.

Proposition 2.2. *The map* $\bar{J}_M$ *extends to a meromorphic map from* $\mathscr{M}_{M^\perp}^*$ *to* $\mathscr{A}_{g(M)}^*$. *When* $r(M) \geqq 19$, $\bar{J}_M$ *extends to a holomorphic map from* $\mathscr{M}_{M^\perp}^*$ *to* $\mathscr{A}_{g(M)}^*$.

Proof. By [13], $\bar{J}_M$ extends to a holomorphic map from

$$\mathscr{M}_{M^\perp}^* \backslash (\mathrm{Sing}\, \mathscr{M}_{M^\perp}^* \cup \mathrm{Sing}\, \bar{\mathscr{D}}_{M^\perp})$$

to $\mathscr{A}_{g(M)}^*$. Since $\mathscr{M}_{M^\perp}^*$ is normal, we get $\dim(\mathrm{Sing}\, \mathscr{M}_{M^\perp}^* \cup \mathrm{Sing}\, \bar{\mathscr{D}}_{M^\perp}) \leqq \dim \mathscr{M}_{M^\perp}^* - 2$ when $r(M) \leqq 18$, so that $\bar{J}_M$ extends to a meromorphic map from $\mathscr{M}_{M^\perp}^*$ to $\mathscr{A}_{g(M)}^*$ by [57] in this case. If $r(M) = 19$, the result follows from [13] because $\mathscr{M}_{M^\perp}^*$ is a compact Riemann surface and $\mathscr{M}_{M^\perp}^* \backslash \mathscr{M}_{M^\perp}^o$ is a finite subset of $\mathscr{M}_{M^\perp}^*$. If $r(M) = 20$, the result is trivial because $\mathscr{M}_{M^\perp}^*$ is a point. □

2.2. Degenerations of 2-elementary $K3$ surfaces. Let $\Delta \subset \mathbf{C}$ be the unit disc and set $\Delta^* := \Delta \backslash \{0\}$. Let $\mathscr{Z}$ be a smooth complex threefold. Let $p : \mathscr{Z} \to \Delta$ be a proper surjective holomorphic function without critical points on $\mathscr{Z} \backslash p^{-1}(0)$. Let $\imath : \mathscr{Z} \to \mathscr{Z}$ be a holomorphic involution preserving the fibers of p. We set $Z_t = p^{-1}(t)$ and $\imath_t = \imath|_{Z_t}$ for $t \in \Delta$. Then $p : (\mathscr{Z}, \imath) \to \Delta$ is called an *ordinary singular family* of 2-elementary $K3$ surfaces of type M if p has a unique, non-degenerate critical point on Z_0 and if $(Z_t, \imath_t)$ is a 2-elementary $K3$ surface of type M for all $t \in \Delta^*$. Since Z_0 is a singular $K3$ surface, $\imath_0 \in \mathrm{Aut}(Z_0)$ extends to an anti-symplectic holomorphic involution $\tilde{\imath}_0$ on the minimal resolution $\tilde{Z}_0$ of Z_0, i.e., $(\tilde{\imath}_0)^* = -1$ on $H^0(\tilde{Z}_0, \Omega^2_{\tilde{Z}_0})$. Let $o \in \mathscr{Z}$ be the unique critical point of p. There exists a system of coordinates $(\mathscr{U}, (z_1, z_2, z_3))$ centered at o such that

$$\imath(z) = (-z_1, -z_2, -z_3) \text{ or } (z_1, z_2, -z_3), \quad z \in \mathscr{U}.$$

If $\imath(z) = (-z_1, -z_2, -z_3)$ on $\mathscr{U}$, $\imath$ is said to be of *type* $(0,3)$. If $\imath(z) = (z_1, z_2, -z_3)$ on $\mathscr{U}$, $\imath$ is said to be of *type* $(2,1)$.

Theorem 2.3. *Let $d \in \Delta_{M^\perp}$ and let $\overline{H}^o_d := \Pi_{M^\perp}(H^o_d)$ be the image of H^o_d by the natural projection $\Pi_{M^\perp} : \Omega_{M^\perp} \to \mathscr{M}_{M^\perp}$. Let $\gamma : \Delta \to \mathscr{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}^o_d$ transversally at $\gamma(0)$. Then there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathscr{Z}} : (\mathscr{Z}, \imath) \to \Delta$ of type M with Griffiths period map γ satisfying the following properties*:

(1) *$p_{\mathscr{Z}}$ is a projective morphism and the minimal resolution $(\tilde{Z}_0, \tilde{\imath}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$.*

(2) *If $d \in \Delta'_{M^\perp}$, then $\imath$ is of type $(2,1)$ and $(\tilde{Z}_0)^{\tilde{\imath}_0}$ is the normalization of $Z_0^{\imath_0}$ with total genus $g(M) - 1$.*

Proof. By [62], Theorem 2.6, there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathscr{Z}} : (\mathscr{Z}, \imath) \to \Delta$ of type M with Griffiths period map γ such that $p_{\mathscr{Z}}$ is projective. We prove that $(\tilde{Z}_0, \tilde{\imath}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$.

Let $o_{\mathscr{Z}} \in Z_0$ be the unique critical point of $p_{\mathscr{Z}}$. Let $p_{\mathscr{Y}} : (\mathscr{Y}, \imath_{\mathscr{Y}}) \to \Delta$ be the family induced from $p_{\mathscr{Z}} : (\mathscr{Z}, \imath) \to \Delta$ by the map $\Delta \ni t \to t^2 \in \Delta$. Then $\mathscr{Y} = \mathscr{Z} \times_\Delta \Delta$ and $p_{\mathscr{Y}} = \mathrm{pr}_2$. The projection pr_1 induces an identification between $(Y_t, \imath_{\mathscr{Y}}|_{Y_t})$ and $(Z_{t^2}, \imath_{t^2})$ for all $t \in \Delta$. Since the Picard–Lefschetz transformation for the family of $K3$ surfaces $p_{\mathscr{Y}}|_{\Delta^*} : \mathscr{Y}|_{\Delta^*} \to \Delta^*$ is trivial, there exists a marking $\beta : R^2(p_{\mathscr{Y}}|_{\Delta^*})_* \mathbf{Z} \cong \mathbb{L}_{K3, \Delta^*}$. Let $o_{\mathscr{Y}}$ be the unique singular point of $\mathscr{Y}$ with $\mathrm{pr}_2(o_{\mathscr{Y}}) = o_{\mathscr{Z}}$. Since $(\mathscr{Y}, o_{\mathscr{Y}})$ is a three-dimensional ordinary double point, there exist two different resolutions $\pi : (\mathscr{X}, E) \to (\mathscr{Y}, o_{\mathscr{Y}})$ and $\pi' : (\mathscr{X}', E') \to (\mathscr{Y}, o_{\mathscr{Y}})$, which satisfy the following properties (cf. [62], Theorem 2.1, proof of Theorem 2.6 and the references therein):

(i) Set $p := p_{\mathscr{Y}} \circ \pi$ and $p' := p_{\mathscr{Y}} \circ \pi'$. Then $p : \mathscr{X} \to \Delta$ and $p' : \mathscr{X}' \to \Delta$ are simultaneous resolutions of $p_{\mathscr{Y}} : \mathscr{Y} \to \Delta$, and they are smooth families of $K3$ surfaces. The marking β induces a marking α for $p : \mathscr{X} \to \Delta$ and a marking α' for $p' : \mathscr{X}' \to \Delta$.

(ii) $E = \pi^{-1}(o_{\mathscr{Y}})$ is a smooth rational curve on X_0, and $E' = (\pi')^{-1}(o_{\mathscr{Y}})$ is a smooth rational curve on X'_0. The marked family $(p' : \mathscr{X}' \to \Delta, \alpha')$ is the elementary modification of $(p : \mathscr{X} \to \Delta, \alpha)$ with center E (cf. [62], Section 2.1). Replacing β by $g \circ \beta$, $g \in \Gamma(M) := \{g \in O(\mathbb{L}_{K3}); gI_M = I_M g\}$ if necessary, one has $d = \alpha(c_1([E]))$.

(iii) Let $e : \mathscr{X} \backslash E \to \mathscr{X}' \backslash E'$ be the isomorphism defined as

$$e := (\pi'|_{X' \backslash E'})^{-1} \circ (\pi|_{X \backslash E}).$$

Then e is an isomorphism of fiber spaces over Δ^* and the isomorphism

$$e|_{X_0 \backslash E} : X_0 \backslash E \to X_0' \backslash E'$$

extends to an isomorphism $\tilde{e}_0 : X_0 \to X_0'$ with

$$\alpha_0 \circ (\tilde{e}_0)^* \circ (\alpha_0')^{-1} = s_d. \tag{2.1}$$

(iv) There exists an isomorphism $\varphi_{K3}(I_M) : \mathscr{X} \to \mathscr{X}'$ of fiber spaces over Δ such that the following diagrams are commutative (cf. [62], Eqs. (1.6), (2.8)):

$$\begin{array}{ccccc} (\mathscr{X}, E) & \xrightarrow{\pi} & (\mathscr{Y}, o) & \xrightarrow{\mathrm{pr}_1} & (\mathscr{Z}, o) \\ \downarrow{\scriptstyle \varphi_{K3}(I_M)} & & \downarrow{\scriptstyle \iota_{\mathscr{Y}}} & & \downarrow{\scriptstyle \iota} \\ (\mathscr{X}', E') & \xrightarrow{\pi'} & (\mathscr{Y}, o) & \xrightarrow{\mathrm{pr}_1} & (\mathscr{Z}, o), \end{array} \qquad \begin{array}{ccc} R^2 p'_* \mathbf{Z} & \xrightarrow{\varphi_{K3}(I_M)^*} & R^2 p_* \mathbf{Z} \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ \mathbb{L}_{K3,\Delta} & \xrightarrow{I_M} & \mathbb{L}_{K3,\Delta}. \end{array} \tag{2.2}$$

We define $\theta := (\tilde{e}_0)^{-1} \circ \varphi_{K3}(I_M)|_{X_0} \in \mathrm{Aut}(X_0)$. Since $\pi' \circ \tilde{e}_0 = \pi|_{X_0}$ by (iii) and hence $\pi'|_{X_0' \backslash E'} = (\pi|_{X_0 \backslash E}) \circ (\tilde{e}_0)^{-1}|_{X_0' \backslash E'}$, we get by the first diagram of (2.2)

$$\begin{aligned}(\pi|_{X_0 \backslash E}) \circ (\theta|_{X_0 \backslash E}) &= (\pi|_{X_0 \backslash E}) \circ (\tilde{e}_0)^{-1}|_{X_0' \backslash E'} \circ \varphi_{K3}(I_M)|_{X_0 \backslash E} \\ &= (\pi'|_{X_0' \backslash E'}) \circ \varphi_{K3}(I_M)|_{X_0 \backslash E} \\ &= (\iota_{\mathscr{Y}}|_{Y_0 \backslash \{o\}}) \circ \pi|_{X_0 \backslash E},\end{aligned}$$

which implies $(\pi|_{X_0}) \circ \theta = (\iota_{\mathscr{Y}}|_{Y_0}) \circ (\pi|_{X_0})$. Since X_0 is the minimal resolution of Z_0, i.e., $X_0 = \tilde{Z}_0$ and since $(Y_0, \iota_{\mathscr{Y}}|_{Y_0}) = (Z_0, \iota_0)$, the equality $(\pi|_{X_0}) \circ \theta = (\iota_{\mathscr{Y}}|_{Y_0}) \circ (\pi|_{X_0})$ implies that θ is the involution on X_0 induced from ι_0. Thus $\theta = \tilde{\iota}_0$.

By (1.1), (2.1) and the second diagram of (2.2), we get

$$\alpha_0 \theta^* \alpha_0^{-1} = \alpha_0 \varphi_{K3}(I_M)^* (\alpha_0')^{-1} \circ \alpha_0' (\tilde{e}_0^{-1})^* \alpha_0^{-1} = I_M \circ s_d = I_{[M \perp d]}. \tag{2.3}$$

By (2.3), $\theta = \tilde{\iota}_0$ is an anti-symplectic involution of type $[M \perp d]$. This proves (1).

Let $d \in \Delta'_{M^\perp}$. If ι is of type $(0,3)$, then $g([M \perp d]) = g(M)$ by [62], Proposition 2.5. Since $d \in \Delta'_{M^\perp}$ implies $g([M \perp d]) = g(M) - 1$ by Lemma 11.5 below, we get a contradiction. Hence ι must be of type $(2,1)$. Since $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$, $(\tilde{Z}_0)^{\tilde{\iota}_0}$ has total genus $g([M \perp d]) = g(M) - 1$ by Lemma 11.5. Since $\tilde{Z}_0 \to Z_0$ is the blow-up at the ordinary double point $o_{\mathscr{Z}}$, it follows from the local description $\iota(z) = (z_1, z_2, -z_3)$ near $o_{\mathscr{Z}}$ that the set of fixed points $(\tilde{Z}_0)^{\tilde{\iota}_0}$ is the normalization of $Z_0^{\iota_0}$. This proves (2). □

Let $\mathscr{C}$ be a (possibly disconnected) smooth complex surface. Let $p : \mathscr{C} \to \Delta$ be a proper, surjective holomorphic function without critical points on $\mathscr{C} \setminus p^{-1}(0)$. Then $p : \mathscr{C} \to \Delta$ is called an ordinary singular family of curves if p has a unique, non-degenerate critical point on $p^{-1}(0)$. We set $C_t := p^{-1}(t)$ for $t \in \Delta$.

Lemma 2.4. *Let $p : \mathscr{C} \to \Delta$ be an ordinary singular family of curves and let $g = \dim H^0(C_t, \Omega^1_{C_t})$ for $t \neq 0$. Let $J : \Delta^* \to \mathscr{A}_g$ be the holomorphic map defined as $J(t) := \Omega(C_t)$ for $t \in \Delta^*$. Then J extends to a holomorphic map from Δ to $\mathscr{A}_g^*$ by setting $J(0) := \Omega(\widehat{C_0})$, where $\hat{C}_0$ is the normalization of C_0.*

Proof. Since the result is obvious when $g = 0$, we assume $g > 0$. The extendability of J follows from, e.g., [4], Chapter III, Theorem 16.1. Assume that p has connected fibers. Either C_0 is the join of two smooth curves A and B intersecting transversally at $\operatorname{Sing} C_0$ or C_0 is irreducible. The result follows from, e.g., [20], Corollaries 3.2 and 3.8.

Assume that $\mathscr{C}$ is not connected. Let $\mathscr{C} = \mathscr{C}_0 \amalg \cdots \amalg \mathscr{C}_k$ be the decomposition into the connected components and set $p_i := p|_{\mathscr{C}_i}$. Since the period matrix of C_t is the direct sum of those of the curves $p_i^{-1}(t)$, the result follows from the case where p has connected fibers and [4], Chapter III, Theorem 16.1. □

Theorem 2.5. *For $d \in \Delta_{M^\perp}$, the following equality holds:*

$$J_M|_{H_d^o} = J^o_{[M \perp d]}|_{H_d^o}.$$

Proof. Let $\mathfrak{p} \in \overline{H}_d^o$ and let $\gamma : \Delta \to \mathscr{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}_d^o$ transversally at $\mathfrak{p} = \gamma(0)$. Let $p_{\mathscr{Z}} : (\mathscr{Z}, \iota) \to \Delta$ be an ordinary singular family of 2-elementary $K3$ surfaces of type M with Griffiths period map γ, such that $p_{\mathscr{Z}}$ is projective and such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$ (cf. Theorem 2.3). Let $o \in \mathscr{Z}$ be the unique critical point of $p_{\mathscr{Z}}$. Since $\bar{J}_M(\mathfrak{p}) = \bar{J}_M(\gamma(0)) = \lim_{t \to 0} \bar{J}_M(\gamma(t))$ by the continuity of $\bar{J}_M$ and since

$$\bar{J}^o_{[M \perp d]}(\mathfrak{p}) = \bar{J}^o_{[M \perp d]}(\tilde{Z}_0, \tilde{\iota}_0) = \Omega((\tilde{Z}_0)^{\tilde{\iota}_0})$$

by Theorem 2.3, it suffices to prove

$$\bar{J}_M(\mathfrak{p}) = \lim_{t \to 0} \bar{J}_M(\gamma(t)) = \Omega((\tilde{Z}_0)^{\tilde{\iota}_0}) = \bar{J}^o_{[M \perp d]}(\mathfrak{p}). \tag{2.4}$$

Set $\mathscr{Z}^\iota := \{z \in \mathscr{Z}; \iota(z) = z\}$.

Assume that ι is of type $(0, 3)$. By [62], Proposition 2.5 (1), $\mathscr{C} := \mathscr{Z}^\iota \setminus \{o\}$ is a smooth complex surface and $p|_{\mathscr{C}} : \mathscr{C} \to \Delta$ is a proper holomorphic submersion. Then

$$\lim_{t \to 0} \bar{J}^o_M(Z_t, \iota_t) = \lim_{t \to 0} \Omega(C_t) = \Omega(C_0). \tag{2.5}$$

Since $Z_0^{\iota_0} = C_0 \amalg \{o\}$, we get $(\tilde{Z}_0)^{\tilde{\iota}_0} = C_0 \amalg \mathbf{P}^1$, which yields that

$$\Omega(C_0) = \Omega((\tilde{Z}_0)^{\tilde{\iota}_0}). \tag{2.6}$$

Equation (2.4) follows from (2.5) and (2.6) in this case.

Assume that ι is of type $(2,1)$. By [62], Proposition 2.5 (2), $p|_{\mathscr{X}^\iota} : \mathscr{X}^\iota \to \Delta$ is an ordinary singular family of curves. Let $W \to Z_0^{\iota_0}$ be the normalization. We get

$$\lim_{t\to 0} \bar{J}_M^o(Z_t, \iota_t) = \lim_{t\to 0} \Omega(Z_t^{\iota_t}) = \Omega(W) \in \mathscr{A}_{g(M)}^* \tag{2.7}$$

by Lemma 2.4. In the same manner as in the proof of Theorem 2.3 (2), we get $W = (\tilde{Z}_0)^{\tilde{\iota}_0}$, which together with (2.7) yields (2.4) in this case. Since $\mathfrak{p}$ is an arbitrary point of $\overline{H}_d^o$, we get the result. □

The following propositions shall be used in the proof of Proposition 4.2 (3).

Proposition 2.6. *If $g(M) = 1$ and $d \in \Delta'_{M^\perp}$, then $J_M(H_d^o) = \mathscr{A}_0 = \mathscr{A}_1^* \backslash \mathscr{A}_1$.*

Proof. By Lemma 11.5 below, $g([M \perp d]) = g(M) - 1 = 0$. By Theorem 2.5, we get $J_M(H_d^o) = J_{[M\perp d]}^o(H_d^o) = \mathscr{A}_0 = \mathscr{A}_1^* \backslash \mathscr{A}_1$. □

Proposition 2.7. *If $g(M) = 1$, then $\overline{J_M^o(\Omega_{M^\perp}^o)} = \mathscr{A}_1^*$.*

Proof. By Proposition 2.2, $\bar{J}_M$ extends to a meromorphic map from $\mathscr{M}_{M^\perp}^*$ to $\mathscr{A}_1^*$. Since $J_M^o(\Omega_{M^\perp}^o) = \bar{J}_M(\mathscr{M}_{M^\perp}^o)$ and since $\dim \mathscr{A}_1^* = 1$, we have $\overline{J_M^o(\Omega_{M^\perp}^o)} = \mathscr{A}_1^*$ if $\bar{J}_M^o$ is non-constant. We see that $\bar{J}_M^o$ is non-constant.

Since $g(M) = 1$, we get by [48], p. 1434, Table 1, or [21], Appendix, Table 2,

$$M^\perp \cong \mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus m-1}\ (1 \leqq m \leqq 10), \quad \mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{D}_4, \quad \mathbb{U} \oplus \mathbb{U}(2). \tag{2.8}$$

By (2.8), $\Delta'_{M^\perp} \neq \emptyset$. Let $d \in \Delta'_{M^\perp}$. By Proposition 2.6, we get $J_M(H_d^o) = \mathscr{A}_0 = \mathscr{A}_1^* \backslash \mathscr{A}_1$. Since $J_M(\Omega_{M^\perp}^o) \subset \mathscr{A}_1$, this implies that $\bar{J}_M$ is non-constant. □

Proposition 2.8. *If $g(M) = 1$ and $d \in \Delta''_{M^\perp}$, then $J_M(H_d^o) \subset \mathscr{A}_1$.*

Proof. Since $d \in \Delta''_{M^\perp}$, we get $g([M \perp d]) = g(M) = 1$ by Lemma 11.5 below. By Theorem 2.5, we get $J_M(H_d^o) = J_{[M\perp d]}^o(H_d^o) \subset J_{[M\perp d]}^o(\Omega_{[M\perp d]^\perp}^o) \subset \mathscr{A}_1$. □

Proposition 2.9. *If $g(M) = 2$ and $d \in \Delta'_{M^\perp}$, then $\overline{J_M(H_d^o)} = \mathscr{A}_2^* \backslash \mathscr{A}_2$.*

Proof. By Lemma 11.5 below, $g([M \perp d]) = 1$. By Theorem 2.5, we get

$$\overline{J_M(H_d^o)} = \overline{J_{[M\perp d]}^o(H_d^o)} = \overline{J_{[M\perp d]}(\Omega_{[M\perp d]^\perp}^o)} = \mathscr{A}_1^* = \mathscr{A}_2^* \backslash \mathscr{A}_2,$$

where the third equality follows from Proposition 2.7. □

We define the divisor $\mathscr{N}_2 \subset \mathscr{A}_2$ as

$$\mathscr{N}_2 := \{\Omega(E_1 \times E_2) \in \mathscr{A}_2;\, E_1, E_2 \text{ are elliptic curves}\}.$$

Proposition 2.10. *Let $g(M)=2$ and $d\in\Delta''_{M^\perp}$. Then $J_M(H_d^o)\cap\mathscr{N}_2\neq\emptyset$ if and only if $M\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$ and $d/2\equiv\mathbf{1}_{M^\perp}\bmod M^\perp$. In particular, if either $M\not\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$ or $d/2\not\equiv\mathbf{1}_{M^\perp}\bmod M^\perp$, then*

$$J_M(H_d^o)\subset\mathscr{A}_2\backslash\mathscr{N}_2.$$

Proof. Assume $J_M(H_d^o)\cap\mathscr{N}_2\neq\emptyset$. By Theorem 2.5,

$$J_{[M\perp d]}(\Omega^o_{[M\perp d]^\perp})\cap\mathscr{N}_2\supset J_{[M\perp d]}(H_d^o)\cap\mathscr{N}_2=J_M(H_d^o)\cap\mathscr{N}_2\neq\emptyset.$$

Let (X,ι) be a 2-elementary $K3$ surface of type $[M\perp d]$ such that $J_{[M\perp d]}(X,\iota)\in\mathscr{N}_2$. If $[M\perp d]\not\cong\mathbb{U}\oplus\mathbb{E}_8(2),\mathbb{U}(2)\oplus\mathbb{E}_8(2)$, there exists an irreducible smooth curve C of genus $g([M\perp d])$ with $J_{[M\perp d]}(X,\iota)=\Omega(C)$ by Proposition 2.1. By $d\in\Delta''_{M^\perp}$ and Lemma 11.5 below, we get $g([M\perp d])=2$. However, the period of an irreducible smooth curve of genus 2 lies in $\mathscr{A}_2\backslash\mathscr{N}_2$. This contradicts the condition $\Omega(C)\in\mathscr{N}_2$. Thus $[M\perp d]\cong\mathbb{U}\oplus\mathbb{E}_8(2)$ or $[M\perp d]\cong\mathbb{U}(2)\oplus\mathbb{E}_8(2)$. If $[M\perp d]\cong\mathbb{U}(2)\oplus\mathbb{E}_8(2)$, then $C=\emptyset$ by Proposition 2.1 (1), which contradicts the condition $\Omega(C)\in\mathscr{N}_2$. We get $[M\perp d]\cong\mathbb{U}\oplus\mathbb{E}_8(2)$ and hence $M^\perp\cap d^\perp=[M\perp d]^\perp\cong\mathbb{U}^{\oplus 2}\oplus\mathbb{E}_8(2)$.

Since $d\in\Delta''_{M^\perp}$, we get $\big(r(M),l(M)\big)=\big(r([M\perp d])-1,l([M\perp d])+1\big)=(9,9)$ by Proposition 11.6 below. Since $r(M)=9$, we get $\delta(M)=1$. All together, we get $\big(r(M),l(M),\delta(M)\big)=(9,9,1)$ and hence $M\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$. Set $L:=\mathbf{Z}d\subset M^\perp$. Then $L\cong\mathbb{A}_1$. Since $\big(r(M^\perp),l(M^\perp),\delta(M^\perp)\big)=(13,9,1)$, we get the decomposition

$$M^\perp\cong(M^\perp\cap d^\perp)\oplus L\cong\mathbb{U}^{\oplus 2}\oplus\mathbb{E}_8(2)\oplus L$$

by comparing the triplets (r,l,δ), which implies $\mathbf{1}_{M^\perp}=\mathbf{1}_{M^\perp\cap d^\perp}\oplus\mathbf{1}_L=\mathbf{1}_L=d/2$ in $A_{M^\perp}$.

Conversely, assume $M\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$ and $d/2\equiv\mathbf{1}_{M^\perp}\bmod M^\perp$. Since $d/2\in(M^\perp)^\vee$, we get $d\in\Delta''_{M^\perp}$. Since $\big(r(M^\perp),l(M^\perp)\big)=\big(r(M^\perp\cap d^\perp)+1,l(M^\perp\cap d^\perp)+1\big)$ by Proposition 11.6 (3), (4) below, we get

$$r(M^\perp\cap d^\perp)=r(M^\perp)-1=12\quad\text{and}\quad l(M^\perp\cap d^\perp)=l(M^\perp)-1=8.$$

Since $\delta(M^\perp)=1$, we get $M^\perp=(M^\perp\cap d^\perp)\oplus L$ by comparing (r,l,δ). Let us see that $\delta(M^\perp\cap d^\perp)=0$. Let $x\in(M^\perp\cap d^\perp)^\vee$ be an arbitrary element and let $k\in\mathbf{Z}$. Set $y=x+k(d/2)\in(M^\perp)^\vee=(M^\perp\cap d^\perp)^\vee\oplus L^\vee$. Since $\mathbf{1}_{M^\perp}\equiv d/2\bmod M^\perp$, we get by the definition of $\mathbf{1}_{M^\perp}$

$$-k/2=\langle y,d/2\rangle\equiv\langle y,\mathbf{1}_{M^\perp}\rangle\equiv\langle y,y\rangle\equiv\langle x,x\rangle-k^2/2\bmod\mathbf{Z}.$$

Hence $x^2\equiv k(k-1)/2\equiv 0\bmod\mathbf{Z}$, which implies $\delta(M^\perp\cap d^\perp)=0$. Since

$$(r,l,\delta)=(12,8,0)\quad\text{for } M^\perp\cap d^\perp,$$

we get $M^\perp\cap d^\perp\cong\mathbb{U}^{\oplus 2}\oplus\mathbb{E}_8(2)$ and hence $[M\perp d]\cong\mathbb{U}\oplus\mathbb{E}_8(2)$. By Theorem 2.5, $J_M(H_d^o)\subset J^o_{[M\perp d]}(\Omega^o_{[M\perp d]})\subset\mathscr{N}_2$, where the last inclusion follows from Proposition 2.1 (2). Hence $J_M(H_d^o)\cap\mathscr{N}_2\neq\emptyset$. $\square$

3. Log del Pezzo surfaces and 2-elementary $K3$ surfaces

In this section, we recall the notion of log del Pezzo surfaces of index $\leqq 2$ and DNP surfaces, for which we refer the reader to [1] and [44]. In Section 3, the canonical divisor of a normal complex surface S is denoted by K_S. Hence the canonical line bundle of S is denoted by $\mathcal{O}_S(K_S)$ in stead of Ω_S^2 in this section.

3.1. Log del Pezzo surfaces of index 2 and DNP surfaces. A normal projective surface S is a *log del Pezzo surface* if it has only log terminal singularities and if its anti-canonical divisor $-K_S$ is an ample $\mathbf{Q}$-Cartier divisor. The *index* of S is the smallest integer $v \in \mathbf{Z}_{>0}$ such that $-vK_S$ is Cartier, [1], Section 1.

A smooth projective surface Y is a *DNP surface* if $h^1(Y) = 0$, $K_Y \not\sim 0$ and if there exists an effective divisor $C \in |-2K_Y|$ with only simple singularities, [1], Section 2.1. A DNP surface Y is rational if $|-2K_Y| \neq \emptyset$. If Y is a DNP surface and if $C \in |-2K_Y|$ is a smooth divisor, the pair (Y, C) is called a right DNP pair.

Let S be a log del Pezzo surface of index 2. By [1], Theorem 1.5, $|-2K_S|$ contains a smooth curve. Let $C \in |-2K_S|$ be smooth. To the pair (S, C), one can associate a right DNP pair and a 2-elementary $K3$ surface as follows ([1], Section 2.1, [44], Section 6.6).

Let $\alpha : \tilde{S} \to S$ be the minimal resolution. Since S has only log terminal singularities of index 2, we deduce from [1], Section 1.2, the existence of a non-zero α-exceptional simple normal crossing divisor E on $\tilde{S}$ such that $-2K_{\tilde{S}} \sim \alpha^*(-2K_S) + E$. If D is a connected component of E, the germ $\big(S, \alpha(D)\big) \in \operatorname{Sing} S$ is isomorphic to one of the singularities K_n in [1], Section 1.2, [44], Example 4.17.

Let $\beta : Y \to \tilde{S}$ be the blowing-up at the nodes of E. By [1], Section 1.2, the proper transform E_Y of E is the disjoint union of (-4)-curves on Y and the total transform $\beta^* E$ is the disjoint union of the configurations in [1], Section 1.5 (9). Set $p := \alpha \circ \beta$. The birational morphism $p : Y \to S$ is called the *right resolution* of S.

Let $C_Y := p^{-1}(C) \subset Y$ be the total transform of C with respect to the birational morphism $p : Y \to S$. Since $C \in |-2K_S|$ is smooth and hence $C \cap \operatorname{Sing} S = \emptyset$, $p|_{C_Y} : C_Y \to C$ is an isomorphism. By [1], Section 2.1, [44], p. 415, Eq. (6-1), Y is a DNP surface and the pair $(Y, C_Y + E_Y)$ is a right DNP pair. We call $(Y, C_Y + E_Y)$ *the right DNP pair associated to* (S, C).

Since $C_Y + E_Y \in |-2K_Y|$, there exists a double covering $\pi : X \to Y$ with branch divisor $C_Y + E_Y$. Let $\iota : X \to X$ be the non-trivial covering transformation of $\pi : X \to Y$. By [1], Section 2.1, [44], Section 6.6, (X, ι) is a 2-elementary $K3$ surface such that $X^\iota \cong C_Y + E_Y$. We call (X, ι) *the* 2*-elementary* $K3$ *surface associated to* (S, C). In this case, $g(C) = g(X^\iota) \geqq 2$ by [1], Theorem 4.1.

Conversely, if (X, ι) is a 2-elementary $K3$ surface with $g(X^\iota) \geqq 2$, then $(X/\iota, X^\iota)$ is a right DNP pair. By [1], Theorem 4.1, there exists a unique pair (S, C), where S is a log del Pezzo surface of index $\leqq 2$ and $C \in |-2K_S|$ is a smooth member, such that (X, ι) is associated to (S, C).

3.2. Some properties of the main component of X^ι. Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Assume that M is non-exceptional and that $g(M) \geqq 1$. Recall that if (X, ι) is a 2-elementary $K3$ surface of type M and if $X^\iota = C \amalg E_1 \amalg \cdots \amalg E_k$ denotes the decomposition into the connected components with $g(C) = g(M) \geqq 1$ and $E_i \cong \mathbf{P}^1$ for $1 \leqq i \leqq k$, then C is called the *main component* of X^ι.

Proposition 3.1. *Assume that* $r(M) > 10$ *or* $(r(M), \delta(M)) = (10, 1)$. *Then*

$$0 \leqq g(M) \leqq 5$$

and the following hold:

(1) *If* $g(M) = 3$, *then there exists a* 2-*elementary* $K3$ *surface* (X, ι) *of type* M *such that the main component of* X^ι *is non-hyperelliptic.*

(2) *If* $g(M) = 4$, *then there exists a* 2-*elementary* $K3$ *surface* (X, ι) *of type* M *such that the main component of* X^ι *is isomorphic to the complete intersection of a smooth quadric and a* (*possibly singular*) *cubic in* $\mathbf{P}^3$.

(3) *If* $g(M) = 5$, *then there exists a* 2-*elementary* $K3$ *surface* (X, ι) *of type* M *such that the main component of* X^ι *is the normalization of an irreducible plane quintic with one node.*

Proof. By [48], p. 1434, Table 1, and the assumption on M, we get $0 \leqq g(M) \leqq 5$.

In what follows, we use Nakayama's notation [44], p. 410, Table 6, and [44], pp. 494–495, Table 14, for the type of log del Pezzo surfaces of index $\leqq 2$. See [44], p. 410, Tables 9, for the relation between the type of a log del Pezzo surface and the type of the associated 2-elementary $K3$ surface.

Let S be a log del Pezzo surface and let $\Gamma \in |-2K_S|$ be a non-singular member. Let $M \subset \mathbb{L}_{K3}$ be the type of the 2-elementary $K3$ surface (X, ι) associated to (S, Γ). The main component of X^ι is isomorphic to Γ by construction.

(1) Since $g(M) = 3$, $r(M) \geqq 10$ and $(r(M), \delta(M)) \neq (10, 0)$, the type of S is one of $[2]_+(b)$ $(0 \leqq b \leqq 4)$ by [44], p. 410, Table 6 and p. 444 Table 9. By [44], pp. 494–495, Table 14, S is a hypersurface of $\mathbf{P}(1, 1, 1, 2)$ defined by the following equation:

Lattice $M^\perp$	Type of S	Equation defining S
$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{\oplus 4}$	$[2]_+(0)$	$xyu = z^4 + F_3(x, z)x + G_3(y, z)y,$
$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{\oplus 4-b}$	$[2]_+(b)$ $(1 \leqq b \leqq 4)$	$xyu = F_{4-b}(x, z)x^b + G_3(y, z)y,$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(u) = 2$ and $F_k(x, y) \in \mathbf{C}[x, y]$ is a homogeneous polynomial of degree k and $G_3(x, y) \in \mathbf{C}[x, y]$ is a homogeneous polynomial of degree 3. Since $\mathcal{O}_S(-2K_S) \cong \mathcal{O}_{\mathbf{P}(1,1,1,2)}(2)|_S$ by the adjunction formula and hence

$$u - U(x, y, z) \in H^0(S, \mathcal{O}_S(-2K_S))$$

for a homogeneous polynomial $U(x,y,z) \in \mathbf{C}[x,y,z]$ of degree 2, a general member of the linear system $|-2K_S|$ is a hypersurface of $\mathbf{P}^2$ defined by the following equation:

Type of S	Equation defining Γ
$[2]_+(0)$	$xyU(x,y,z) = z^4 + F_3(x,z)x + G_3(y,z)y,$
$[2]_+(b)$ $(1 \leqq b \leqq 4)$	$xyU(x,y,z) = F_{4-b}(x,z)x^b + G_3(y,z)y.$

In particular, $\Gamma \in |-2K_S|$ is a smooth plane quartic if $F_k(x,y)$, $G_3(x,y)$ and $U(x,y,z)$ are sufficiently general. Since a smooth plane quartic curve is non-hyperelliptic, we get (1).

(2) Since $g(M) = 4$, $r(M) \geqq 10$ and $(r(M), \delta(M)) \neq (10,0)$, the type of S is one of $[0;1,1]_+(b)$ $(1 \leqq b \leqq 3)$ by [44], p. 410, Table 6 and p. 444, Table 9. By [44], pp. 494–495, Table 14, S is a complete intersection of $\mathbf{P}(1,1,1,1,2)$ defined by the following equations:

Lattice $M^\perp$	Type of S	Equations defining S
$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus 2}$	$[0;1,1]_+(1)$	$\begin{cases} xw = yz, \\ xu = (w+cz)zw + (w+c'y)yw, \end{cases}$
$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus 3-b}$	$[0;1,1]_+(b)$ $(b \geqq 2)$	$\begin{cases} xw = yz, \\ xu = (w+cz)zw + w^{3-b}y^b, \end{cases}$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = \mathrm{wt}(w) = 1$, $\mathrm{wt}(u) = 2$ and $c, c' \in \mathbf{C}$ are constants. Since $\mathcal{O}_S(-2K_S) \cong \mathcal{O}_{\mathbf{P}(1,1,1,1,2)}(2)|_S$ by the adjunction formula or by Lemma 3.2 below and hence $u - U(x,y,z,w) \in H^0(S, \mathcal{O}_S(-2K_S))$ for a homogeneous polynomial

$$U(x,y,z,w) \in \mathbf{C}[x,y,z,w]$$

of degree 2, $\Gamma \in |-2K_S|$ is a complete intersection of $\mathbf{P}^3$ defined by the following equations:

Type of S	Equations defining Γ
$[0;1,1]_+(1)$	$xw = yz, \quad xU(x,y,z,w) = (w+cz)zw + (w+c'y)yw,$
$[0;1,1]_+(b)$ $(2 \leqq b \leqq 3)$	$xw = yz, \quad xU(x,y,z,w) = (w+cz)zw + w^{3-b}y^b.$

By choosing c, c', $U(x,y,z,w)$ sufficiently general, Γ is a complete intersection in $\mathbf{P}^3$ of a smooth quadric and a (possibly singular) cubic. This proves (2).

(3) *Step* 1. Since $g(M) = 5$, $r(M) \geqq 10$ and $(r(M), \delta(M)) \neq (10,0)$, the type of S is one of $[1;1,1]_+(1,b)$ $(2 \leqq b \leqq 3)$ by [44], p. 410, Table 6 and p. 444, Table 9. By [44], pp. 494–495, Table 14, S is a subvariety of the weighted projective space $\mathbf{P}(1,1,2,2,4)$ defined by the following equations:

Lattice $M^\perp$	Type of S	Equations defining S
$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus 3-b}$	$[1;1,1]_+(b)$ $(2 \leqq b \leqq 3)$	$\begin{cases} xw = yz, \\ z^2w = (xu - y^{2b-1}w^{3-b})x, \\ zw^2 = (xu - y^{2b-1}w^{3-b})y, \end{cases}$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = \mathrm{wt}(w) = 2$, $\mathrm{wt}(u) = 4$. Notice that S is *not* a complete intersection in $\mathbf{P}(1,1,2,2,4)$.

Since $\mathcal{O}_S(-2K_S) \cong \mathcal{O}_{\mathbf{P}(1,1,2,2,4)}(4)|_S$ by Lemma 3.2 below and hence

$$u - U(x,y,z,w) \in H^0\big(S, \mathcal{O}_S(-2K_S)\big)$$

for a weighted homogeneous polynomial $U(x,y,z,w) \in \mathbf{C}[x,y,z,w]$ of degree 4, Γ is a subvariety of $\mathbf{P}(1,1,2,2)$ defined by the following equations:

Type of S	Equations defining Γ
$[1;1,1]_+(b)$ $(2 \leqq b \leqq 3)$	$\begin{cases} xw = yz, \\ z^2 w = (xU(x,y,z,w) - y^{2b-1}w^{3-b})x, \\ zw^2 = (xU(x,y,z,w) - y^{2b-1}w^{3-b})y. \end{cases}$

Step 2. Set $\Sigma_1 := \{(x:y:z:w) \in \mathbf{P}(1,1,2,2); xw = yz\}$. By [44], Lemma 7.6, $\Sigma_1 \cong \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)\big)$ is a Hirzebruch surface, which contains Γ as an irreducible divisor. The projection $p : \Sigma_1 \to \mathbf{P}^1$ is given by the formula

$$p : \Sigma_1 \ni (x:y:z:w) \to (x:y) = (z:w) \in \mathbf{P}^1.$$

By [44], Lemma 7.6, the negative section σ of $p : \Sigma_1 \to \mathbf{P}^1$ is given by

$$\sigma : \mathbf{P}^1 \ni (z:w) \to (0:0:z:w) \in \Sigma_1.$$

Let ℓ and C be the divisors on Σ_1 defined as

$$\ell := \{(0:y:0:w) \in \Sigma_1; (y:w) \in \mathbf{P}(1,2)\} = p^{-1}(0:1) \subset \Sigma_1,$$

$$C := \big\{(x:y:z:w) \in \Sigma_1; z^2 w = \big(xU(x,y,z,w) - y^{2b-1}w^{3-b}\big)x\big\} \subset \Sigma_1.$$

We have the equation of divisors $C = \Gamma + \ell$. Since $\mathcal{O}_{\mathbf{P}(1,1,2,2)}(2)|_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}(\sigma + 2\ell)$ by [44], Lemma 7.6, and since $z^2w - \{xU(x,y,z,w) - y^{2b-1}w^{3-b}\}x$ is an element of $H^0\big(\mathbf{P}(1,1,2,2), \mathcal{O}_{\mathbf{P}(1,1,2,2)}(6)\big)$, we get

$$\Gamma + \ell = C = \mathrm{div}\big(z^2w - \{xU(x,y,z,w) - y^{2b-1}w^{3-b}\}x\big) \in \big|\mathcal{O}_{\Sigma_1}\big(3(\sigma + 2\ell)\big)\big|.$$

Hence $\Gamma = C - \ell \in |\mathcal{O}_{\Sigma_1}(3\sigma + 5\ell)|$. Regard $H^2(\Sigma_1, \mathbf{Z})$ as the Neron–Severi lattice of Σ_1. Then we have the equations $\sigma \cdot \sigma = -1$, $\ell \cdot \ell = 0$ and $\sigma \cdot \ell = 1$. Since Γ is linearly equivalent to $3\sigma + 5\ell$, we get $\Gamma \cdot \Gamma = 21$, $\Gamma \cdot \sigma = 2$.

Step 3. Let $\pi : \Sigma_1 \to \mathbf{P}^2$ be the blowing-down of the (-1)-curve σ and set $\bar{\Gamma} := \pi(\Gamma)$. Let $\mu := \mathrm{mult}_{\pi(\sigma)} \bar{\Gamma}$ be the multiplicity of $\bar{\Gamma}$ at $\pi(\sigma)$. Then $\mu = \Gamma \cdot \sigma = 2$, so that $\pi(\sigma)$ is a double point of $\bar{\Gamma}$. Since Γ is smooth and since $\pi : \Gamma \backslash \sigma \to \bar{\Gamma} \backslash \{\pi(\sigma)\}$ is an isomorphism, $\bar{\Gamma}$ has a unique singular point at $\pi(\sigma)$ and $\pi|_{\Gamma} : \Gamma \to \bar{\Gamma}$ is the normalization. Since $(\deg \bar{\Gamma})^2 = \bar{\Gamma} \cdot \bar{\Gamma} = \Gamma \cdot \Gamma + \mu^2 = 25$, we get $\deg \bar{\Gamma} = 5$.

Since $\pi : \Sigma_1 \to \mathbf{P}^2$ is the blowing-down of σ and since $\bar{\Gamma} = \pi(\Gamma)$, $\pi(\sigma)$ is a node of $\bar{\Gamma}$ if and only if Γ intersects σ transversally at two different points. Since $\Gamma \cdot \sigma = 2$, $\pi(\sigma)$ is a node of $\bar{\Gamma}$ if and only if $\#(\Gamma \cap \sigma) = 2$. By the definitions of Γ and σ,

$$\#(\Gamma \cap \sigma) = \#\{(x : y : z : w) \in \Gamma; x = y = 0\}$$
$$= \#\{(0 : 0 : z : w) \in \mathbf{P}(1,1,2,2); zw = 0\} = 2.$$

Hence $\overline{\Gamma} \subset \mathbf{P}^2$ is a quintic with one node, and Γ is the normalization of $\overline{\Gamma}$. □

3.3. Some log del Pezzo surfaces of index 2. Let n be an integer with $0 \leqq n \leqq 3$. Let $(x : y : z : w : u)$ be the system of homogeneous coordinates of the weighted projective space $\mathbf{P}(1,1,n+1,n+1,2(n+1))$ with weights

$$\mathrm{wt}(x) = \mathrm{wt}(y) = 1, \quad \mathrm{wt}(z) = \mathrm{wt}(w) = n+1, \quad \mathrm{wt}(u) = 2(n+1).$$

Set

$$W := \{(x : y : z : w : u) \in \mathbf{P}(1,1,n+1,n+1,2(n+1)); xw = yz\}.$$

In [44], Proposition 7.13, Nakayama gave a system of homogeneous polynomials that defines a log del Pezzo surface of index 2 as a subvariety of W.

Lemma 3.2. *Let $S \subset W$ be a log del Pezzo surface of index* 2 *as in* [44], *Proposition* 7.13. *Then the following isomorphism of holomorphic line bundles on S holds*:

$$\mathcal{O}_{\mathbf{P}(1,1,n+1,n+1,2(n+1))}(2(n+1))|_S \cong \mathcal{O}_S(-2K_S).$$

Proof. Let $\mathscr{F}$ be the vector bundle of rank 2 over $\mathbf{P}(1,1,n+1,n+1)$ defined as

$$\mathscr{F} := \mathcal{O}_{\mathbf{P}(1,1,n+1,n+1)} \oplus \mathcal{O}_{\mathbf{P}(1,1,n+1,n+1)}(2(n+1)).$$

Let $\mathbf{P}(\mathscr{F}) \to \mathbf{P}(1,1,n+1,n+1)$ be the $\mathbf{P}^1$-bundle associated with $\mathscr{F}$ and let $\mathcal{O}_{\mathbf{P}(\mathscr{F})}(1) \to \mathbf{P}(\mathscr{F})$ be the tautological quotient line bundle. Let

$$\Psi : \mathbf{P}(\mathscr{F}) \to \mathbf{P}(1,1,n+1,n+1,2(n+1))$$

be the birational morphism as in [44], Lemma 7.5.

Set $\Sigma_n := \{(x : y : z : w) \in \mathbf{P}(1,1,n+1,n+1); xw = yz\}$. By [44], Lemma 7.6, $\Sigma_n \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n+1))$ is a Hirzebruch surface. We set

$$\mathscr{E} := \mathscr{F}|_{\Sigma_n}, \quad \mathbf{P}(\mathscr{E}) := \mathbf{P}(\mathscr{F})|_{\Sigma_n}, \quad \mathcal{O}_{\mathbf{P}(\mathscr{E})}(1) := \mathcal{O}_{\mathbf{P}(\mathscr{F})}(1)|_{\mathbf{P}(\mathscr{E})}.$$

Then $\mathbf{P}(\mathscr{E}) \to \Sigma_n$ is the $\mathbf{P}^1$-bundle associated with $\mathscr{E}$, and $\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1) \to \mathbf{P}(\mathscr{E})$ is the tautological quotient line bundle. Set $\Phi := \Psi|_{\mathbf{P}(\mathscr{E})}$. Then $\Phi(\mathbf{P}(\mathscr{E})) = W$ by [44], Proposition 7.13. By [44], Proposition 7.8, $\Phi : \mathbf{P}(\mathscr{E}) \to W$ is a birational morphism. By [44], p. 461, l. 10, we have $\Psi^*\mathcal{O}_{\mathbf{P}(1,1,n+1,n+1,2(n+1))}(2(n+1)) \cong \mathcal{O}_{\mathbf{P}(\mathscr{F})}(1)$ and hence

$$\text{(3.1)} \qquad \Phi^*\mathcal{O}_{\mathbf{P}(1,1,n+1,n+1,2(n+1))}(2(n+1))|_W \cong \mathcal{O}_{\mathbf{P}(\mathscr{F})}(1)|_{\mathbf{P}(\mathscr{E})} = \mathcal{O}_{\mathbf{P}(\mathscr{E})}(1).$$

Let $V \subset \mathbf{P}(\mathscr{E})$ be the proper transform of S with respect to the birational morphism $\Phi : \mathbf{P}(\mathscr{E}) \to W$. We set $\varphi := \Phi|_V$ (cf. [44], p. 465, l. 15). Then $\varphi : V \to S$ is a birational morphism. By [44], p. 464, ll. 1–11, we have

$$\varphi^* \mathscr{O}_S(-2K_S) \cong \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)|_V. \tag{3.2}$$

By (3.1) and (3.2), we have an isomorphism of holomorphic line bundles on V:

$$\varphi^* \mathscr{O}_{\mathbf{P}(1,1,n+1,n+1,2(n+1))}\big(2(n+1)\big)|_S \cong \varphi^* \mathscr{O}_S(-2K_S). \tag{3.3}$$

Since $\varphi|_{V\backslash\varphi^{-1}(\operatorname{Sing} S)} : V\backslash\varphi^{-1}(\operatorname{Sing} S) \to S\backslash\operatorname{Sing} S$ is an isomorphism by [44], p. 464, ll. 9–10, and since S is normal, the desired isomorphism follows from (3.3). □

3.4. Even theta-characteristics on the main component of X^ι. Recall that a *theta-characteristic* on a compact Riemann surface C is a half canonical line bundle on C, i.e., a holomorphic line bundle on C whose square is the canonical line bundle of C. A theta-characteristic L is *even* if $h^0(L) \equiv 0 \bmod 2$. A theta-characteristic L is effective if $h^0(L) > 0$. If $g(C)$ denotes the genus of C, there are exactly $2^{g(C)-1}(2^{g(C)}+1)$ even theta-characteristics on C.

Proposition 3.3. *Let C be a compact Riemann surface of genus $g(C)$.*

(1) *If $g(C) \leqq 2$, C has no effective even theta-characteristics.*

(2) *When $g(C) = 3$, C has no effective even theta-characteristics if and only if C is non-hyperelliptic.*

(3) *When $g(C) = 4$, C has no effective even theta-characteristics if and only if C is a complete intersection of a smooth quadric and a cubic in $\mathbf{P}^3$.*

(4) *If C is the normalization of an irreducible plane quintic with one node, then $g(C) = 5$ and C has no effective even theta-characteristics.*

Proof. (1) When $g(C) = 0$, the result is trivial. When $g(C) = 1, 2$, the result follows from, e.g., [43], Chapter IIIa, Proposition 6.1 (iv), since C is hyperelliptic.

(2) The result follows from, e.g., [35], p. 58.

(3) We may assume C to be non-hyperelliptic by [25], p. 258, [43], Chapter IIIa, Proposition 6.1 (iv), Corollary 6.7. The result follows from, e.g., [2], p. 196, A-3, p. 206 and p. 232.

(4) The result follows from [58], Lemma 0.18 (i), (ii). □

Proposition 3.4. *Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. If $r(M) > 10$ or $\big(r(M), \delta(M)\big) = (10, 1)$, then there exists a 2-elementary K3 surface (X, ι) of type M such that X^ι has no effective even theta-characteristics.*

Proof. The result follows from Propositions 3.1 and 3.3. □

4. Automorphic forms on the period domain

4.1. Igusa's Siegel modular form and its pull-back on $\Omega_{M^\perp}$. Let $\mathscr{F}_g$ be the Hodge line bundle on $\mathscr{A}_g$. Then $\mathscr{F}_g$ is an ample line bundle on $\mathscr{A}_g$ in the sense of orbifolds. There is an integer $v \in \mathbf{N}$ such that $\mathscr{F}_g^v$ is a line bundle on $\mathscr{A}_g$ in the ordinary sense and such that $\mathscr{F}_g^{mv}$ extends to a very ample line bundle on $\mathscr{A}_g^*$ for $m \gg 0$. In this case, let $\bar{\mathscr{F}}_g^{mv}$ denote the holomorphic extension of $\mathscr{F}_g^{mv}$ to $\mathscr{A}_g^*$. An element of $H^0(\mathscr{A}_g, \mathscr{F}_g^k)$ is identified with a Siegel modular form on $\mathfrak{S}_g$ for $\mathrm{Sp}_{2g}(\mathbf{Z})$ of weight k. For $g > 0$, we define

$$\chi_g(\Sigma) := \prod_{(a,b)\ \text{even}} \theta_{a,b}(\Sigma), \quad \Sigma \in \mathfrak{S}_g,$$

where $a, b \in \left\{0, \frac{1}{2}\right\}^g$ and $\theta_{a,b}(\Sigma) := \sum_{n \in \mathbf{Z}^g} \exp\{\pi i^t(n+a)\Sigma(n+a) + 2\pi i^t(n+a)b\}$ is the corresponding theta constant. Here (a,b) is *even* if $4^t ab \equiv 0 \bmod 2$. When $g = 0$, we define $\chi_0 := 1$. By [31], Lemma 10, χ_g^8 is a Siegel modular form of weight $2^{g+1}(2^g+1)$. Let $\theta_{\mathrm{null},g}$ be the reduced divisor on $\mathscr{A}_g$ defined as

$$\theta_{\mathrm{null},g} := \{[\Sigma] \in \mathscr{A}_g; \chi_g(\Sigma) = 0\}.$$

It is classical that $\theta_{\mathrm{null},2} = \mathscr{N}_2$. In Section 9, χ_g^8 shall play a crucial role.

Define the *Petersson metric* on $\mathscr{F}_g$ by

$$\|\xi\|^2(\Sigma) := (\det \operatorname{Im} \Sigma)|\xi|^2, \quad (\Sigma, \xi) \in \mathfrak{S}_g \times \mathbf{C}. \tag{4.1}$$

Since χ_g^8 is a Siegel modular form, $\|\chi_g^8\|^2 = (\det \operatorname{Im} \Sigma)^{w(g)} |\chi_g(\Sigma)^8|^2$, $w(g) = 2^{g+1}(2^g+1)$, is a C^∞ function on $\mathscr{A}_g$ in the sense of orbifolds.

Lemma 4.1. *Let $p : \mathscr{C} \to \Delta$ be an ordinary singular family of curves of genus $g > 0$ such that C_0 is irreducible. Let $\mathfrak{o} := \operatorname{Sing} C_0$.*

(1) *There exists a holomorphic function $h(t) \in \mathcal{O}(\Delta)$ such that*

$$\log\|\chi_g(\Omega(C_t))^8\|^2 = 2^{2g-2}\log|t|^2 + \log|h(t)|^2 + O(\log\log|t|^{-1}) \quad (t \to 0).$$

(2) *If $g = 1$ or $g = 2$, then $h(0) \neq 0$.*

Proof. We follow [41], p. 370, Section 3. For $\Sigma \in \mathfrak{S}_g$, we write $\Sigma = \begin{pmatrix} z & {}^t\omega \\ \omega & Z \end{pmatrix}$, where $z \in \mathfrak{H}$, $\omega \in \mathbf{C}^{g-1}$, $Z \in \mathfrak{S}_{g-1}$.

(1) Since C_0 is an irreducible curve of arithmetic genus $g > 0$ with one node, the normalization of C_0 is a smooth curve of genus $g-1$. By [20], Corollary 3.8, there exists a holomorphic function $\psi(t)$ on Δ with values in complex symmetric $g \times g$-matrices such that

$$\Omega(C_t) = \left[\frac{\log t}{2\pi i} A + \psi(t)\right] \in \mathscr{A}_g, \quad A = \begin{pmatrix} 1 & {}^t\mathbf{0}_{g-1} \\ \mathbf{0}_{g-1} & O_{g-1} \end{pmatrix}. \tag{4.2}$$

Write $\psi(0) = \begin{pmatrix} \psi_0 & {}^t\omega_0 \\ \omega_0 & Z_0 \end{pmatrix}$. Then $Z_0 \in \mathfrak{S}_{g-1}$ and $\lim_{t \to 0} \Omega(C_t) = [Z_0] \in \mathscr{A}_{g-1} \subset \mathscr{A}_g^*$.

For $a, b \in \left\{0, \frac{1}{2}\right\}^g$, write $a = (a_1, a')$, $b = (b_1, b')$, where

$$a_1, b_1 \in \left\{0, \frac{1}{2}\right\}, \quad a', b' \in \left\{0, \frac{1}{2}\right\}^{g-1}.$$

Let $a_1 = \frac{1}{2}$. There is a holomorphic function $f_{a',b'}(\zeta, \omega, Z)$ such that

$$\begin{aligned}(4.3)\qquad \theta_{a,b}(\Sigma) &= \sum_{n=(n_1,n')\in\mathbf{Z}\times\mathbf{Z}^{g-1}} e^{\pi i\left(n_1+\frac{1}{2}\right)^2 z + 2\pi i\left(n_1+\frac{1}{2}\right){}^t\omega(n'+a') + \pi i{}^t(n'+a')Z(n'+a') + 2\pi i{}^t(n+a)b} \\ &= e^{\frac{\pi i z}{4}}\{e^{-\pi i b_1}\theta_{a',b'}(-\omega/2, Z) + e^{\pi i b_1}\theta_{a',b'}(\omega/2, Z) \\ &\qquad + e^{2\pi i z} f_{a',b'}(e^{2\pi i z}, \omega, Z)\} \\ &= e^{\frac{\pi i z}{4}}\{2 i^{2b_1}\theta_{a',b'}(\omega/2, Z) + e^{2\pi i z} f_{a',b'}(e^{2\pi i z}, \omega, Z)\},\end{aligned}$$

where we used $4{}^t ab \in 2\mathbf{Z}$ and the identity $\theta_{a',b'}(-\omega/2, Z) = (-1)^{4{}^t a'b'}\theta_{a',b'}(\omega/2, Z)$ to get the third equality; see [42], p. 167, Proposition 3.14. The number of even (a, b) with $a_1 = 1/2$ is given by $2^{2(g-1)}$.

Similarly, let $a_1 = 0$. Then the pair (a', b') must be even. There is a holomorphic function $g_{a',b'}(\zeta, \omega, Z)$ such that

$$\begin{aligned}(4.4)\qquad \theta_{a,b}(\Sigma) &= \sum_{n=(n_1,n')\in\mathbf{Z}\times\mathbf{Z}^{g-1}} e^{\pi i n_1^2 z + 2\pi i n_1{}^t\omega(n'+a') + \pi i{}^t(n'+a')Z(n'+a') + 2\pi i{}^t(n+a)b} \\ &= (-1)^{2{}^t a'b'}\theta_{a',b'}(Z) + e^{\pi i z} g_{a',b'}(e^{\pi i z}, \omega, Z).\end{aligned}$$

By (4.3), (4.4), there is a holomorphic function $F(\zeta, \omega, Z)$ such that

$$\begin{aligned}(4.5)\qquad \chi_g(\Sigma)^8 &= \prod_{(a,b)\text{ even}} \theta_{a,b}(\Sigma)^8 = (e^{\frac{\pi i z}{4}})^{8\cdot 2^{2(g-1)}} F(e^{\pi i z}, \omega, Z) \\ &= (e^{2\pi i z})^{2^{2g-2}} F(e^{\pi i z}, \omega, Z).\end{aligned}$$

Since χ_g^8 is a Siegel modular form and hence $\chi_g(\Omega + A)^8 = \chi_g(\Omega)^8$, we have that $F(\zeta, \omega, Z)$ is an even function in ζ. By (4.2), $z = (\log t)/2\pi i + \psi_{11}(t)$ for some $\psi_{11}(t) \in \mathcal{O}(\Delta)$. Hence $\exp(2\pi i z) = t\exp\bigl(2\pi i\psi_{11}(t)\bigr)$. By (4.5), there exists $h(t) \in \mathcal{O}(\Delta)$ such that

$$(4.6)\qquad \chi_g\left(\frac{\log t}{2\pi i}A + \psi(t)\right)^8 = t^{2^{2g-2}} h(t).$$

Since

$$\operatorname{Im}\left(\frac{\log t}{2\pi i}A + \psi(t)\right) = \left(-\frac{1}{2\pi}\log|t|\right)A + \operatorname{Im}\psi(0) + O(|t|)$$

with $\psi(0) = \begin{pmatrix} \psi_0 & {}^t\omega_0 \\ \omega_0 & Z_0 \end{pmatrix}$, $Z_0 \in \mathfrak{S}_{g-1}$, we get

$$\det \operatorname{Im}\left(\frac{\log t}{2\pi i} A + \psi(t)\right) = -\frac{\det \operatorname{Im} Z_0}{2\pi} \log|t| + O(1). \tag{4.7}$$

By (4.2), (4.6), (4.7), we get (1).

(2) Let $g = 1$. Since $p : \mathscr{C} \to \Delta$ is an ordinary singular family of elliptic curves, $(\Delta, 0)$ is regarded as a local coordinate of $\mathscr{A}_1^*$ centered at the cusp $+i\infty$. Since $\chi_1^8(\tau) = \eta(\tau)^{24}$ vanishes of order 1 at the cusp of $\mathscr{A}_1^*$, we get (2) in this case.

Let $g = 2$. Then $\omega_0 \in \mathbf{C}$, $Z_0 \in \mathfrak{H}$ and $\theta_{a',b'}(Z_0) \neq 0$ in (4.4). Set $\Lambda_0 := \mathbf{Z} + Z_0\mathbf{Z}$. By (4.3), the assertion (2) follows if $\theta_{a,b}(\omega_0/2, Z_0) \neq 0$ for all $(a,b) \in \{0, 1/2\}$. Since $\operatorname{div} \theta_{a,b}(\cdot, Z_0) = \left[\left(a + \frac{1}{2}\right)Z_0 + \left(b + \frac{1}{2}\right)\right] \in \mathbf{C}/\Lambda_0$ by [42], Lemma 4.1, it suffices to prove that $\frac{\omega_0}{2} \notin \left(\frac{1}{2}\mathbf{Z}\right)Z_0 + \frac{1}{2}\mathbf{Z}$, i.e., $\omega_0 \notin \Lambda_0$. Let $\iota : \hat{C}_0 \to C_0$ be the normalization. Since $\mathfrak{o}$ is the node of C_0, we can write $\iota^{-1}(\mathfrak{o}) = \{\hat{\mathfrak{o}}_1, \hat{\mathfrak{o}}_2\}$ with $\hat{\mathfrak{o}}_1 \neq \hat{\mathfrak{o}}_2$. By [20], p. 53, Corollary 3.8, there exist a symplectic basis $\{\alpha, \beta\}$ of $H_1(\hat{C}_0, \mathbf{Z})$ and a holomorphic 1-form v on $\hat{C}_0$ such that $\int_\alpha v = 1$, $\int_\beta v = Z_0$ and $\int_{\hat{\mathfrak{o}}_1}^{\hat{\mathfrak{o}}_2} v = \omega_0$. Since $\hat{\mathfrak{o}}_1 \neq \hat{\mathfrak{o}}_2$, we get $\omega_0 \notin \Lambda_0$. This proves (2). $\square$

Let $\omega_{\mathfrak{S}_g}$ be the $\mathrm{Sp}_{2g}(\mathbf{Z})$-invariant Kähler form on $\mathfrak{S}_g$ defined as

$$\omega_{\mathfrak{S}_g}(\Sigma) := -dd^c \log \det \operatorname{Im} \Sigma, \quad \Sigma \in \mathfrak{S}_g.$$

Let $\omega_{\mathscr{A}_g}$ be the Kähler form on $\mathscr{A}_g$ in the sense of orbifolds induced from $\omega_{\mathfrak{S}_g}$. Then

$$\omega_{\mathscr{A}_g} = c_1(\mathscr{F}_g, \|\cdot\|).$$

Let $\mathscr{I}(M) \subset \mathbf{Z}$ be the ideal defined as follows: $q \in \mathscr{I}(M)$ if and only if there exists $\overline{\mathscr{F}}_{g(M)}^q \in H^1(\mathscr{A}_{g(M)}^*, \mathscr{O}_{\mathscr{A}_{g(M)}^*}^*)$ with $\overline{\mathscr{F}}_{g(M)}^q|_{\mathscr{A}_{g(M)}} = \mathscr{F}_{g(M)}^q$.

Let $i : \Omega_{M^\perp}^o \cup \mathscr{D}_{M^\perp}^o \hookrightarrow \Omega_{M^\perp}$ be the inclusion. For $q \in \mathscr{I}(M)$, we set

$$\lambda_M^q := i_* \mathscr{O}_{\Omega_{M^\perp}^o \cup \mathscr{D}_{M^\perp}^o}(J_M^* \overline{\mathscr{F}}_{g(M)}^q).$$

By [62], Lemma 3.6, and by Proposition 2.2, the $\mathscr{O}_{\Omega_{M^\perp}}$-module λ_M^q is an invertible sheaf on $\Omega_{M^\perp}$. We identify λ_M^q with the corresponding holomorphic line bundle on $\Omega_{M^\perp}$. By [62], Lemma 3.7, and by Proposition 2.2, the $O(M^\perp)$-action on $\lambda_M^q|_{\Omega_{M^\perp}^o \cup \mathscr{D}_{M^\perp}^o}$ induced from the $O(M^\perp)$-equivariant map J_M extends to the one on λ_M^q. Hence λ_M^q is equipped with the structure of an $O(M^\perp)$-equivariant line bundle on λ_M^q.

Let $\|\cdot\|_{\lambda_M^q}$ be the $O(M^\perp)$-invariant Hermitian metric on $\lambda_M^q|_{\Omega_{M^\perp}^o}$ defined as

$$\|\cdot\|_{\lambda_M^q} := (J_M^o)^* \|\cdot\|.$$

By (4.1), $(J_M^o)^*\omega_{\mathscr{A}_{g(M)}}$ is a C^∞ closed semi-positive $(1,1)$-form on $\Omega_{M^\perp}^o$ such that

$$q(J_M^o)^*\omega_{\mathscr{A}_{g(M)}} = c_1(\lambda_M^q|_{\Omega_{M^\perp}^o}, \|\cdot\|_{\lambda_M^q}).$$

Since $\dim \Omega_{M^\perp} \backslash (\Omega_{M^\perp}^o \cup \mathscr{D}_{M^\perp}^o) \leqq \dim \Omega_{M^\perp} - 2$ when $r(M) \leqq 18$, we can define the closed positive $(1,1)$-current $J_M^*\omega_{\mathscr{A}_{g(M)}}$ on $\Omega_{M^\perp}$ as the trivial extension of $(J_M^o)^*\omega_{\mathscr{A}_{g(M)}}$ from $\Omega_{M^\perp}^o$ to $\Omega_{M^\perp}$ by [56], p. 53, Theorem 1, and [62], Theorem 3.9. When $r(M) = 19$, $(J_M^o)^*\omega_{\mathscr{A}_{g(M)}}$ extends trivially to a closed positive $(1,1)$-current $J_M^*\omega_{\mathscr{A}_{g(M)}}$ on $\Omega_{M^\perp}$, because $(J_M^o)^*\omega_{\mathscr{A}_{g(M)}}$ has Poincaré growth along $\mathscr{D}_{M^\perp}$ by [62], Proposition 3.8. By [56], p. 53, Theorem 1, and [62], Theorem 3.13, the Hermitian metric $\|\cdot\|_{\lambda_M^q}$ on $\lambda_M^q|_{\Omega_{M^\perp}^o}$ extends to a singular Hermitian metric on λ_M^q with curvature current

$$c_1(\lambda_M^q, \|\cdot\|_{\lambda_M^q}) = qJ_M^*\omega_{\mathscr{A}_{g(M)}}. \tag{4.8}$$

Let $\ell \in \mathbf{Z}_{>0}$ be such that $2^{g(M)+1}(2^{g(M)}+1)\ell \in \mathscr{I}(M)$. Then $\mathscr{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$ extends to a holomorphic line bundle on $\mathscr{A}_{g(M)}^*$. Since $\chi_{g(M)}^{8\ell}$ is a holomorphic section of $\mathscr{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$, $J_M^*\chi_{g(M)}^{8\ell}$ is an $O(M^\perp)$-invariant holomorphic section of $\lambda_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}$. If $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null},g(M)}$, we define

$$\mathfrak{D} := \mathrm{div}(J_M^*\chi_{g(M)}^{8\ell}).$$

Since J_M is $O(M^\perp)$-equivariant with respect to the trivial $O(M^\perp)$-action on $\mathscr{A}_{g(M)}^*$, $\mathfrak{D}$ is an $O(M^\perp)$-invariant effective divisor on $\Omega_{M^\perp}$. By [56], p. 53, Theorem 1, [62], Theorem 3.13, and (4.8), $\log\|J_M^*\chi_{g(M)}^8\|$ lies in $L^1_{\mathrm{loc}}(\Omega_{M^\perp})$ and satisfies the following equation of currents on $\Omega_{M^\perp}$:

$$-dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 = 2^{g(M)+1}(2^{g(M)}+1)\ell J_M^*\omega_{\mathscr{A}_{g(M)}} - \delta_{\mathfrak{D}}. \tag{4.9}$$

Recall that the divisor $\mathscr{D}'_{M^\perp}$ was defined in Section 1.4.

Proposition 4.2. *Assume that* $r(M) > 10$ *or* $(r(M), \delta(M)) = (10,1)$ *and that* $g(M) > 0$. *Hence* M *is non-exceptional. Let* $\ell \in \mathbf{Z}_{>0}$ *be such that*

$$2^{g(M)+1}(2^{g(M)}+1)\ell \in \mathscr{I}(M).$$

Then the following hold:

(1) $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null},g(M)}$.

(2) *There exist an integer* $a \in \mathbf{Z}_{\geqq 0}$ *and an* $O(M^\perp)$-*invariant* (*possibly empty*) *effective divisor* E *on* $\Omega_{M^\perp}$ *such that* $\dim(E \cap \mathscr{D}'_{M^\perp}) < \dim \mathscr{D}'_{M^\perp}$ *and*

$$\mathfrak{D} = 2(2^{2g(M)-2}+a)\ell\mathscr{D}'_{M^\perp} + E.$$

In particular, *the following equation of currents on* $\Omega_{M^\perp}$ *holds*:

$$-dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 = 2^{g(M)+1}(2^{g(M)}+1)\ell J_M^*\omega_{\mathscr{A}_{g(M)}} - 2(2^{2g(M)-2}+a)\ell\delta_{\mathscr{D}'_{M^\perp}} - \delta_E.$$

(3) *If* $g(M) = 1$ *or* $g(M) = 2$, *then* $a = 0$ *and* $E = 0$ *in* (2).

Proof. (1) Let (X, ι) be a 2-elementary $K3$ surface of type M and let C be the main component of X^{ι}. By Riemann's theorem [25], p. 338, and Riemann's singularities theorem [40], [25], p. 348, C has an effective even theta-characteristic if and only if $J_M^o(X, \iota) \in \theta_{\mathrm{null}, g(M)}$. Since $r(M) > 10$ or $\big(r(M), \delta(M)\big) = (10, 1)$, there exists by Proposition 3.4 a 2-elementary $K3$ surface (X, ι) of type M such that the main component of X^{ι} has *no* effective even theta-characteristics, i.e., $J_M^o(X, \iota) \notin \theta_{\mathrm{null}, g(M)}$. This proves (1).

(2) Since $\mathfrak{D}$ is an $O(M^{\perp})$-invariant effective divisor on $\Omega_{M^{\perp}}$, we can write

$$\mathfrak{D} = \sum_{d \in \Delta'_{M^{\perp}}} m(d) H_d + E,$$

where $m(d) \in \mathbf{Z}_{\geqq 0}$ and E is an effective divisor on $\Omega_{M^{\perp}}$ with

$$\dim(\mathscr{D}'_{M^{\perp}} \cap E) \leqq \dim \mathscr{D}'_{M^{\perp}} - 1.$$

Since $g(H_d) = H_{g(d)}$ for all $g \in O(M^{\perp})$ and $d \in \Delta'_{M^{\perp}}$, the $O(M^{\perp})$-invariance of $\mathfrak{D}$ implies that $m\big(g(d)\big) = m(d)$ for all $g \in O(M^{\perp})$ and $d \in \Delta'_{M^{\perp}}$. Since $O(M^{\perp})$ acts transitively on $\Delta'_{M^{\perp}}$ by [21], Proposition 3.3, and Proposition 11.6 (5) below, there exists $\alpha \in \mathbf{Z}_{\geqq 0}$ with

$$\mathfrak{D} = \alpha \mathscr{D}'_{M^{\perp}} + E. \tag{4.10}$$

Let $d \in \Delta'_{M^{\perp}}$ and $\mathfrak{p} \in \overline{H}{}_d^o$. Let $\gamma : \Delta \to \mathscr{M}_{M^{\perp}}$ be a holomorphic curve intersecting $\overline{H}{}_d^o$ transversally at $\gamma(0) = \mathfrak{p}$ such that $\gamma(\Delta \backslash \{0\}) \subset \mathscr{M}_M \backslash (\overline{\mathscr{D}}_{M^{\perp}} \cup \mathfrak{D})$. By Theorem 2.3 (1), there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathscr{X}} : (\mathscr{X}, \iota) \to \Delta$ of type M with Griffiths period map γ, such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$.

Since the natural projection $\Pi_{M^{\perp}} : \Omega_{M^{\perp}} \to \mathscr{M}_{M^{\perp}}$ is doubly ramified along H_d^o by [62], Proposition 1.9 (4), there exists a holomorphic curve $c : \Delta \to \Omega_{M^{\perp}}$ intersecting H_d^o transversally at $c(0) \in H_d^o$ such that $\Pi_{M^{\perp}}\big(c(t)\big) = \gamma(t^2)$. Hence we have

$$J_M\big(c(t)\big) = \Omega(Z_{t^2}^{\iota_{t^2}}). \tag{4.11}$$

Since $d \in \Delta'_{M^{\perp}}$, by Theorem 2.3 (2), ι is of type $(2, 1)$. By [62], Proposition 2.5, $p|_{\mathscr{X}^{\iota}} : \mathscr{X}^{\iota} \to \Delta$ is an ordinary singular family of curves. Let $\mathscr{C} \subset \mathscr{X}^{\iota}$ be the connected component such that $C_t := \mathscr{C} \cap Z_t^{\iota_t}$ is the main component of $Z_t^{\iota_t}$ for all $t \in \Delta \backslash \{0\}$. Since the normalization of $Z_0^{\iota_0}$ is given by $(\tilde{Z}_0)^{\tilde{\iota}_0}$, the normalization of C_0 has genus $g(M) - 1$ by Theorem 2.3 (2). Hence C_0 is singular and $p|_{\mathscr{C}} : \mathscr{C} \to \Delta$ is an ordinary singular family of curves. Since the normalization of C_0 has genus $g(M) - 1$ and since C_0 has a unique node as its singular set, C_0 is irreducible.

We apply Lemma 4.1 to the ordinary singular family $p|_{\mathscr{C}} : \mathscr{C} \to \Delta$ with irreducible C_0. Since $\Omega(C_t) = \Omega(Z_t^{\iota_t})$ for all $t \in \Delta \backslash \{0\}$, there exists $h(t) \in \mathscr{O}(\Delta)$ by Lemma 4.1 (1) such that

$$\log \big\| \chi_{g(M)}\big(\Omega(Z_t^{\iota_t})\big)^8 \big\|^2 = 2^{2g(M)-2} \log|t|^2 + \log|h(t)|^2 + O(\log\log|t|^{-1}). \tag{4.12}$$

Since $\gamma(\Delta\backslash\{0\})\cap\mathfrak{D}=\emptyset$ by the choice of γ, $h(t)$ does not vanish identically on Δ by (4.6). Let $a\in\mathbf{Z}_{\geqq 0}$ be the multiplicity of $h(t)$ at $t=0$. By (4.11), (4.12), we get

$$(4.13)\qquad \log\|\chi_{g(M)}(J_M(c(t)))^{8\ell}\|^2=2(2^{2g(M)-2}+a)\ell\log|t|^2+O(\log\log|t|^{-1}),$$

which yields that $H_d\subset\operatorname{supp}\mathfrak{D}$ for $d\in\Delta'_{M^\perp}$. Comparing (4.9), (4.10) and (4.13), we get $\alpha=2(2^{2g(M)-2}+a)\ell$ in (4.10). Since $\mathfrak{D}$ and $\mathscr{D}'_{M^\perp}$ are $O(M^\perp)$-invariant, so is E by (4.10). This proves (2).

(3) Let $g(M)=1$ or $g(M)=2$. By Proposition 3.3 (1), we get the inclusion $\mathfrak{D}\subset\mathscr{D}_{M^\perp}$. This, together with (4.10), implies the inclusion $E\subset\mathscr{D}''_{M^\perp}$. Since $r(M)\geqq 10$ and hence $M\ncong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$, there exists by Propositions 2.8 and 2.10 a dense Zariski open subset U of $\mathscr{D}''_{M^\perp}$ with $J_M(U)\subset\mathscr{A}_{g(M)}\backslash\theta_{\mathrm{null},g(M)}$. By the inclusion $E\subset\mathscr{D}''_{M^\perp}$, we get $J_M(E\cap U)\subset\mathscr{A}_{g(M)}\backslash\theta_{\mathrm{null},g(M)}$. If $E\neq 0$, $J_M^*\chi_{g(M)}^{8\ell}$ would not vanish on the non-empty dense Zariski open subset $E\cap U$ of E, which contradicts the fact that $E\subset\mathfrak{D}=\operatorname{div}(J_M^*\chi_{g(M)}^{8\ell})$. This proves that $E=0$. The equality $a=0$ follows from (4.12), (4.13) and the non-vanishing $h(0)\neq 0$ in Lemma 4.1 (2). This proves the proposition. □

Lemma 4.3. *Let $p:\mathscr{C}\to\Delta$ be an ordinary singular family of curves of genus 2 such that C_0 is the join of two elliptic curves intersecting at one point transversally. Then*

$$\log\|\chi_2(\Omega(C_t))^8\|^2=8\log|t|^2+O(\log\log|t|^{-1})\quad(t\to 0).$$

Proof. Since $g=2$ and C_0 is reducible, we deduce from [20], Corollary 3.8, the existence of a holomorphic map $\psi:\Delta\to\mathfrak{S}_2$ with

$$\Omega(C_t)=[\psi(t)],\quad \psi(0)=\begin{pmatrix}\psi_1&0\\0&\psi_2\end{pmatrix},\quad \psi'(0)=\begin{pmatrix}0&a\\a&0\end{pmatrix},\quad \psi_1,\psi_2\in\mathfrak{H},\quad a\neq 0.$$

The result follows from, e.g., [61], Eq. (A.24). □

Proposition 4.4. *Let $g(M)=2$ and $r(M)<10$, i.e., $M\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$. Let $\mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp},-\frac{1}{2}\right)$ be the Heegner divisor defined as*

$$\mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp},-\frac{1}{2}\right):=\sum_{\{\lambda\in\mathbf{1}_{M^\perp}+M^\perp;\,\lambda^2=-\frac{1}{2}\}/\pm 1}H_\lambda=\sum_{d\in\Delta''_{M^\perp}/\pm 1,\,d/2\in\mathbf{1}_{M^\perp}+M^\perp}H_d.$$

Then the following equation of divisors on $\Omega_{M^\perp}$ holds:

$$\operatorname{div}(J_M^*\chi_2^{8\ell})=8\ell\mathscr{D}'_{M^\perp}+16\ell\mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp},-\frac{1}{2}\right).$$

In particular, the following equations of currents on $\Omega_{M^\perp}$ holds:

$$-dd^c\log\|J_M^*\chi_2^{8\ell}\|^2=40\ell J_M^*\omega_{\mathscr{A}_2}-8\ell\delta_{\mathscr{D}'_{M^\perp}}-16\ell\delta_{\mathscr{H}_{M^\perp}(\mathbf{1}_{M^\perp},-\frac{1}{2})}.$$

Proof. Let $d\in\Delta''_{M^\perp}$ and $d/2=\mathbf{1}_{M^\perp}$. Since $M\cong\mathbb{A}_1^+\oplus\mathbb{A}_1^{\oplus 8}$, we get

$$[M\perp d]\cong\mathbb{U}\oplus\mathbb{E}_8(2)$$

by the proof of Proposition 2.10. Let $\mathfrak{p} \in \overline{H}^o_d$. Let $\gamma : \Delta \to \mathscr{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}^o_d$ transversally at $\gamma(0) = \mathfrak{p}$ such that $\gamma(\Delta \backslash \{0\}) \subset \mathscr{M}_M \backslash (\overline{\mathscr{D}}_{M^\perp} \cup \mathfrak{D})$. By Theorem 2.3 (1), there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathscr{Z}} : (\mathscr{Z}, \iota) \to \Delta$ of type M with Griffiths period map γ, such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d] \cong \mathbb{U} \oplus \mathbb{E}_8(2)$ with Griffiths period $\gamma(0)$. As in the proof of Proposition 4.2 (2), there exists a holomorphic curve $c : \Delta \to \Omega_{M^\perp}$ intersecting H^o_d transversally at $c(0) \in H^o_d$ and satisfying (4.11). If ι is of type $(0, 3)$, then $Z_0^{\iota_0}$ is the disjoint union of a smooth curve of genus 2 and an isolated point by [62], Proposition 2.5, which implies that $J_M\big(c(0)\big) \in \mathscr{A}_2 \backslash \mathscr{N}_2$. By Theorem 2.5, this leads to the contradiction

$$J_M\big(c(0)\big) = J^o_{[M \perp d]}\big(c(0)\big) = J^o_{\mathbb{U} \oplus \mathbb{E}_8(2)}\big(c(0)\big) \in \mathscr{N}_2,$$

where the last inclusion follows from Proposition 2.1 (2). Hence ι is of type $(2, 1)$.

By [62], Proposition 2.5, $p|_{\mathscr{Z}^\iota} : \mathscr{Z}^\iota \to \Delta$ is an ordinary singular family of curves. Since the normalization of $(Z_0)^{\iota_0}$ is isomorphic to $(\tilde{Z}_0)^{\tilde{\iota}_0}$ by Theorem 2.3 (2) and since $(\tilde{Z}_0, \tilde{\iota}_0)$ is of type $[M \perp d] \cong \mathbb{U} \oplus \mathbb{E}_8(2)$, we deduce from Proposition 2.1 (2) that $(Z_0)^{\iota_0}$ is the join of two elliptic curves intersecting at one point transversally. By Lemma 4.3, we get

$$\log\big\|\chi_2\big(\Omega(Z_t^{\iota_t})\big)^8\big\|^2 = 8 \log|t|^2 + O(\log\log|t|^{-1}) \quad (t \to 0). \tag{4.14}$$

By (4.11) and (4.14), we get

$$\log\big\|\chi_2\big(J_M\big(c(t)\big)\big)^8\big\|^2 = 16 \log|t|^2 + O(\log\log|t|^{-1}) \quad (t \to 0). \tag{4.15}$$

By Proposition 2.1 (3), we get $J_M(\Omega^o_{M^\perp}) = J^o_M(\Omega^o_{M^\perp}) \subset \mathscr{A}_2 \backslash \theta_{\mathrm{null},2}$. By Proposition 2.10, we get $J_M\Bigg(\bigcup_{d \in \Delta''_{M^\perp}, d/2 \not\equiv \mathbf{1}_{M^\perp}} H^o_d\Bigg) \subset \mathscr{A}_2 \backslash \theta_{\mathrm{null},2}$. By these two inclusions,

$$J_M\Bigg(\Omega^o_{M^\perp} \cup \bigcup_{d \in \Delta''_{M^\perp}, d/2 \not\equiv \mathbf{1}_{M^\perp}} H^o_d\Bigg) \subset \mathscr{A}_2 \backslash \theta_{\mathrm{null},2},$$

which implies that $J^*_M \chi_2^{8\ell}$ does not vanish on $\Omega^o_{M^\perp} \cup \bigcup_{d \in \Delta''_{M^\perp}, d/2 \not\equiv \mathbf{1}_{M^\perp}} H^o_d$. Hence

$$\begin{aligned}(\Omega^o_{M^\perp} \cup \mathscr{D}^o_{M^\perp}) \cap \mathfrak{D} &\subset (\Omega^o_{M^\perp} \cup \mathscr{D}^o_{M^\perp}) \Bigg\backslash \Bigg(\Omega^o_{M^\perp} \cup \bigcup_{d \in \Delta''_{M^\perp}, d/2 \not\equiv \mathbf{1}_{M^\perp}} H^o_d\Bigg) \\ &= \mathscr{D}^o_{M^\perp} \Bigg\backslash \bigcup_{d \in \Delta''_{M^\perp}, d/2 \not\equiv \mathbf{1}_{M^\perp}} H^o_d \\ &\subset \mathscr{D}'_{M^\perp} \cup \mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp}, -\frac{1}{2}\right).\end{aligned}$$

Since $\Omega_{M^\perp} \backslash (\Omega^o_{M^\perp} \cup \mathscr{D}^o_{M^\perp})$ is an analytic subset of codimension 2 in $\Omega_{M^\perp}$, we get

$$\mathfrak{D} \subset \mathscr{D}'_{M^\perp} \cup \mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp}, -\frac{1}{2}\right). \tag{4.16}$$

Since the proof of Proposition 4.2 (2) works in the case $M \cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$, (4.13) remains valid. Moreover, we get $a = 0$ in (4.13) by Lemma 4.1 (2). The desired formula follows from (4.10), (4.13) with $a = 0$, (4.15), (4.16). $\square$

4.2. Automorphic forms on Ω_Λ^+. Let Λ be a lattice of signature $(2, r(\Lambda) - 2)$. We fix a vector $l_\Lambda \in \Lambda \otimes \mathbf{R}$ with $\langle l_\Lambda, l_\Lambda \rangle \geqq 0$, and we set

$$j_\Lambda(\gamma, [\eta]) := \frac{\langle \gamma(\eta), l_\Lambda \rangle}{\langle \eta, l_\Lambda \rangle}, \quad [\eta] \in \Omega_\Lambda^+, \quad \gamma \in O^+(\Lambda).$$

Since $H_{l_\Lambda} = \emptyset$, $j_\Lambda(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on Ω_Λ^+.

Let $\Gamma \subset O^+(\Lambda)$ be a cofinite subgroup. A holomorphic function $f \in \mathcal{O}(\Omega_\Lambda^+)$ is called an *automorphic form on* Ω_Λ^+ *for* Γ *of weight* p if

$$f(\gamma \cdot [\eta]) = \chi(\gamma) j_\Lambda(\gamma, [\eta])^p f([\eta]), \quad [\eta] \in \Omega_\Lambda^+, \quad \gamma \in \Gamma,$$

where $\chi : \Gamma \to \mathbf{C}^*$ is a unitary character. For an automorphic form f on Ω_Λ^+ for Γ of weight p, the Petersson norm $\|f\|$ is the function on Ω_Λ^+ defined as

$$\|f([\eta])\|^2 := K_\Lambda([\eta])^p |f([\eta])|^2, \quad K_\Lambda([\eta]) := \frac{\langle \eta, \bar{\eta} \rangle}{|\langle \eta, l_\Lambda \rangle|^2}.$$

If $r(\Lambda) \geqq 5$, then $\|f\|^2$ is a Γ-invariant C^∞ function on Ω_Λ^+, because the group $\Gamma/[\Gamma, \Gamma]$ is finite and Abelian and hence χ is finite in this case.

We also consider automorphic forms on $\Omega_{M^\perp}^+$ with values in the sheaf λ_M^q. Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let χ be a character of $O^+(M^\perp)$. Let $p, q \in \mathbf{Z}$. Then $\Psi \in H^0(\Omega_{M^\perp}^+, \lambda_M^q)$ is called an automorphic form on $\Omega_{M^\perp}^+$ for $O^+(M^\perp)$ of weight (p, q) if for all $\gamma \in O^+(M^\perp)$,

$$\Psi(\gamma \cdot [\eta]) = \chi(\gamma) j_{M^\perp}(\gamma, [\eta])^p \gamma\big(\Psi([\eta])\big), \quad [\eta] \in \Omega_{M^\perp}^+.$$

For an automorphic form Ψ on $\Omega_{M^\perp}^+$ for $O^+(M^\perp)$ of weight (p, q), the Petersson norm of Ψ is a C^∞ function on $\Omega_{M^\perp}^+$ defined as

$$\|\Psi([\eta])\|^2 := K_{M^\perp}([\eta])^p \cdot \|\Psi([\eta])\|^2_{\lambda_M^q}, \quad [\eta] \in \Omega_{M^\perp}^+. \tag{4.17}$$

5. The invariant τ_M of 2-elementary $K3$ surfaces of type M

Let (X, ι) be a 2-elementary $K3$ surface of type M. Identify $\mathbf{Z}_2$ with the subgroup of $\mathrm{Aut}(X)$ generated by ι. Let κ be a $\mathbf{Z}_2$-invariant Kähler form on X. Set

$$\mathrm{Vol}(X, \kappa) := (2\pi)^{-2} \int_X \kappa^2/2!.$$

Let η be a nowhere vanishing holomorphic 2-form on X. The L^2-norm of η is defined as

$$\|\eta\|_{L^2}^2 := (2\pi)^{-2} \int_X \eta \wedge \bar{\eta}.$$

Let $\square_{0,q} = (\bar{\partial} + \bar{\partial}^*)^2$ be the $\bar{\partial}$-Laplacian acting on $C^\infty(0,q)$-forms on X. Let $\sigma(\square_{0,q})$ be the spectrum of $\square_{0,q}$. For $\lambda \in \sigma(\square_{0,q})$, let $E_{0,q}(\lambda)$ be the eigenspace of $\square_{0,q}$ with respect to the eigenvalue λ. Since $\mathbf{Z}_2$ preserves κ, $E_{0,q}(\lambda)$ is a finite-dimensional representation of $\mathbf{Z}_2$. For $s \in \mathbf{C}$, set

$$\zeta_{0,q}(\imath)(s) := \sum_{\lambda \in \sigma(\square_{0,q}) \setminus \{0\}} \mathrm{Tr}(\imath|_{E_{0,q}(\lambda)})\lambda^{-s}.$$

Then $\zeta_{0,q}(\imath)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to the complex plane $\mathbf{C}$, and is holomorphic at $s = 0$. The *equivariant analytic torsion* of the trivial Hermitian line bundle on (X, κ) is defined as

$$\tau_{\mathbf{Z}_2}(X, \kappa)(\imath) := \exp\Big[- \sum_{q \geqq 0} (-1)^q q \zeta'_{0,q}(\imath)(0) \Big].$$

We refer to [52], [6], [7], [24], [5], [38], [32] for more about equivariant and non-equivariant analytic torsion.

Let $X^\imath = \sum_i C_i$ be the decomposition of the fixed point set of $\imath$ into the connected components. Let $c_1(C_i, \kappa|_{C_i})$ be the Chern form of $(TC_i, \kappa|_{C_i})$ and let $\tau(C_i, \kappa|_{C_i})$ be the analytic torsion of the trivial Hermitian line bundle on $(C_i, \kappa|_{C_i})$. We define

$$\tau_M(X, \imath) := \mathrm{Vol}\big(X, (2\pi)^{-1}\kappa\big)^{\frac{14 - r(M)}{4}} \tau_{\mathbf{Z}_2}(X, \kappa)(\imath) \prod_i \mathrm{Vol}\big(C_i, (2\pi)^{-1}\kappa|_{C_i}\big) \tau(C_i, \kappa|_{C_i})$$
$$\times \exp\left[\frac{1}{8} \int_{X^\imath} \log\left(\frac{\eta \wedge \bar{\eta}}{\kappa^2/2!} \cdot \frac{\mathrm{Vol}\big(X, (2\pi)^{-1}\kappa\big)}{\|\eta\|^2_{L^2}} \right)\Bigg|_{X^\imath} c_1(X^\imath, \kappa|_{X^\imath}) \right],$$

which is independent of the choice of κ by [62], Theorem 5.7. Hence $\tau_M(X, \imath)$ is an invariant of the pair $(X, \imath)$, so that τ_M descends to a function on $\mathscr{M}^o_{M^\perp}$.

Theorem 5.1. *There exist an integer $v \in \mathbf{Z}_{>0}$ and an automorphic form Φ_M on $\Omega_{M^\perp}$ for $O^+(M^\perp)$ of weight $\big(v(r(M) - 6), 4v\big)$ with zero divisor $v\mathscr{D}_{M^\perp}$ such that for every 2-elementary $K3$ surface $(X, \imath)$ of type M,*

$$\tau_M(X, \imath) = \big\| \Phi_M\big(\varpi_M(X, \imath)\big) \big\|^{-\frac{1}{2v}}.$$

Proof. See [62], Main Theorem, [66], Theorem 1.1, and Proposition 11.2 below. □

6. Borcherds products

6.1. Modular forms for $\mathrm{Mp}_2(\mathbf{Z})$. Recall that $\mathfrak{H} \subset \mathbf{C}$ is the complex upper half-plane. Let $\mathrm{Mp}_2(\mathbf{Z})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbf{Z})$ (cf. [11], Section 2), which is generated by the two elements

$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \in \mathrm{Mp}_2(\mathbf{Z})$ and $\tau \in \mathfrak{H}$, we define

$$j(\gamma, \tau) := \sqrt{c\tau + d} \quad \text{and} \quad \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Let M be an even lattice, $\mathbf{C}[A_M]$ be the group ring of the discriminant group A_M, and $\{\boldsymbol{e}_\gamma\}_{\gamma \in A_M}$ be the standard basis of $\mathbf{C}[A_M]$. The Weil representation

$$\rho_M : \mathrm{Mp}_2(\mathbf{Z}) \to \mathrm{GL}(\mathbf{C}[A_M])$$

is defined as follows ([11], Section 2):

$$\rho_M(T)\boldsymbol{e}_\gamma := e^{\pi i \gamma^2} \boldsymbol{e}_\gamma, \quad \rho_M(S)\boldsymbol{e}_\gamma := \frac{i^{-\sigma(M)/2}}{|A_M|^{1/2}} \sum_{\delta \in A_M} e^{-2\pi i \langle \gamma, \delta \rangle} \boldsymbol{e}_\delta. \tag{6.1}$$

A $\mathbf{C}[A_M]$-valued holomorphic function $F(\tau)$ on $\mathfrak{H}$ is a *modular form of type* ρ_M *with weight* $w \in \frac{1}{2}\mathbf{Z}$ if the following conditions (a) and (b) are satisfied:

(a) For $\gamma \in \mathrm{Mp}_2(\mathbf{Z})$ and $\tau \in \mathfrak{H}$, $F(\gamma \cdot \tau) = j(\gamma, \tau)^{2w} \rho_M(\gamma) \cdot F(\tau)$.

(b) $F(\tau) = \sum_{\gamma \in A_M} \boldsymbol{e}_\gamma \sum_{k \in \frac{1}{l}\mathbf{Z}} c_\gamma(k) e^{2\pi i k \tau}$, where l is the level of M, $c_\gamma(k) \in \mathbf{Z}$ for all $k \in \frac{1}{l}\mathbf{Z}$ and $c_\gamma(k) = 0$ for $k \ll 0$.

By the first condition of (6.1), [14], Eq. (1.4), and condition (a), we get

$$c_\gamma(k) = \begin{cases} 0 & \text{if } k \notin \gamma^2/2 + \mathbf{Z}, \\ c_{-\gamma}(k) & \text{if } k \in \gamma^2/2 + \mathbf{Z}. \end{cases} \tag{6.2}$$

The group $O(M)$ acts on $\mathbf{C}[A_M]$ by $g(\boldsymbol{e}_\gamma) := \boldsymbol{e}_{\bar{g}(\gamma)}$, where $\bar{g} \in O(q_M)$ is the element induced by $g \in O(M)$. For a modular form F of type ρ_M, we define

$$\mathrm{Aut}(M, F) := \{g \in O(M); g(F) = F\}.$$

Then $\mathrm{Aut}(M, F)$ is a cofinite subgroup of $O(M)$, since $\mathrm{Aut}(M, F) \supset \ker\{O(M) \to O(q_M)\}$.

6.2. Borcherds products. Let Λ be an even lattice of signature $(2, r(\Lambda) - 2)$. Assume that $\Lambda = \mathbb{U}(N) \oplus L$, for simplicity. A vector of $\Lambda \otimes \mathbf{Q}$ is denoted by (m, n, v), where $m, n \in \mathbf{Q}$ and $v \in L \otimes \mathbf{Q}$. We write a vector of A_Λ in the same manner. If $F(\tau) = \sum_{\gamma \in A_\Lambda} f_\gamma(\tau) \boldsymbol{e}_\gamma$ is a modular form of type ρ_Λ, then $F(\tau)$ induces a modular form $F|_L(\tau)$ of type ρ_L with the same weight as follows ([10], Theorem 5.3):

$$F|_L(\tau) := \sum_{\lambda \in A_L} f_{L+\lambda}(\tau) \boldsymbol{e}_\lambda, \quad f_{L+\lambda}(\tau) := \sum_{n=0}^{N-1} f_{(\frac{n}{N}, 0, \bar{\lambda})}(\tau). \tag{6.3}$$

Write $F|_L(\tau) = \sum_{\gamma \in A_L} \boldsymbol{e}_\gamma \sum_{k \in \frac{\gamma^2}{2}+\mathbf{Z}} c_{L,\gamma}(k) e^{2\pi i k\tau}$. By [10], Section 6, p. 517, $F|_L(\tau)$ induces a chamber structure of $\mathscr{C}_L^+$:

$$(6.4)\qquad (\mathscr{C}_L^+)^0_{F|_L} := \mathscr{C}_L^+ \setminus \bigcup_{\lambda \in L^\vee,\, \lambda^2 < 0,\, c_{L,\bar{\lambda}}(\lambda^2/2) \neq 0} h_\lambda = \coprod_{\alpha \in A} \mathscr{W}_\alpha,$$

where $h_\lambda = \lambda^\perp = \{v \in L \otimes \mathbf{R}; \langle v, \lambda \rangle = 0\}$ and $\{\mathscr{W}_\alpha\}_{\alpha \in A}$ is the set of connected components of $(\mathscr{C}_L^+)^0_{F|_L}$. Each component $\mathscr{W}_\alpha$ is called a *Weyl chamber* of $F|_L(\tau)$. If $\lambda \in L \otimes \mathbf{R}$ satisfies $\langle \lambda, w \rangle > 0$ for all $w \in \mathscr{W}_\alpha$, we write $\lambda \cdot \mathscr{W}_\alpha > 0$.

Theorem 6.1. *Let $F(\tau) = \sum_{\gamma \in A_\Lambda} \boldsymbol{e}_\gamma \sum_{k \in \frac{\gamma^2}{2}+\mathbf{Z}} c_\gamma(k) e^{2\pi i k\tau}$ be a modular form of type ρ_Λ with weight $\sigma(\Lambda)/2$. Then there exists a meromorphic automorphic form $\Psi_\Lambda(z, F)$ on Ω_Λ^+ for* $\mathrm{Aut}(\Lambda, F) \cap O^+(\Lambda)$ *of weight $c_0(0)/2$ such that*

$$\operatorname{div}\bigl(\Psi_\Lambda(\cdot, F)\bigr) = \frac{1}{2} \sum_{\lambda \in \Lambda^\vee,\, \lambda^2 < 0} c_{\bar{\lambda}}(\lambda^2/2) H_\lambda = \sum_{\lambda \in \Lambda^\vee/\pm 1,\, \lambda^2 < 0} c_{\bar{\lambda}}(\lambda^2/2) H_\lambda.$$

If $\mathscr{W}$ is a Weyl chamber of $F|_L$, then there exists a vector $\varrho(L, F|_L, \mathscr{W}) \in L \otimes \mathbf{Q}$ such that $\Psi_\Lambda(z, F)$ is expressed as the following infinite product near the cusp under the identification (1.2): *For $z \in L \otimes \mathbf{R} + i\mathscr{W}$ with $(\operatorname{Im} z)^2 \gg 0$,*

$$\Psi_\Lambda(z, F) = e^{2\pi i \langle \varrho(L, F|_L, \mathscr{W}), z \rangle} \prod_{\lambda \in L^\vee,\, \lambda \cdot \mathscr{W} > 0} \prod_{n \in \mathbf{Z}/N\mathbf{Z}} \bigl(1 - e^{2\pi i (\langle \lambda, z \rangle + \frac{n}{N})}\bigr)^{c\left(\frac{n}{N}, 0, \bar{\lambda}\right)(\lambda^2/2)}.$$

Proof. See [10], Theorem 13.3, and [14], Theorem 3.22. $\square$

The automorphic form $\Psi_\Lambda(z, F)$ is called the *Borcherds product* or the *Borcherds lift* of $F(\tau)$, and the vector $\varrho(L, F|_L, \mathscr{W})$ is called the *Weyl vector* of $\Psi_\Lambda(z, F)$. See [10], Theorem 10.4, and [11], p. 321, Correction, for an explicit formula for $\varrho(L, F|_L, \mathscr{W})$.

7. 2-elementary lattices and elliptic modular forms

Throughout Section 7, we assume that Λ *is an even*, 2-*elementary lattice*. Set

$$M\Gamma_0(4) := \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \in \mathrm{Mp}_2(\mathbf{Z}); c \equiv 0 \bmod 4 \right\}.$$

Let $w \in \frac{1}{2}\mathbf{Z}$ and let $\chi : M\Gamma_0(4) \to \mathbf{C}^*$ be a character. A holomorphic function $f(\tau)$ on $\mathfrak{H}$ is a modular form for $M\Gamma_0(4)$ of weight w with character χ if the following conditions are satisfied:

(a) $f(\gamma \cdot \tau) = j(\gamma, \tau)^{2w} \chi(\gamma) f(\tau)$ for all $\gamma \in M\Gamma_0(4)$ and $\tau \in \mathfrak{H}$.

(b) $f(\tau) = \sum_{k \in \frac{1}{4}\mathbf{Z}} c(k) e^{2\pi i k\tau}$ with $c(k) = 0$ for $k \ll 0$.

Set $q = e^{2\pi i\tau}$ for $\tau \in \mathfrak{H}$. Let $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind η-function and let $\vartheta_2(\tau)$, $\vartheta_3(\tau)$, $\vartheta_4(\tau)$ be the Jacobi theta functions:

$$\vartheta_2(\tau) = \sum_{n\in\mathbf{Z}} q^{\left(n+\frac{1}{2}\right)^2/2}, \quad \vartheta_3(\tau) = \sum_{n\in\mathbf{Z}} q^{n^2/2}, \quad \vartheta_4(\tau) = \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2/2}.$$

Notice that we use the notation $q = e^{2\pi i\tau}$ while $q = e^{\pi i\tau}$ in [16], Chapter 4. Recall that $\mathbb{A}_1$ is the negative-definite one-dimensional A_1-lattice $\langle -2\rangle$ and $\mathbb{A}_1^+ = \langle 2\rangle$. For $d \in \{0,1\}$, let $\theta_{\mathbb{A}_1^+ + d/2}(\tau)$ be the theta function of $\mathbb{A}_1^+$:

$$\theta_{\mathbb{A}_1^+}(\tau) := \vartheta_3(2\tau), \quad \theta_{\mathbb{A}_1^+ + 1/2}(\tau) := \vartheta_2(2\tau).$$

By [11], Lemma 5.2, there exists a character $\chi_\theta : \mathrm{M}\Gamma_0(4) \to \{\pm 1, \pm i\}$ such that $\theta_{\mathbb{A}_1^+}(\tau)$ is a modular form for $\mathrm{M}\Gamma_0(4)$ of weight $1/2$ with character χ_θ.

For $k \in \mathbf{Z}$, define $f_k^{(0)}(\tau), f_k^{(1)}(\tau) \in \mathcal{O}(\mathfrak{H})$ and $\{c_k^{(0)}(l)\}_{l\in\mathbf{Z}}$, $\{c_k^{(1)}(l)\}_{l\in\mathbf{Z}+k/4}$ by

$$f_k^{(0)}(\tau) := \frac{\eta(2\tau)^8 \theta_{\mathbb{A}_1^+}(\tau)^k}{\eta(\tau)^8 \eta(4\tau)^8} = \sum_{l\in\mathbf{Z}} c_k^{(0)}(l) q^l = q^{-1} + 8 + 2k + O(q),$$

$$f_k^{(1)}(\tau) := -16\frac{\eta(4\tau)^8 \theta_{\mathbb{A}_1^+ + \frac{1}{2}}(\tau)^k}{\eta(2\tau)^{16}} = \sum_{l\in\frac{k}{4}+\mathbf{Z}} 2c_k^{(1)}(l) q^l$$

$$= -2^{k+4} q^{k/4}\{1 + (k+16)q^2 + O(q^4)\}.$$

We define holomorphic functions $g_k^{(i)}(\tau) \in \mathcal{O}(\mathfrak{H})$, $i \in \mathbf{Z}/4\mathbf{Z}$, by

$$g_k^{(i)}(\tau) := \sum_{l\equiv i \bmod 4} c_k^{(0)}(l) q^{l/4}.$$

By definition,

$$\sum_{i\in\mathbf{Z}/4\mathbf{Z}} g_k^{(i)}(\tau) = \frac{\eta(\tau/2)^8 \theta_{\mathbb{A}_1^+}(\tau/4)^k}{\eta(\tau)^8 \eta(\tau/4)^8} = f_k^{(0)}(\tau/4).$$

For a modular form $\phi(\tau)$ of weight l for $\mathrm{M}\Gamma_0(4)$ and for $g \in \mathrm{Mp}_2(\mathbf{Z})$, we define

$$\phi|_g(\tau) := \phi(g\cdot\tau) j(g,\tau)^{-2l}.$$

The following key construction of modular forms of type ρ_Λ is due to Borcherds.

Proposition 7.1. *Let $\phi(\tau)$ be a modular form for $\mathrm{M}\Gamma_0(4)$ of weight l with character $\chi_\theta^{\sigma(\Lambda)}$ and set*

$$\mathscr{B}_\Lambda[\phi](\tau) := \sum_{g\in\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})} \phi|_g(\tau)\rho_\Lambda(g^{-1})\boldsymbol{e}_0.$$

Then the following hold:

(1) $\rho_\Lambda(g)\boldsymbol{e}_0 = \chi_\theta(g)^{\sigma(\Lambda)}\boldsymbol{e}_0$ *for all* $g \in \mathrm{M}\Gamma_0(4)$.

(2) $\mathscr{B}_\Lambda[\phi](\tau)$ *is independent of the choice of representatives of* $\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})$. *Moreover,* $\mathscr{B}_\Lambda[\phi](\tau)$ *is a modular form for* $\mathrm{Mp}_2(\mathbf{Z})$ *of type* ρ_Λ *of weight l.*

Proof. (1) Let χ_{A_Λ} be the character of $\mathrm{M}\Gamma_0(4)$ defined in [11], Theorem 5.4. Let $k \in \mathbf{Z}_{>0}$ be such that $\sigma(\Lambda) + 8k \geqq 0$. Since $|A_\Lambda| \cdot 2^{\sigma(\Lambda)+8k} = 2^{2\{2+4k+(l(\Lambda)-r(\Lambda))/2\}}$, we get $\left(\dfrac{-1}{|A|}\right) = 1$ and $\chi_{|A_\Lambda|\cdot 2^{\sigma(\Lambda)+8k}} \equiv 1$ by the definitions of the character χ_n and the symbol $\left(\dfrac{c}{d}\right)$ in [11], p. 328. Hence we get $\chi_{A_\Lambda} = \chi_\theta^{\sigma(\Lambda)}$ by [11], Theorem 5.4.

Set $\mathrm{M}\Gamma(4)' := \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) \in \mathrm{Mp}_2(\mathbf{Z}); b \equiv c \equiv 0 \bmod 4\right\}$. By [11], Theorem 5.4, we get $\rho_\Lambda(g)\boldsymbol{e}_0 = \chi_\theta(g)^{\sigma(\Lambda)}\boldsymbol{e}_0$ for all $g \in \mathrm{M}\Gamma(4)'$, because $\chi_{A_\Lambda} = \chi_\theta^{\sigma(\Lambda)}$. Since the coset $\mathrm{M}\Gamma_0(4)/\mathrm{M}\Gamma(4)'$ is represented by $\{1, T, T^2, T^3\}$, any $g \in \mathrm{M}\Gamma_0(4)$ can be expressed as $g = T^a g_0$, where $a \in \mathbf{Z}$ and $g_0 \in \mathrm{M}\Gamma(4)'$. Since $\rho_\Lambda(T)\boldsymbol{e}_0 = \boldsymbol{e}_0$ by (6.1) and $\chi_\theta(T) = 1$ by [11], Lemma 5.2, we get

$$\rho_\Lambda(g)\boldsymbol{e}_0 = \rho(T)^a\rho(g_0)\boldsymbol{e}_0 = \chi_\theta(g_0)^{\sigma(\Lambda)}\boldsymbol{e}_0 = \chi_\theta(T^a g_0)^{\sigma(\Lambda)}\boldsymbol{e}_0 = \chi_\theta(g)^{\sigma(\Lambda)}\boldsymbol{e}_0.$$

Since $g \in \mathrm{M}\Gamma_0(4)$ is an arbitrary element, we get (1).

(2) By (1), the result follows from [55], Theorem 6.2. See also [10], Lemma 2.6, and [11], proof of Lemma 11.1. □

Lemma 7.2. *The function* $f_k^{(0)}(\tau)$ *is a modular form for* $\mathrm{M}\Gamma_0(4)$ *of weight* $-4 + \dfrac{k}{2}$ *with character* χ_θ^k.

Proof. The result follows from [11], Lemma 5.2 and Theorem 6.2. □

Set $Z := S^2 = \left(-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i\right)$ and $V := S^{-1}T^2S = \left(\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \sqrt{-2\tau + 1}\right)$.

Lemma 7.3. *The coset* $\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})$ *is represented by* $\{1, S, ST, ST^2, ST^3, V\}$.

Proof. Since none of two elements of $\{1, S, ST, ST^2, ST^3, V\}$ represent the same element of $\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})$ and $\#\mathrm{M}\Gamma_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z}) = 6$, we get the result. □

Recall that $\mathbf{1}_\Lambda \in A_\Lambda$ was defined in Section 1.2. Define $\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbf{C}[A_\Lambda]$ by

$$\boldsymbol{v}_k := \sum_{\delta \in A_\Lambda, \delta^2 \equiv k/2 \bmod 2} \boldsymbol{e}_\delta.$$

Lemma 7.4. *The following identities hold*:

(1) $\rho_\Lambda((ST^l)^{-1})\boldsymbol{e}_0 = i^{\frac{\sigma(\Lambda)}{2}} 2^{-\frac{l(\Lambda)}{2}} \sum_{k=0}^{3} i^{-lk}\boldsymbol{v}_k$,

(2) $\rho_\Lambda(V^{-1})\boldsymbol{e}_0 = \boldsymbol{e}_{\mathbf{1}_\Lambda}$.

Proof. (1) Since $S^{-1} = SZ^3$ and since $\rho_\Lambda(Z)\boldsymbol{e}_\gamma = i^{-\sigma(\Lambda)}\boldsymbol{e}_{-\gamma}$ by (6.1), we get

$$\rho_\Lambda(S^{-1})\boldsymbol{e}_0 = \rho_\Lambda(S)\rho_\Lambda(Z^3)\boldsymbol{e}_0 = i^{\sigma(\Lambda)}\frac{i^{-\frac{\sigma(\Lambda)}{2}}}{|A_\Lambda|^{1/2}}\sum_{\delta\in A_\Lambda}\boldsymbol{e}_\delta = i^{\frac{\sigma(\Lambda)}{2}}2^{-\frac{l(\Lambda)}{2}}\sum_{\delta\in A_\Lambda}\boldsymbol{e}_\delta.$$

This, together with the first equation of (6.1), yields (1).

(2) By [11], p. 325, l. 16, we get

$$\rho_\Lambda(ST^{-2}S)\boldsymbol{e}_0 = i^{-\sigma(\Lambda)}|A_\Lambda|^{-1}\sum_{\gamma,\delta\in A_\Lambda}e^{2\pi i\{\langle\gamma,\delta\rangle+\gamma^2\}}\boldsymbol{e}_\delta = i^{-\sigma(\Lambda)}\boldsymbol{e}_{\mathbf{1}_\Lambda},$$

where we used the identity

$$\sum_{\gamma\in A_\Lambda}e^{2\pi i\langle\gamma,\varepsilon+\gamma\rangle} = \sum_{\gamma\in A_\Lambda}e^{2\pi i\langle\gamma,\varepsilon+\mathbf{1}_\Lambda\rangle} = |A_\Lambda|\delta_{\mathbf{1}_\Lambda,\varepsilon}$$

(cf. [11], Lemma 3.1) to get the second equality. Since $S^{-1} = S^7 = Z^3S$, we get

$$\begin{aligned}\rho_\Lambda(V^{-1})\boldsymbol{e}_0 &= \rho_\Lambda(Z)^3\rho_\Lambda(ST^{-2}S)\boldsymbol{e}_0\\ &= i^{-\sigma(\Lambda)}\rho_\Lambda(Z)^3\boldsymbol{e}_{\mathbf{1}_\Lambda}\\ &= i^{-\sigma(\Lambda)}i^{-3\sigma(\Lambda)}\boldsymbol{e}_{\mathbf{1}_\Lambda} = \boldsymbol{e}_{\mathbf{1}_\Lambda}.\end{aligned}$$

This proves (2). □

Lemma 7.5. *The following identities hold*:

(1) $f_k^{(0)}|_{ST^l}(\tau) = 2^{\frac{8-k}{2}}i^{-\frac{k}{2}}f_k^{(0)}\left(\frac{\tau+l}{4}\right)$,

(2) $f_k^{(0)}|_V(\tau) = f_k^{(1)}(\tau)$.

Proof. We apply [10], Theorem 5.1, to the lattice $\mathbb{A}_1^+ = \langle 2\rangle$. Since

$$A_{\mathbb{A}_1^+} = \langle 2\rangle^\vee/\langle 2\rangle = \left\{0,\frac{1}{2}\right\},$$

the group ring $\mathbf{C}[A_{\mathbb{A}_1^+}]$ is equipped with the standard basis $\{\boldsymbol{e}_0,\boldsymbol{e}_{1/2}\}$. Set

$$\Theta_{\mathbb{A}_1^+}(\tau) := \theta_{\mathbb{A}_1^+}(\tau)\boldsymbol{e}_0 + \theta_{\mathbb{A}_1^++1/2}(\tau)\boldsymbol{e}_{1/2}.$$

By applying [10], Theorem 5.1, to $\mathbb{A}_1^+$, we get

$$\Theta_{\mathbb{A}_1^+}(g\cdot\tau) = j(g,\tau)\rho_{\mathbb{A}_1^+}(g)\Theta_{\mathbb{A}_1^+}(\tau),\quad g\in\mathrm{Mp}_2(\mathbf{Z}). \tag{7.1}$$

By (6.1) and (7.1), we have

$$\begin{aligned}\Theta_{\mathbb{A}_1^+}(ST^l\cdot\tau) &= j(ST^l,\tau)\left\{\frac{\boldsymbol{e}_0+\boldsymbol{e}_{1/2}}{\sqrt{2i}}\theta_{\mathbb{A}_1^+}(\tau) + i^l\frac{\boldsymbol{e}_0-i\boldsymbol{e}_{1/2}}{\sqrt{2i}}\theta_{\mathbb{A}_1^++1/2}(\tau)\right\},\\ \Theta_{\mathbb{A}_1^+}(V\cdot\tau) &= j(V,\tau)\{\boldsymbol{e}_0\theta_{\mathbb{A}_1^++1/2}(\tau) + \boldsymbol{e}_{1/2}\theta_{\mathbb{A}_1^+}(\tau)\}.\end{aligned}$$

Comparing the coefficients of $\boldsymbol{e}_0$, we get

$$(7.2)\quad \theta_{\mathbb{A}_1^+}|_{ST^l}(\tau) = (2i)^{-\frac{1}{2}}\{\theta_{\mathbb{A}_1^+}(\tau) + i^l\theta_{\mathbb{A}_1^+ + 1/2}(\tau)\} = (2i)^{-\frac{1}{2}}\theta_{\mathbb{A}_1^+}\left(\frac{\tau+l}{4}\right),$$

$$(7.3)\quad \theta_{\mathbb{A}_1^+}|_V(\tau) = \theta_{\mathbb{A}_1^+ + 1/2}(\tau).$$

Here we get the second equality of (7.2) as follows:

$$\theta_{\mathbb{A}_1^+}\left(\frac{\tau+l}{4}\right) = \sum_{n \text{ even}} e^{2\pi i n^2(\tau+l)/4} + \sum_{n \text{ odd}} e^{2\pi i n^2(\tau+l)/4} = \theta_{\mathbb{A}_1^+}(\tau) + i^l\theta_{\mathbb{A}_1^+ + 1/2}(\tau).$$

Set $\eta_{1^{-8}2^84^{-8}}(\tau) := \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$, which is a modular form for $M\Gamma_0(4)$ by Lemma 7.2. Since $ST^l = \left(\begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix}, \sqrt{\tau+l}\right)$ and since $\eta(-\tau^{-1})^8 = \tau^4\eta(\tau)^8$ by [11], Lemma 6.1, we get

$$\begin{aligned}
\eta_{1^{-8}2^84^{-8}}|_{ST^l}(\tau) &= (\tau+l)^{\frac{8}{2}}\eta_{1^{-8}2^84^{-8}}\left(-\frac{1}{\tau+l}\right) \\
&= (\tau+l)^4\eta\left(-\frac{1}{\tau+l}\right)^{-8}\eta\left(-\frac{2}{\tau+l}\right)^8\eta\left(-\frac{4}{\tau+l}\right)^{-8} \\
&= (\tau+l)^4(\tau+l)^{-4}\left(\frac{\tau+l}{2}\right)^4\left(\frac{\tau+l}{4}\right)^{-4} \\
&\quad \times \eta(\tau+l)^{-8}\eta\left(\frac{\tau+l}{2}\right)^8\eta\left(\frac{\tau+l}{4}\right)^{-8} \\
&= 2^4\eta_{1^{-8}2^84^{-8}}\left(\frac{\tau+l}{4}\right),
\end{aligned}$$

which, together with (7.2), yields (1).

Since $V = \left(\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \sqrt{-2\tau+1}\right)$ and since $\eta_{1^{-8}2^84^{-8}}(\tau)$ has weight -4, we get

$$\begin{aligned}
\eta_{1^{-8}2^84^{-8}}|_V(\tau) &= (-2\tau+1)^4\eta\left(\frac{\tau}{-2\tau+1}\right)^{-8}\eta\left(\frac{2\tau}{-2\tau+1}\right)^8\eta\left(\frac{4\tau}{-2\tau+1}\right)^{-8} \\
&= (-2\tau+1)^4\left(2-\frac{1}{\tau}\right)^{-4}\left(1-\frac{1}{2\tau}\right)^4\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-4} \\
&\quad \times \eta\left(2-\frac{1}{\tau}\right)^{-8}\eta\left(1-\frac{1}{2\tau}\right)^8\eta\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-8} \\
&= 2^4\tau^4\eta\left(2-\frac{1}{\tau}\right)^{-8}\eta\left(1-\frac{1}{2\tau}\right)^8\eta\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-8}.
\end{aligned}$$

We define $h(\tau) := \eta\left(\tau+\frac{1}{2}\right)^{-8}\eta(2\tau+1)^8\eta(4\tau+2)^{-8}$ for $\tau \in \mathfrak{H}$. Then

$$(7.4)\qquad \eta_{1^{-8}2^84^{-8}}|_V(\tau) = 16\tau^4 h\left(-\frac{1}{4\tau}\right).$$

Set $\zeta := \exp(2\pi i/48)$. Since $h(\tau)$ is equal to

$$\zeta^{-8+16-32}\left\{q^{-\frac{8}{24}}\prod_{n=1}^{\infty}(1-(-q)^n)^{-8}\right\}\left\{q^{\frac{16}{24}}\prod_{n=1}^{\infty}(1-q^{2n})^8\right\}\left\{q^{-\frac{32}{24}}\prod_{n=1}^{\infty}(1-q^{4n})^{-8}\right\}$$

$$= -q^{-1}\prod_{n=1}^{\infty}\{(1-q^{2n})^{-8}(1+q^{2n-1})^{-8}\}\cdot(1-q^{2n})^8\cdot\{(1-q^{2n})^{-8}(1+q^{2n})^{-8}\}$$

$$= -q^{-1}\prod_{n=1}^{\infty}(1-q^{2n})^{-8}(1+q^{2n})^{-8}(1+q^{2n-1})^{-8}$$

and since we have the identities

$$\vartheta_2(2\tau) = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2$$

and

$$\vartheta_3(2\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2, \quad \vartheta_4(2\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2 \tag{7.5}$$

by [16], p. 105, Eqs. (32)–(36), we get

$$\vartheta_2(2\tau)^4\vartheta_3(2\tau)^4 = 2^4 q\prod_{n=1}^{\infty}(1-q^{2n})^8(1+q^{2n})^8(1+q^{2n-1})^8 = -2^4 h(\tau)^{-1}. \tag{7.6}$$

By [16], p. 104, Eq. (20), we have

$$\vartheta_2(-\tau^{-1})^4 = -\tau^2\vartheta_4(\tau)^4, \quad \vartheta_3(-\tau^{-1})^4 = -\tau^2\vartheta_3(\tau)^4,$$

which, together with (7.6), yield the equality

$$\begin{aligned}
h\left(-\frac{1}{4\tau}\right) &= -2^4\vartheta_2\left(-\frac{1}{2\tau}\right)^{-4}\vartheta_3\left(-\frac{1}{2\tau}\right)^{-4} \\
&= -\tau^{-4}\vartheta_3(2\tau)^{-4}\vartheta_4(2\tau)^{-4} \\
&= -\tau^{-4}\left\{\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^2(1-q^{2n-1})^2\right\}^{-4} \\
&= -\tau^{-4}\left\{\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{4n})(1-q^{4n-2})}{(1-q^{4n})}\right\}^{-8} \\
&= -\tau^{-4}\left\{\frac{\prod_{n=1}^{\infty}(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1-q^{4n})}\right\}^{-8} \\
&= -\tau^{-4}\eta(2\tau)^{-16}\eta(4\tau)^8.
\end{aligned} \tag{7.7}$$

Here we used (7.5) to get the third equality. We deduce from (7.4) and (7.7) that

$$\eta_{1^{-8}2^{8}4^{-8}}|_{V}(\tau) = -16\eta(2\tau)^{-16}\eta(4\tau)^{8}. \tag{7.8}$$

We get (2) from (7.3) and (7.8). □

Definition 7.6. For a 2-elementary lattice Λ, define a $\mathbf{C}[A_\Lambda]$-valued holomorphic function $F_\Lambda(\tau)$ on $\mathfrak{H}$ by

$$\begin{aligned} F_\Lambda(\tau) &:= f^{(0)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{l=0}^{3} g^{(l)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{v}_l + f^{(1)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{e}_{\mathbf{1}_\Lambda} \\ &= f^{(0)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{\gamma\in A_\Lambda} g^{(2\gamma^2)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{e}_\gamma + f^{(1)}_{8+\sigma(\Lambda)}(\tau)\boldsymbol{e}_{\mathbf{1}_\Lambda}. \end{aligned}$$

By the Fourier expansions of $f^{(0)}_k(\tau)$ and $f^{(1)}_k(\tau)$ at $q=0$, we get the following Fourier expansion of $F_\Lambda(\tau)$ at $q=0$:

$$\begin{aligned} F_\Lambda(\tau) = {}& \{q^{-1} + 24 + 2\sigma(\Lambda) + O(q)\}\boldsymbol{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\{24 + 2\sigma(\Lambda) + O(q)\}\boldsymbol{v}_0 \\ & + O(q^{1/4})\boldsymbol{v}_1 + O(q^{1/2})\boldsymbol{v}_2 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\{q^{-1/4} + O(q^{3/4})\}\boldsymbol{v}_3 \\ & - 2^{12+\sigma(\Lambda)}q^{\frac{8+\sigma(\Lambda)}{4}}\{1 + (24+\sigma(\Lambda))q^2 + O(q^4)\}\boldsymbol{e}_{\mathbf{1}_\Lambda}. \end{aligned} \tag{7.9}$$

Theorem 7.7. (1) $F_\Lambda(\tau) = \mathscr{B}_\Lambda[\eta_{1^{-8}2^{8}4^{-8}}\theta^{8+\sigma(\Lambda)}_{\mathbb{A}_1^+}](\tau)$. *In particular, $F_\Lambda(\tau)$ is a modular form for* $\mathrm{Mp}_2(\mathbf{Z})$ *of type ρ_Λ with weight $\sigma(\Lambda)/2$.*

(2) *The group $O(\Lambda)$ preserves F_Λ, i.e.,* $\mathrm{Aut}(F_\Lambda,\Lambda) = O(\Lambda)$.

(3) *If $b^+(\Lambda) \leqq 2$ and $\sigma(\Lambda) \geqq -12$, then $F_\Lambda(\tau)$ has integral Fourier coefficients.*

Proof. (1) Set $k = 8+\sigma(\Lambda)$ and $\phi(\tau) = f^{(0)}_k(\tau)$ in Proposition 7.1. Since $f^{(0)}_k(\tau)$ is a modular form for $\mathrm{M}\Gamma_0(4)$ of weight $(k-8)/2 = \sigma(\Lambda)/2$ with character $\chi^k_\theta = \chi^{\sigma(\Lambda)}_\theta$ by Lemma 7.2, $\mathscr{B}_\Lambda[f^{(0)}_k](\tau)$ is a modular form for $\mathrm{Mp}_2(\mathbf{Z})$ of type ρ_Λ with weight $\sigma(\Lambda)/2$ by Proposition 7.1. We prove that $F_\Lambda = \mathscr{B}_\Lambda[f^{(0)}_k]$. Since $k = 8+\sigma(\Lambda)$ and $|A_\Lambda| = 2^{l(\Lambda)}$, we deduce from Lemmas 7.4 (1) and 7.5 (1) that

$$\begin{aligned} &\sum_{l=0}^{3} f^{(0)}_k|_{ST^l}(\tau)\rho_\Lambda\big((ST^l)^{-1}\big)\boldsymbol{e}_0 \\ &= \sum_{l=0}^{3} 2^{\frac{8-k}{2}} i^{-\frac{k}{2}} i^{\frac{\sigma(\Lambda)}{2}} |A_\Lambda|^{-\frac{1}{2}} \sum_{j=0}^{3} f^{(0)}_k\left(\frac{\tau+l}{4}\right) i^{-lj}\boldsymbol{v}_j \\ &= 2^{\frac{-\sigma(\Lambda)-l(\Lambda)}{2}} \sum_{j=0}^{3}\sum_{l=0}^{3} f^{(0)}_k\left(\frac{\tau+l}{4}\right) i^{-lj}\boldsymbol{v}_j \\ &= 2^{-\frac{\sigma(\Lambda)+l(\Lambda)}{2}} \sum_{j=0}^{3}\sum_{l=0}^{3}\sum_{s\in\mathbf{Z}/4\mathbf{Z}} g^{(s)}_k(\tau+l) i^{-lj}\boldsymbol{v}_j. \end{aligned} \tag{7.10}$$

Recall that $f_k^{(0)}(\tau) = \sum_{n=-1}^{\infty} c_k^{(0)}(n)q^n$. Since $g_k^{(s)}(\tau) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)q^{n/4}$, we get

$$g_k^{(s)}(\tau+l) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)e^{2\pi in(\tau+l)/4} = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)i^{sl}q^{n/4},$$

which yields that

$$\sum_{l=0}^{3} i^{-jl}g_k^{(s)}(\tau+l) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)\sum_{l=0}^{3} i^{(s-j)l}q^{n/4} = 4\delta_{js}g_k^{(s)}(\tau).$$

Hence we get

$$\sum_{l=0}^{3}\sum_{s\in\mathbf{Z}/4\mathbf{Z}} i^{-jl}g_k^{(s)}(\tau+l) = \sum_{s\in\mathbf{Z}/4\mathbf{Z}} 4\delta_{sj}g_k^{(s)}(\tau) = 4g_k^{(j)}(\tau),$$

which, together with (7.10), yields that

$$\sum_{l=0}^{3} f_k^{(0)}|_{ST^l}(\tau)\cdot\rho_\Lambda\big((ST^l)^{-1}\big)\boldsymbol{e}_0 = 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{j=0}^{3} g_k^{(j)}(\tau)\boldsymbol{v}_j. \tag{7.11}$$

Similarly, we get by Lemmas 7.4 (2) and 7.5 (2)

$$f_k^{(0)}|_V(\tau)\rho_\Lambda(V^{-1})\boldsymbol{e}_0 = f_k^{(1)}(\tau)\boldsymbol{e}_{\mathbf{1}_\Lambda}. \tag{7.12}$$

By (7.11) and (7.12), we get $F_\Lambda = \mathscr{B}_\Lambda[f_k^{(0)}]$.

(2) Since $g(\boldsymbol{e}_\gamma) = \boldsymbol{e}_{\bar g(\gamma)}$ for $g\in O(\Lambda)$ and $\gamma\in A_\Lambda$, we get $g(\boldsymbol{e}_0) = \boldsymbol{e}_0$ and $g(\boldsymbol{v}_i) = \boldsymbol{v}_i$ for all $g\in O(\Lambda)$ by the definition of $\boldsymbol{v}_i$. Since $\mathbf{1}_\Lambda$ is $O(q_\Lambda)$-invariant by its uniqueness, we get $\bar g(\mathbf{1}_\Lambda) = \mathbf{1}_\Lambda$ for all $g\in O(\Lambda)$. This proves $\mathrm{Aut}(F_\Lambda,\Lambda) = O(\Lambda)$.

(3) Since $f_k^{(0)}(\tau)$, $g_k^{(j)}(\tau)$, $f_k^{(1)}(\tau)$ have integral Fourier coefficients for $k \geqq -4$, i.e., $\sigma(\Lambda)\geqq -12$, it suffices to prove by Definition 7.6 that $2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\in\mathbf{Z}$ when $b^+(\Lambda)\leqq 2$. Since $\sigma(\Lambda) = 2b^+(\Lambda) - r(\Lambda)$, $r(\Lambda)\geqq l(\Lambda)$ and $r(\Lambda)\equiv l(\Lambda) \bmod 2$, we get

$$4-\sigma(\Lambda)-l(\Lambda) = 2\big(2-b^+(\Lambda)\big) + r(\Lambda) - l(\Lambda) \geqq 0$$

and $4-\sigma(\Lambda)-l(\Lambda)\equiv 0 \bmod 2$. □

Recall that F_Λ induces a modular form $F_\Lambda|_L$ of type ρ_L when $\Lambda = \mathbb{U}(N)\oplus L$ (cf. Section 6.2). Since Λ is 2-elementary, $N\in\{1,2\}$ and L is 2-elementary in this case.

Lemma 7.8. *If $\Lambda = \mathbb{U}(N)\oplus L$, then $F_\Lambda|_L = F_L$.*

Proof. Write $F_\Lambda|_L(\tau) = \sum_{\gamma \in A_L} (F_\Lambda|_L)_\gamma(\tau)\mathbf{e}_\gamma$. Since $\mathbf{1}_{\mathbb{U}(N)} = (0,0)$ for $N = 1,2$, we get $\mathbf{1}_\Lambda = \big((0,0), \mathbf{1}_L\big)$. Since $\big((n/N,0),\gamma\big)^2 = \gamma^2 \bmod 2$ for $\gamma \in A_L$, it follows from Definition 7.6 and the definition of $(F_\Lambda|_L)(\tau)$ (cf. (6.3)) that

$$(7.13)\qquad (F_\Lambda|_L)_\gamma(\tau) = \begin{cases} N 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} g^{(2\gamma^2)}_{8+\sigma(\Lambda)}(\tau) & \text{if } \gamma \neq 0, \mathbf{1}_L, \\ f^{(0)}_{8+\sigma(\Lambda)}(\tau) + N 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} g^{(0)}_{8+\sigma(\Lambda)}(\tau) & \text{if } \gamma = 0, \\ f^{(1)}_{8+\sigma(\Lambda)}(\tau) + N 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} g^{(\sigma(\Lambda))}_{8+\sigma(\Lambda)}(\tau) & \text{if } \gamma = \mathbf{1}_L. \end{cases}$$

In the last equality, we used the formula $\mathbf{1}_\Lambda^2 \equiv \dfrac{\sigma(\Lambda)}{2} \bmod 2$, which follows from (6.2) and (7.9). If $N = 1$, then $A_\Lambda = A_L$ and hence $F_\Lambda|_L = F_\Lambda = F_L$ by Definition 7.6 and (7.13). Assume $N = 2$. Since $\sigma(\Lambda) = \sigma(L)$ and $l(\Lambda) = l(L) + 2$, we get $F_\Lambda|_L = F_L$ by comparing the definition of F_L with (7.13). This proves the lemma. □

Lemma 7.9. *Let L be a* 2*-elementary Lorentzian lattice. If* $r(L) \leqq 10$, *then a subset of* $\mathscr{C}_L^+$ *is a Weyl chamber of L if and only if it is a Weyl chamber of* F_L.

Proof. Write $F_L(\tau) = \sum_{\gamma \in A_L} \mathbf{e}_\gamma \sum_{k \in \frac{\gamma^2}{2} + \mathbf{Z}} c_{L,\gamma}(k) q^k$. By (6.4), it suffices to prove that if $\lambda \in L^\vee$, $\lambda^2 < 0$ and $c_{L,\bar{\lambda}}(\lambda^2/2) \neq 0$, then $h_\lambda = h_d$ for some $d \in \Delta_L$. Since $8 + \sigma(L) \geqq 0$, this follows from (7.9). □

8. Borcherds products for 2-elementary lattices

Throughout this section, we assume that Λ is an even 2-elementary lattice with

$$\mathrm{sign}(\Lambda) = \big(2, r(\Lambda) - 2\big).$$

Recall that $\mathscr{D}'_\Lambda$ and $\mathscr{D}''_\Lambda$ were defined in Section 1.4. We have the splitting

$$\mathscr{D}''_\Lambda = \sum_{\lambda/2 \not\equiv \mathbf{1}_\lambda,\, \lambda \in \Delta''_\Lambda/\pm 1} H_\lambda + \mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right)$$

when $r(\Lambda) \equiv 1 \bmod 4$.

Theorem 8.1. *If* $r(\Lambda) \leqq 12$ *or if* $r(\Lambda) = 13$ *and* $l(\Lambda) \leqq 7$, *then the Borcherds lift* $\Psi_\Lambda(\cdot, F_\Lambda)$ *is a holomorphic automorphic form on* Ω_Λ^+ *for* $O^+(\Lambda)$. *The zero divisor of* $\Psi_\Lambda(\cdot, F_\Lambda)$ *is given by*

$$\mathrm{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big) = \mathscr{D}'_\Lambda + (2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1)\mathscr{D}''_\Lambda$$

for $r(\Lambda) \leqq 12$ *and by*

$$\mathrm{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big) = \mathscr{D}'_\Lambda + (2^{\frac{13-l(\Lambda)}{2}} + 1) \sum_{\lambda/2 \not\equiv \mathbf{1}_\lambda,\, \lambda \in \Delta''_\Lambda/\pm 1} H_\lambda + (2^{\frac{13-l(\Lambda)}{2}} - 7)\mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right)$$

for $r(\Lambda) = 13$ *and* $l(\Lambda) \leqq 7$. *The weight* $w(\Lambda)$ *of* $\Psi_\Lambda(\cdot, F_\Lambda)$ *is given by the formula*

$$w(\Lambda) = \begin{cases} (16 - r(\Lambda))(2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1) - 8(1 - \delta(\Lambda)) & \text{if } r(\Lambda) = 12, \\ (16 - r(\Lambda))(2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1) & \text{if } r(\Lambda) \neq 12. \end{cases}$$

In particular, $\Psi_\Lambda(\cdot, F_\Lambda)$ *is reflective in the sense of Gritsenko–Nikulin* [28], I, *Definition* 2.1.2, *for* $r(\Lambda) \leqq 13$ *and* $l(\Lambda) \leqq 7$.

Proof. Assume $r(\Lambda) \leqq 12$. Since $\operatorname{sign}(\Lambda) = (2, r(\Lambda) - 2)$, we get $\sigma(\Lambda) = 4 - r(\Lambda)$ and $8 + \sigma(\Lambda) \geqq 0$. By Theorem 7.7 (2), we get $\operatorname{Aut}(\Lambda, F_\Lambda) = O(\Lambda)$. Write

$$F_\Lambda(\tau) = \sum_{\gamma \in A_\Lambda} \boldsymbol{e}_\gamma \sum_{k \in \frac{\gamma^2}{2} + \mathbf{Z}} c_{\Lambda,\gamma}(k) q^k.$$

By (7.9), we see that $c_{\Lambda,\gamma}(k) \geqq 0$ if $k < 0$ and that the coefficient of $\boldsymbol{e}_{\mathbf{1}_\Lambda}$, i.e., $f^{(1)}_{8+\sigma(\Lambda)}(\tau)$, is regular at $q = 0$. By Theorem 6.1 and (7.9), $\Psi_\Lambda(\cdot, F_\Lambda)$ is an automorphic form for $O^+(\Lambda)$ such that

$$\text{(8.1)} \qquad \begin{aligned} \operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda)) &= \sum_{\lambda \in \Lambda^\vee/\pm 1, \lambda^2 < 0} c_{\Lambda,\bar\lambda}(\lambda^2/2) H_\lambda \\ &= \sum_{\lambda \in \Lambda/\pm 1, \lambda^2/2 = -1} c_{\Lambda,\bar 0}(\lambda^2/2) H_\lambda \\ &\quad + \sum_{\lambda \in \Lambda^\vee/\pm 1, \lambda^2/2 = -1/4} c_{\Lambda,\bar\lambda}(\lambda^2/2) H_\lambda \\ &= \sum_{\lambda \in \Delta_\Lambda/\pm 1} H_\lambda + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} \sum_{\lambda \in \Delta''_\Lambda/\pm 1} H_\lambda \\ &= \mathscr{D}'_\Lambda + (2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1)\mathscr{D}''_\Lambda. \end{aligned}$$

By Theorem 6.1, $w(\Lambda) = c_{\Lambda,\bar 0}(0)/2$. If $r(\Lambda) = 12$ and $\delta(\Lambda) = 0$, then $\mathbf{1}_\Lambda = 0$, which, substituted into (7.9), implies that

$$\text{(8.2)} \qquad \begin{aligned} F_\Lambda(\tau) &= \{q^{-1} + 24 + 2\sigma(\Lambda) + O(q)\}\boldsymbol{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\{24 + 2\sigma(\Lambda) + O(q)\}\boldsymbol{v}_0 \\ &\quad + O(q^{1/4})\boldsymbol{v}_1 + O(q^{1/2})\boldsymbol{v}_2 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\{q^{-1/4} + O(q^{3/4})\}\boldsymbol{v}_3 \\ &\quad + \{-16 + O(q)\}\boldsymbol{e}_0. \end{aligned}$$

Since $\boldsymbol{v}_0$ contains $\boldsymbol{e}_0$ with multiplicity one and since $\sigma(\Lambda) = 4 - r(\Lambda)$, we deduce from (8.2) that

$$\begin{aligned} w(\Lambda) = \frac{c_{\Lambda,0}(0)}{2} &= 12 + \sigma(\Lambda) + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}(12 + \sigma(\Lambda)) - 8 \\ &= (16 - r(\Lambda))(2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1) - 8. \end{aligned}$$

This proves the formula for $w(\Lambda)$ when $r(\Lambda) = 12$ and $\delta(\Lambda) = 0$.

If $r(\Lambda) < 12$ or $\big(r(\Lambda),\delta(\Lambda)\big) = (12,1)$, the coefficient of $\boldsymbol{e}_{\mathbf{1}_\Lambda}$ does not contribute to $c_{\Lambda,0}(0)$ by (7.9), so that

$$\text{(8.3)}\qquad w(\Lambda) = \frac{c_{\Lambda,0}(0)}{2} = 12 + \sigma(\Lambda) + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\big(12+\sigma(\Lambda)\big)$$
$$= \big(16 - r(\Lambda)\big)\big(2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1\big)$$

in this case. This proves the theorem when $r(\Lambda) \leqq 12$.

Assume that $r(\Lambda) = 13$ and $l(\Lambda) \leqq 7$. By (7.9), the principal part of $F_\Lambda(\tau)$ is given by the formula

$$\text{(8.4)}\qquad \mathscr{P}(F_\Lambda) = q^{-1}\boldsymbol{e}_0 + 2^{\frac{13-l(\Lambda)}{2}} q^{-\frac{1}{4}}\boldsymbol{v}_3 - 8q^{-\frac{1}{4}}\boldsymbol{e}_{\mathbf{1}_\Lambda}$$
$$= q^{-1}\boldsymbol{e}_0 + 2^{\frac{13-l(\Lambda)}{2}} q^{-\frac{1}{4}} \sum_{\gamma^2 \equiv 3/2,\, \gamma \neq \mathbf{1}_\Lambda} \boldsymbol{e}_\gamma + \big(2^{\frac{13-l(\Lambda)}{2}} - 8\big) q^{-\frac{1}{4}}\boldsymbol{e}_{\mathbf{1}_\Lambda}.$$

Since $l(\Lambda) \leqq 7$, we get $c_{\Lambda,\gamma}(k) \geqq 0$ for $k < 0$ by (8.4). The formula for $\operatorname{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big)$ follows from Theorem 6.1 and (8.4) in the same manner as (8.1). Since $\mathbf{1}_\Lambda \neq 0$ when $r(\Lambda) = 13$, the coefficient of $\boldsymbol{e}_{\mathbf{1}_\Lambda}$ does not contribute to $c_{\Lambda,0}(0)$ by (7.9), so that $w(\Lambda)$ is given by (8.3). This completes the proof of Theorem 8.1. □

Corollary 8.2. *If* $r(\Lambda) \leqq 12$ *and* $\Delta''_\Lambda = \emptyset$, *then* $\operatorname{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big) = \mathscr{D}_\Lambda$.

Proof. Since $\Delta''_\Lambda = \emptyset$, the result follows from Theorem 8.1. □

If $\Lambda \subset \mathbb{L}_{K3}$ is *primitive* and $r(\Lambda) = 13$, $l(\Lambda) \geqq 9$, then we get

$$\big(r(\Lambda), l(\Lambda), \delta(\Lambda)\big) = (13, 9, 1)$$

because $l(\Lambda) \leqq \min\{r(\Lambda), 22 - r(\Lambda)\} = 9$ and $\delta(\Lambda) = 1$. Since $\Lambda^\perp$ has invariants $(r, l, \delta) = (9, 9, 1)$ in this case, we get $\Lambda^\perp \cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$ if $\Lambda \subset \mathbb{L}_{K3}$ is *primitive* and $r(\Lambda) = 13$, $l(\Lambda) \geqq 9$.

Corollary 8.3. *The moduli space of* 2*-elementary K*3 *surfaces of type M is quasi-affine if* $r(M) \geqq 9$ *and* $M \not\cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$.

Proof. Set $\Lambda := M^\perp$. Since $\Lambda \subset \mathbb{L}_{K3}$ is primitive, either $r(\Lambda) \leqq 12$ or $r(\Lambda) = 13$ and $l(\Lambda) \leqq \min\{r(\Lambda), 22 - r(\Lambda)\} = 9$ by the assumption $r(M) \geqq 9$. Since $M \not\cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$ by assumption, we get $\big(r(\Lambda), l(\Lambda)\big) \neq (13, 9)$. Hence either $r(\Lambda) \leqq 12$ or $r(\Lambda) = 13$, $l(\Lambda) \leqq 7$. By Theorem 8.1, $\Psi_\Lambda(\cdot, F_\Lambda)$ is a holomorphic automorphic form on Ω_Λ. Recall that an automorphic form on Ω_Λ is identified with a holomorphic section of an ample line bundle over $\mathscr{M}_\Lambda^*$ by Baily–Borel [3]. Hence $\mathscr{M}_\Lambda \backslash \operatorname{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big)$ is quasi-affine. Since $\operatorname{supp}\operatorname{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big) = \mathscr{D}_\Lambda$ by Theorem 8.1 and hence $\mathscr{M}_\Lambda^o = \mathscr{M}_\Lambda \backslash \operatorname{div}\big(\Psi_\Lambda(\cdot, F_\Lambda)\big)$, we get the result. □

In [47], Section 2, [1], Section 2.2, and [18], Sections 1–3, the notion of lattice polarized *K*3 surface was introduced. We follow the definition in [18].

Corollary 8.4. *If $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice with $r(M) \geqq 9$ and $M \not\cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$, then the moduli space of ample M-polarized $K3$ surfaces is quasi-affine.*

Proof. Set $O_0^+(M^\perp) := \ker\{O^+(M^\perp) \to O(q_{M^\perp})\}$, where $O^+(M^\perp) \to O(q_{M^\perp})$ denotes the natural homomorphism. By [17], p. 2607, the coarse moduli space of ample M-polarized $K3$ surfaces is isomorphic to the analytic space $\Omega_{M^\perp}^o / O_0^+(M^\perp)$. By this description, the proof of the corollary is similar to that of Corollary 8.3. □

For the table of isometry classes of primitive 2-elementary Lorentzian sublattices $M \subset \mathbb{L}_{K3}$ with $r(M) \geqq 9$ and $M \not\cong \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus 8}$, see [21], Appendix, Tables 1–3; there are 53 isometry classes. There are some examples of lattices Λ with $b^+(\Lambda) = 2$ admitting an automorphic form on Ω_Λ^+ with zero divisor $\mathscr{D}_\Lambda$. See, e.g., [8], Section 16, Examples 1–3, [9], [11], Section 12, [12], Examples 2.1, 2.2, [28], II, Theorem 5.2.1, [34], Theorem 6.4, [55], Section 10.

As a related result, we mention the following theorem.

Theorem 8.5. *The moduli space of 2-elementary $K3$ surfaces of type M contains no complete curves if $r(M) \geqq 7$. The same is true for the moduli space of ample M-polarized $K3$ surfaces if M is 2-elementary and $r(M) \geqq 7$.*

Proof. By [62], Theorem 5.9, τ_M is a strongly pluri-subharmonic function on $\mathscr{M}_{M^\perp}^o$ if $r(M) \geqq 7$. Hence $\mathscr{M}_{M^\perp}^o$ contains no complete curves when $r(M) \geqq 7$. Since the moduli space of ample M-polarized $K3$ surfaces $\Omega_{M^\perp}^o / O_0^+(M^\perp)$ is a finite covering of $\mathscr{M}_{M^\perp}^o$, the second assertion follows from the first one. □

Question 8.6. The existence of a strongly pluri-subharmonic function on a quasi-projective variety X does *not* necessarily imply the quasi-affinity of X (see [29], p. 232, Example 3.2, for a counter example). If $r(M) \geqq 7$, is $\mathscr{M}_{M^\perp}^o$ quasi-affine?

The referee suggested an interesting approach to the problem of quasi-affinity of $\mathscr{M}_{M^\perp}^o$ using the Lefschetz formula (cf. [50] for a similar approach using the Grothendieck–Riemann–Roch formula).

Assume $\Lambda = \mathbb{U}(N) \oplus L$, where L is a 2-elementary Lorentzian lattice with $r(L) \leqq 10$ and $N \in \{1, 2\}$. Hence $r(\Lambda) \leqq 12$, and $F_\Lambda|_L = F_L$ by Lemma 7.8. By [10], Theorem 13.3, by Definition 7.6 and the definitions of $f_k^{(0)}(\tau)$, $f_k^{(1)}(\tau)$ and $g_k^{(i)}(\tau)$, the infinite product for $\Psi_\Lambda(\cdot, F_\Lambda)$ is given explicitly as follows:

$$
\begin{aligned}
\text{(8.5)} \qquad \Psi_\Lambda(z, F_\Lambda) = e^{2\pi i\langle\varrho, z\rangle} & \prod_{\lambda \in L,\, \lambda\cdot\mathscr{W} > 0,\, \lambda^2 \geqq -2} (1 - e^{2\pi i\langle\lambda, z\rangle})^{c_{8+\sigma(\Lambda)}^{(0)}(\lambda^2/2)} \\
& \times \prod_{\lambda \in 2L^\vee,\, \lambda\cdot\mathscr{W} > 0,\, \lambda^2 \geqq -2} (1 - e^{\pi i N\langle\lambda, z\rangle})^{2^{\frac{r(\Lambda) - l(\Lambda)}{2}} c_{8+\sigma(\Lambda)}^{(0)}(\lambda^2/2)} \\
& \times \prod_{\lambda \in (\mathbf{1}_L + L),\, \lambda\cdot\mathscr{W} > 0,\, \lambda^2 \geqq 0} (1 - e^{2\pi i\langle\lambda, z\rangle})^{2c_{8+\sigma(\Lambda)}^{(1)}(\lambda^2/2)},
\end{aligned}
$$

where $z \in L \otimes \mathbf{R} + i\mathscr{W}$ with $(\operatorname{Im} z)^2 \gg 0$, $\mathscr{W} \subset L \otimes \mathbf{R}$ is a Weyl chamber of L by Lemma 7.9 and $\varrho = \varrho(L, F_L, \mathscr{W}) \in L \otimes \mathbf{Q}$ is the Weyl vector of $(L, F_L, \mathscr{W})$.

Example 8.7. Let $\Lambda = \mathbb{U}(2) \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus k}$ with $0 \leqq k \leqq 8$. By [65], Theorem 1.1, $\Psi_\Lambda(\cdot, F_\Lambda)$ is regarded as an automorphic form on the Kähler moduli of a del Pezzo surface of degree $9 - k$, which appears in the formula for the BCOV invariant [19] of certain Borcea–Voisin threefolds. By [65], Propositions 4.1 and 4.3, and [27], proof of Theorem 2.3 (a) and Section 3, there is a Borcherds–Kac–Moody superalgebra with denominator function $\Psi_\Lambda(\cdot, F_\Lambda)$. In [26], Corollaries 3.4 and 3.5, Gritsenko gave a very explicit Fourier series expansion of $\Psi_\Lambda(\cdot, F_\Lambda)$ under an appropriate identification of the domains Ω_Λ^+ and $\Omega_{\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_{k-1}}$.

Example 8.8. Let $\Lambda = \mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 12$ and $w(\Lambda) = 0$. This Λ admits no primitive embedding into $\mathbb{L}_{K3}$ by [46], Theorem 1.12.1. Since $\Delta_\Lambda = \emptyset$, we get $\mathscr{D}_\Lambda = \emptyset$, so that $\Psi_\Lambda(\cdot, F_\Lambda)$ is a constant function. This $F_\Lambda(\tau)$ gives an example of non-trivial elliptic modular form for $\mathrm{Mp}_2(\mathbf{Z})$ whose Borcherds lift becomes trivial.

Example 8.9. Let $\Lambda = \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 10$ and $w(\Lambda) = 4$. Then $\Psi_\Lambda(\cdot, F_\Lambda)$ is the Borcherds Φ-function of dimension 10. See [8], Section 15, Example 4, [9], [10], Example 13.7, [19], Section 13, [23], Section 11, [33], Remark 4.7, Theorem 7.1, [54], [62], Section 8.1, for more about this example and related results.

Example 8.10. Let $\Lambda = \mathbb{U}^2 \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 8$ and $w(\Lambda) = 12$. Then $\Psi_\Lambda(\cdot, F_\Lambda) = \Psi_\Lambda\big(\cdot, \Theta_{\Lambda_{16}^+}(\tau)/\eta(\tau)^{24}\big)$ is the restriction of the Borcherds Φ-function of dimension 26 to Ω_Λ, where $\Theta_{\Lambda_{16}^+}(\tau)$ is the theta function [10], Section 4, for the positive-definite 16-dimensional Barnes–Wall lattice Λ_{16}^+. See [62], Section 8.2.

Example 8.11. Let $\Lambda = \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^2$. We have $l(\Lambda) = 6$ and $w(\Lambda) = 28$. Kondō [34], Theorem 6.4, used $\Psi_\Lambda(\cdot, F_\Lambda)$ to study the projective model of the moduli space of 8 points on $\mathbf{P}^1$. By [34], Theorem 6.7 and its proof, $\Psi_\Lambda(\cdot, F_\Lambda)^{15}$ is expressed as the product of certain 105 *additive* Borcherds lifts ([10], Section 14). See also [23], Section 12.

Example 8.12. Let $\Lambda = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8$. Then $l(\Lambda) = 0$ and $w(\Lambda) = 252$. We get $F_\Lambda(\tau) = E_4(\tau)^2/\eta(\tau)^{24}$, where $E_4(\tau)$ is the Eisenstein series of weight 4. The corresponding Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda) = \Psi_\Lambda\big(\cdot, E_4(\tau)^2/\eta(\tau)^{24}\big)$ was introduced by Borcherds [8], Theorem 10.1, Section 16, Example 1. By Harvey–Moore [30], Sections 4 and 5, $\Psi_\Lambda\big(\cdot, E_4(\tau)^2/\eta(\tau)^{24}\big)$ appears in the formula for the one-loop coupling renormalization. See [30], Eqs. (4.1), (4.5), (4.16), (4.27).

Example 8.13. When $\Lambda = \mathbb{U}^2 \oplus \mathbb{D}_4$, $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form Δ of Freitag–Hermann [22], Theorem 11.6. Notice that the weight of Δ is 72 in our definition (cf. [22], p. 250, ll. 21–23). By [22], proof of Theorem 11.5, $\Psi_\Lambda(\cdot, F_\Lambda)$ is expressed as the product of certain 36 theta functions.

Example 8.14. When $\Lambda = (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus 4}$, $\Psi_\Lambda(\cdot, F_\Lambda)$ is the product of all even Freitag theta functions ([59] and [63], Theorem 7.9), so that the structure of $\Psi_\Lambda(\cdot, F_\Lambda)$ is similar to that of $\Psi_{\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^2}(\cdot, F_{\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^2})$, $\Psi_{\mathbb{U}^2 \oplus \mathbb{D}_4}(\cdot, F_{\mathbb{U}^2 \oplus \mathbb{D}_4})$. For the corresponding 2-elementary $K3$ surfaces, see [59].

Example 8.15. When $\Lambda = (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus 3}$, then $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form Δ_{11} of Gritsenko–Nikulin [28], II, Example 3.4 and Theorem 5.2.1. When $\Lambda = \mathbb{U}^2 \oplus \mathbb{A}_1$, then $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form $\Delta_5^4 \Delta_{35}$ of Gritsenko–Nikulin [28], II, Examples 2.4 and 3.9, Theorem 5.2.1.

We study the case where $\Lambda \subset \mathbb{L}_{K3}$ is primitive and $r(\Lambda) = 13$, $l(\Lambda) \geqq 9$. Since $l(\Lambda) \leqq 9$ as in the proof of Corollary 8.3, we get $\big(r(\Lambda), l(\Lambda), \delta(\Lambda)\big) = (13, 9, 1)$ and hence $\Lambda \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1$.

Theorem 8.16. *Let $\Lambda \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1$. Then the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ is a meromorphic automorphic form for $O^+(\Lambda)$ of weight 15 with zero divisor*

$$\mathscr{D}'_\Lambda + 5\mathscr{D}''_\Lambda - 8\mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right).$$

Proof. We have $r(\Lambda) = 13$, $l(\Lambda) = 9$, $\sigma(\Lambda) = -9$ and $\delta(\Lambda) = 1$. By Theorem 6.1 and (7.9), (8.4), the weight of $\Psi_\Lambda(\cdot, F_\Lambda)$ is given by $\big(12 + \sigma(\Lambda)\big)(2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} + 1) = 15$ and the divisor of $\Psi_\Lambda(\cdot, F_\Lambda)$ is given by

$$\mathscr{D}_\Lambda + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\mathscr{D}''_\Lambda - 2^{12+\sigma(\Lambda)}\mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right) = \mathscr{D}'_\Lambda + 5\mathscr{D}''_\Lambda - 8\mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right),$$

where $-2^{12+\sigma(\Lambda)}\mathscr{H}_\Lambda\left(\mathbf{1}_\Lambda, -\frac{1}{2}\right)$ comes from the negative coefficient of $q^{\frac{8+\sigma(\Lambda)}{4}}\boldsymbol{e}_{\mathbf{1}_\Lambda}$ in (7.9), (8.4). This proves the theorem. □

9. An explicit formula for τ_M

Theorem 9.1. *Let M be a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$. If $r(M) > 10$ or $\big(r(M), \delta(M)\big) = (10, 1)$, then there is a constant $C_M > 0$ depending only on M such that for every 2-elementary K3 surface (X, ι) of type M,*

$$\tau_M(X, \iota)^{-2^{g(M)+1}(2^{g(M)}+1)} = C_M \big\|\Psi_{M^\perp}\big(\varpi_M(X, \iota), F_{M^\perp}\big)\big\|^{2^{g(M)}} \big\|\chi_{g(M)}\big(\Omega(X^\iota)\big)\big\|^{16}.$$

In particular, if $\ell \in \mathbf{Z}_{>0}$ is an integer such that $\mathscr{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$ extends to a very ample line bundle on $\mathscr{A}^*_{g(M)}$, then the following equality holds in Theorem 5.1:

$$\Phi_M = C_M^{\ell/2} \Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^* \chi_{g(M)}^{8\ell}.$$

Proof. By our assumption $r(M) \geqq 10$, we get $r(M^\perp) \leqq 12$. If the equality $r(M^\perp) = 12$ holds, then $\delta(M) = 1$. We set $\Lambda = M^\perp$ in Theorem 8.1. Then we have

$$16 - r(\Lambda) = r(M) - 6, \quad \frac{r(\Lambda) - l(\Lambda)}{2} = 11 - \frac{r(M) + l(M)}{2} = g(M).$$

Recall that the Bergman kernel $K_{M^\perp} \in C^\infty(\Omega^+_{M^\perp})$ was defined in Section 4.2. Let $\omega_{M^\perp}$ be the Kähler form of the Bergman metric on $\Omega^+_{M^\perp}$, i.e.,

$$\omega_{M^\perp} := -dd^c \log K_{M^\perp}.$$

By [62], Eq. (7.1), and [66], Theorem 4.1, we have the following equation of currents on $\Omega_{M^\perp}$:

$$(9.1) \qquad dd^c \log \tau_M = \frac{r(M)-6}{4}\omega_{M^\perp} + J_M^*\omega_{\mathscr{A}_{g(M)}} - \frac{1}{4}\delta_{\mathscr{D}_{M^\perp}}.$$

By Theorem 8.1, by (4.17) and the Poincaré–Lelong formula, we get

$$\begin{aligned}(9.2) \qquad & -2^{g(M)-1} dd^c \log\|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|^2 \\ &= 2^{g(M)-1}(2^{g(M)}+1)\big(r(M)-6\big)\omega_{M^\perp} \\ &\quad - 2^{g(M)-1}\delta_{\mathscr{D}'_{M^\perp}} - 2^{g(M)-1}(2^{g(M)}+1)\delta_{\mathscr{D}''_{M^\perp}}.\end{aligned}$$

By Proposition 4.2 (2), there exist $a \in \mathbf{Z}_{\geqq 0}$ and an $O^+(M^\perp)$-invariant effective divisor E on $\Omega^+_{M^\perp}$ such that

$$\begin{aligned}(9.3) \qquad & -dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 \\ &= 2^{g(M)+1}(2^{g(M)}+1)\ell J_M^*\omega_{\mathscr{A}_{g(M)}} - 2(2^{2g(M)-2}+a)\ell\delta_{\mathscr{D}'_{M^\perp}} - \delta_E.\end{aligned}$$

By (9.1), (9.2), (9.3), we get the following equation of currents on $\Omega^+_{M^\perp}$:

$$\begin{aligned}(9.4) \qquad & -dd^c \log[\tau_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}\|\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^*\chi_{g(M)}^{8\ell}\|^2] \\ &= -2a\ell\delta_{\mathscr{D}'_{M^\perp}} - \delta_E.\end{aligned}$$

Since $\log \tau_M$, $\log\|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|$ and $\log\|J_M^*\chi_{g(M)}^{8\ell}\|$ are $O^+(M^\perp)$-invariant L^1_{loc}-functions on $\Omega^+_{M^\perp}$, we deduce from (9.4), [62], Theorem 3.17, and [66], Eq. (4.8), the existences of an integer m and an $O^+(M^\perp)$-invariant meromorphic function φ on $\Omega^+_{M^\perp}$ with divisor $m(2a\ell\mathscr{D}'_{M^\perp} + E)$ such that

$$(9.5) \qquad \tau_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}\|\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^*\chi_{g(M)}^{8\ell}\|^2 = |\varphi|^{2/m}.$$

Since $a \geqq 0$, $\ell > 0$ and E is effective, φ is holomorphic. By the $O^+(M^\perp)$-invariance of φ, there exists a holomorphic function $\tilde{\varphi}$ on $\mathscr{M}_{M^\perp}$ such that

$$\Pi^*_{M^\perp}\tilde{\varphi} = \varphi,$$

where $\Pi_{M^\perp} : \Omega^+_{M^\perp} \to \mathscr{M}_{M^\perp}$ is the projection. Recall that $\mathscr{M}^*_{M^\perp}$ is the Baily–Borel–Satake compactification of $\mathscr{M}_{M^\perp}$. We define $B_{M^\perp} := \mathscr{M}^*_{M^\perp} \setminus \mathscr{M}_{M^\perp}$.

Case 1. Assume that $r(M) \leqq 17$. Since $\mathscr{M}^*_{M^\perp}$ is an irreducible normal projective variety and since $\dim B_{M^\perp} \leqq \dim \mathscr{M}^*_{M^\perp} - 2$ by the condition $r(M) \leqq 17$, $\tilde{\varphi}$ extends to a holomorphic function on $\mathscr{M}^*_{M^\perp}$. Since $\mathscr{M}^*_{M^\perp}$ is compact, $\tilde{\varphi}$ must be a constant function on $\mathscr{M}^*_{M^\perp}$. Hence $a = 0$, $E = 0$ and φ is a constant. Setting $C_M := |\varphi|^{-2/m}$ in (9.5), we get the result.

Case 2. Assume that $r(M) \geqq 18$. Then $g(M) = 11 - \dfrac{r(M) + l(M)}{2} \leqq 2$. By Proposition 4.2 (3), we get $a = 0$ and $E = 0$. Hence $\tilde{\varphi}$ is a *nowhere vanishing* holomorphic function on $\mathcal{M}_{M^\perp}$. By [66], Theorem 1.1, $\tilde{\varphi}$ has at most zeros or poles on $B_{M^\perp}$. In particular, $\tilde{\varphi}$ extends to a meromorphic function on $\mathcal{M}^*_{M^\perp}$ such that $\mathrm{div}(\tilde{\varphi}) \subset B_{M^\perp}$. Since $B_{M^\perp}$ is irreducible when $r(M) \geqq 18$ by Proposition 11.7 below, either $\mathrm{div}(\tilde{\varphi})$ or $-\mathrm{div}(\tilde{\varphi})$ is effective. In both cases, $\tilde{\varphi}$ must be a constant. This completes the proof. □

Remark 9.2. The same proof does not work in the case $r(M) = 9$. Since we get by Theorem 8.1 an extra contribution of the divisor $-8\mathcal{H}_{M^\perp}\left(\mathbf{1}_{M^\perp}, -\dfrac{1}{2}\right)$ in (9.2) in this case, the divisor corresponding to (the minus sign of) the right-hand side of (9.4) may not be effective. As a result, φ in (9.5) may not be a constant.

Table 1 lists all $M^\perp$ such that M is a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$ with $r(M) > 10$ or $\big(r(M), \delta(M)\big) = (10, 1)$.

$g(M)$	$M^\perp$ with $\delta(M^\perp) = 1$		$M^\perp$ with $\delta(M^\perp) = 0$
0	$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus k}$	$(0 \leqq k \leqq 9)$	$\mathbb{U}(2)^{\oplus 2}$
1	$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus k}$	$(0 \leqq k \leqq 9)$	$\mathbb{U}(2)^{\oplus 2} \oplus \mathbb{D}_4$, $\mathbb{U} \oplus \mathbb{U}(2)$
2	$\mathbb{U}^{\oplus 2} \oplus \mathbb{A}_1^{\oplus k}$	$(1 \leqq k \leqq 8)$	$\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4$, $\mathbb{U}^{\oplus 2}$
3	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{\oplus k}$	$(1 \leqq k \leqq 4)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4$
4	$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus k}$	$(0 \leqq k \leqq 2)$	
5	$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus k}$	$(0 \leqq k \leqq 1)$	

Table 1. List of $M^\perp$ with $r(M) > 10$ or $\big(r(M), \delta(M)\big) = (10, 1)$.

When $(r, \delta) = (10, 0)$, the same formula for τ_M as in Theorem 9.1 does not hold.

Proposition 9.3. *If* $\big(r(M), \delta(M)\big) = (10, 0)$ *and* $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, *then*

$$J_M^o(\Omega_{M^\perp}^o) \subset \theta_{\mathrm{null}, g(M)}.$$

Proof. We prove that $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null}, g(M)}$ yields a contradiction. In what follows, assume $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null}, g(M)}$. Since $\delta(M^\perp) = 0$ and $r(M^\perp) = 12$,

$$\varphi := \Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}(2^{g(M)}+1)\ell} \otimes (J_M^* \chi_{g(M)}^{8\ell})^{2^{g(M)}-1}$$

is an automorphic form on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight $2^{g(M)-1}(2^{2g(M)} - 1)\ell(4, 4)$ by Theorem 8.1. Since $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null}, g(M)}$, we get $\varphi \not\equiv 0$. We can put $\nu = 2^{g(M)-1}(2^{g(M)} + 1)\ell$ in Theorem 5.1. Set $\psi := \varphi/\Phi_M^{2^{g(M)}-1}$. Since ψ is an $O^+(M^\perp)$-invariant meromorphic function on $\Omega^+_{M^\perp}$, we identify ψ with the corresponding meromorphic function on $\mathcal{M}^*_{M^\perp}$. We compute the divisor of ψ.

Since $\delta(M^{\perp}) = 0$ implies $\Delta''_{M^{\perp}} = \emptyset$, we get $\mathscr{D}''_{M^{\perp}} = \emptyset$. Since $r(M) = 10$ and $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, we get $g(M) > 0$ by Proposition 2.1. By Proposition 4.2 (2) and Theorem 8.1, we get

$$\begin{aligned}(9.6)\qquad \operatorname{div}(\varphi) &= 2^{g(M)-1}(2^{g(M)}+1)\ell\mathscr{D}'_{M^{\perp}} + (2^{g(M)}-1)\{2(2^{2g(M)-2}+a)\ell\mathscr{D}'_{M^{\perp}} + E\} \\ &= \{2^{g(M)-1}(2^{2g(M)}+1) + 2a(2^{g(M)}-1)\}\ell\mathscr{D}'_{M^{\perp}} + (2^{g(M)}-1)E.\end{aligned}$$

By Theorem 5.1, $\operatorname{div}(\Phi_M) = v\mathscr{D}'_{M^{\perp}} = 2^{g(M)-1}(2^{g(M)}+1)\ell\mathscr{D}'_{M^{\perp}}$, which, together with (9.6), yields that

$$\begin{aligned}(9.7)\qquad \operatorname{div}(\psi) &= \operatorname{div}(\varphi) - (2^{g(M)}-1)\operatorname{div}(\Phi_M) \\ &= \{2^{g(M)} + 2a(2^{g(M)}-1)\}\ell\mathscr{D}'_{M^{\perp}} + (2^{g(M)}-1)E.\end{aligned}$$

Since $\ell \geqq 1$, $a \geqq 0$ and since E is an effective divisor, $\operatorname{div}(\psi)$ is a *non-zero* and *effective* divisor on $\Omega^{+}_{M^{\perp}}$ by (9.7). This contradicts the fact that ψ descends to a meromorphic function on $\mathscr{M}^{*}_{M^{\perp}}$. □

When $2 \leqq g(M) \leqq 5$, one can verify Proposition 9.3 by using the explicit equations defining the corresponding log del Pezzo surfaces [44], pp. 494–495, Table 14.

Theorem 9.4. *If $M \cong \mathbb{A}_1^{+} \oplus \mathbb{A}_1^{\oplus 8}$, then there exists a constant $C_M > 0$ depending only on M such that for every 2-elementary K3 surface (X, ι) of type M,*

$$\tau_M(X,\iota)^{-40} = C_M \big\|\Psi_{M^{\perp}}\big(\varpi_M(X,\iota), F_{M^{\perp}}\big)\big\|^4 \big\|\chi_{g(M)}\big(\Omega(X^{\iota})\big)\big\|^{16}.$$

Proof. Since $M \cong \mathbb{A}_1^{+} \oplus \mathbb{A}_1^{\oplus 8}$, we get $M^{\perp} \cong \mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1$ by, e.g., [21], Appendix, Table 2. By (9.1) and Proposition 4.4, we get

$$\begin{aligned}(9.8)\qquad & dd^c\{-40\ell\log\tau_M - \log\|J_M^{*}\chi_2^{8\ell}\|^2\} \\ &= \ell\{-30\omega_{M^{\perp}} + 10\delta_{\mathscr{D}_{M^{\perp}}} - 8\delta_{\mathscr{D}'_{M^{\perp}}} - 16\delta_{\mathscr{H}_{M^{\perp}}(\mathbf{1}_{M^{\perp}}, -\frac{1}{2})}\} \\ &= \ell\{-30\omega_{M^{\perp}} + 2\delta_{\mathscr{D}'_{M^{\perp}}} + 10\delta_{\mathscr{D}''_{M^{\perp}}} - 16\delta_{\mathscr{H}_{M^{\perp}}(\mathbf{1}_{M^{\perp}}, -\frac{1}{2})}\}.\end{aligned}$$

By (9.8) and [62], Theorem 3.17, there is a meromorphic automorphic form φ_M on $\Omega^{+}_{M^{\perp}}$ for $O^{+}(M^{\perp})$ of weight 30ℓ with

$$(9.9)\qquad \operatorname{div}\varphi_M = \ell\left\{2\mathscr{D}'_{M^{\perp}} + 10\mathscr{D}''_{M^{\perp}} - 16\mathscr{H}_{M^{\perp}}\left(\mathbf{1}_{M^{\perp}}, -\frac{1}{2}\right)\right\}$$

such that

$$(9.10)\qquad 40\ell\log\tau_M + \log\|J_M^{*}\chi_2^{8\ell}\|^2 = -\log\|\varphi_M\|^2.$$

Since $O^{+}(M^{\perp})/[O^{+}(M^{\perp}), O^{+}(M^{\perp})]$ is a finite Abelian group, there exists $v \in \mathbf{Z}_{>0}$ such that φ_M^{v} and $\Psi_{M^{\perp}}(\cdot, F_{M^{\perp}})^{2v}$ are automorphic forms with trivial character. By Theorem 8.16

and (9.9), $\left(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M\right)^\nu$ is an $O^+(M^\perp)$-invariant meromorphic function on $\Omega^+_{M^\perp}$ with

$$(9.11)\qquad \operatorname{div}\left(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M\right)^\nu = \nu\ell\left\{2\mathscr{D}'_{M^\perp} + 10\mathscr{D}''_{M^\perp} - 16\mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp}, -\frac{1}{2}\right)\right\} - \nu\ell\left\{2\mathscr{D}'_{M^\perp} + 10\mathscr{D}''_{M^\perp} - 16\mathscr{H}_{M^\perp}\left(\mathbf{1}_{M^\perp}, -\frac{1}{2}\right)\right\} = 0.$$

Since $\operatorname{div}\left(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M\right)^\nu$ is empty, there exists a non-zero constant C_M with

$$\varphi_M = C_M^{\ell/2}\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}.$$

By (9.10), (9.11), we get the result. □

Question 9.5. Is $\operatorname{div}(J_M^*\chi_{g(M)}^{8\ell})$ a linear combination of Heegner divisors on $\Omega^+_{M^\perp}$? If it is the case and if $M^\perp \cong \mathbb{U}^{\oplus 2} \oplus K$ for some lattice K, $\Phi_M/J_M^*\chi_{g(M)}^{8\ell}$ will be expressed as a Borcherds product by [14], Theorem 0.8. Is there a Siegel modular form ψ on $\mathfrak{S}_{g(M)}$ such that $\operatorname{div}(J_M^*\psi)$ is a linear combination of Heegner divisors on $\Omega^+_{M^\perp}$?

10. Equivariant determinant of the Laplacian on real $K3$ surfaces

In this section, we give an explicit formula for the equivariant determinant of real $K3$ surfaces. We refer to [17], [64] for more details about real $K3$ surfaces.

The pair of a $K3$ surface and an anti-holomorphic involution is called a real $K3$ surface. Let (Y, σ) be a real $K3$ surface. There exists a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$ and a marking α of Y such that $\alpha\sigma^*\alpha^{-1} = I_M$. A holomorphic 2-form η on Y is said to be defined over $\mathbf{R}$ if $\sigma^*\eta = \bar{\eta}$. Let γ be a σ-invariant Ricci-flat Kähler metric on Y with volume 1. Let $\Delta_{(Y,\gamma)}$ be the Laplacian of (Y, γ). Since σ preserves γ, $\Delta_{(Y,\gamma)}$ commutes with the σ-action on $C^\infty(Y)$. We define $C^\infty_\pm(Y) := \{f \in C^\infty(Y); \sigma^* f = \pm f\}$, which are preserved by $\Delta_{(Y,\gamma)}$. We set $\Delta_{(Y,\gamma),\pm} := \Delta_{(Y,\gamma)}|_{C^\infty_\pm(Y)}$. Let $\zeta_\pm(Y, \gamma)(s)$ denote the spectral zeta function of $\Delta_{(Y,\gamma),\pm}$. Then it converges absolutely for $\operatorname{Re} s \gg 0$ and extends meromorphically to the complex plane $\mathbf{C}$, and it is holomorphic at $s = 0$. We define

$$\det{}^*_{\mathbf{Z}_2}\Delta_{(Y,\gamma)}(\sigma) := \exp[-\zeta'_+(Y,\gamma)(0) + \zeta'_-(Y,\gamma)(0)].$$

Let $Y(\mathbf{R}) := \{y \in Y; \sigma(y) = y\}$ be the set of real points of (Y, σ) and let $Y(\mathbf{R}) = \coprod_i C_i$ be the decomposition into the connected components. Then $Y(\mathbf{R})$ is the disjoint union of oriented two-dimensional manifolds. The Riemannian metric $g|_{Y(\mathbf{R})}$ induces a complex structure on $Y(\mathbf{R})$. The period of $Y(\mathbf{R})$ with respect to this complex structure is denoted by $\Omega\left(Y(\mathbf{R}), \gamma|_{Y(\mathbf{R})}\right)$. Let $\Delta_{(C_i,\gamma|_{C_i})}$ be the Laplacian of the Riemannian manifold $(C_i, \gamma|_{C_i})$ and let $\zeta(C_i, \gamma|_{C_i})(s)$ denote the spectral zeta function of $\Delta_{(C_i,\gamma|_{C_i})}$. The regularized determinant of $\Delta_{(C_i,\gamma|_{C_i})}$ is defined as

$$\det{}^*\Delta_{(C_i,\gamma|_{C_i})} := \exp[-\zeta(C_i,\gamma|_{C_i})'(0)].$$

After [64], Definition 4.4, we define

$$\tau(Y,\sigma,\gamma) := \{\det^*_{\mathbf{Z}_2}\Delta_{(Y,\gamma)}(\sigma)\}^{-2}\prod_i \mathrm{Vol}(C_i,\gamma|_{C_i})(\det{}^*\Delta_{(C_i,\gamma|_{C_i})})^{-1}.$$

Theorem 10.1. *Let (Y,σ) be a real K3 surface and let α be a marking of Y such that $\alpha\sigma^*\alpha^{-1} = I_M$. Let γ be a σ-invariant Ricci-flat Kähler metric on Y with volume 1. Let ω_γ be the Kähler form of γ, and let η_γ be a holomorphic 2-form on Y defined over $\mathbf{R}$ such that $\eta_\gamma\wedge\bar\eta_\gamma = 2\omega_\gamma^2$. If $r(M) > 10$ or $\big(r(M),\delta(M)\big) = (10,1)$, then the following identity holds*:

$$-4(2^{g(M)}+1)\log\tau(Y,\sigma,\gamma) = \log\big\|\Psi_{M^\perp}\big(\alpha(\omega_\gamma + i\,\mathrm{Im}\,\eta_\gamma),F_{M^\perp}\big)\big\|^2 + 2^{(4-g(M))}\log\big\|\chi_{g(M)}\big(\Omega\big(Y(\mathbf{R}),\gamma|_{Y(\mathbf{R})}\big)\big)\big\|^2 + C'_M,$$

where $C'_M = 2\log C_M$ and ω_γ, η_γ are identified with their cohomology classes.

Proof. The result follows from Theorem 9.1 and [64], Lemma 4.5, Eq. (4.6). □

11. Appendix

In this section, we prove some technical results about lattices.

11.1. A proof of the equality $\Gamma_M = O(M^\perp)$. Let M be a primitive sublattice of $\mathbb{L}_{K3}$ and set $H_M := \mathbb{L}_{K3}/(M\oplus M^\perp)$. Since $\mathbb{L}_{K3}$ is unimodular, we get

$$M\oplus M^\perp \subset \mathbb{L}_{K3} = \mathbb{L}_{K3}^\vee \subset M^\vee\oplus(M^\perp)^\vee,$$

so that $H_M \subset A_M\oplus A_{M^\perp}$. Let $p_1 : H_M \to A_M$ and $p_2 : H_M \to A_{M^\perp}$ be the homomorphism induced by the projections $A_M\oplus A_{M^\perp}\to A_M$ and $A_M\oplus A_{M^\perp}\to A_{M^\perp}$, respectively. By [46], Propositions 1.5.1 and 1.6.1, the following hold:

(a) p_1 and p_2 are isomorphisms.

(b) $A_M \cong A_{M^\perp}$ via the isomorphism $\gamma^{\mathbb{L}_{K3}}_{M,M^\perp} := p_2\circ p_1^{-1}$.

(c) $q_{M^\perp}\circ\gamma^{\mathbb{L}_{K3}}_{M,M^\perp} = -q_M$.

Recall that $g\in O(M^\perp)$ induces $\bar g\in O(q_{M^\perp})$. For $g\in O(M^\perp)$, we set

$$\psi_g := (\gamma^{\mathbb{L}_{K3}}_{M,M^\perp})^{-1}\circ\bar g\circ\gamma^{\mathbb{L}_{K3}}_{M,M^\perp}.$$

Then $\psi_g\in\mathrm{Aut}(A_M)$.

Lemma 11.1. *The automorphism ψ_g preserves q_M, i.e., $\psi_g\in O(q_M)$.*

Proof. The result follows from condition (c) and the fact that $\bar g\in O(q_{M^\perp})$. □

Assume that $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice. Recall that the isometry $I_M \in O(\mathbb{L}_{K3})$ was defined in Section 1.2. In [62], Section 1.4 (c), we introduced the following subgroup $\Gamma_M \subset O(M^{\perp})$:

$$\Gamma_M := \{g|_{M^{\perp}} \in O(M^{\perp}); g \in O(\mathbb{L}_{K3}), g \circ I_M = I_M \circ g\}.$$

Proposition 11.2. *The following equality holds*:

$$\Gamma_M = O(M^{\perp}).$$

Proof. By the definition of Γ_M, it suffices to prove $O(M^{\perp}) \subset \Gamma_M$. Let $g \in O(M^{\perp})$ be an arbitrary element. Since M is 2-elementary and indefinite, the natural homomorphism $O(M) \to O(q_M)$ is surjective by [46], Theorem 3.6.3, which implies the existence of $\Psi_g \in O(M)$ with $\psi_g = \overline{\Psi_g}$. Define $\tilde{g} := \Psi_g \oplus g \in O(M \oplus M^{\perp})$. Then

$$\gamma^{\mathbb{L}_{K3}}_{M,M^{\perp}} \circ \overline{\Psi_g} = \gamma^{\mathbb{L}_{K3}}_{M,M^{\perp}} \circ \psi_g = \bar{g} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^{\perp}}. \tag{11.1}$$

By (11.1) and the criterion of Nikulin [46], Corollary 1.5.2, we get $\tilde{g} \in O(\mathbb{L}_{K3})$. We have $\tilde{g} \circ I_M = I_M \circ \tilde{g}$ on $M \oplus M^{\perp}$ because for all $(m,n) \in M \oplus M^{\perp}$,

$$\tilde{g} \circ I_M(m,n) = \tilde{g}(m,-n) = \big(\Psi_g(m), -g(n)\big) = I_M\big(\Psi_g(m), g(n)\big) = I_M \circ \tilde{g}(m,n).$$

Since $M \oplus M^{\perp}$ linearly spans $\mathbb{L}_{K3} \otimes \mathbf{Q}$, we have $\tilde{g} \circ I_M = I_M \circ \tilde{g}$ in $O(\mathbb{L}_{K3})$. Hence $\tilde{g} \in \Gamma_M$. This proves the inclusion $O(M^{\perp}) \subset \Gamma_M$. □

11.2. A formula for $g([M \perp d])$.

Lemma 11.3. *Let $d \in \Delta_{M^{\perp}}$. The smallest primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ containing $M \oplus \mathbf{Z}d$ is given by $[M \perp d] = (M^{\perp} \cap d^{\perp})^{\perp}$.*

Proof. Set $L := \mathbf{Z}d \cong \mathbb{A}_1$. Then $[M \perp d]$ is the smallest primitive Lorentzian sublattice of $\mathbb{L}_{K3}$ containing $M \oplus L$. Since $M \oplus L \subset [M \perp d] \subset [M \perp d]^{\vee} \subset M^{\vee} \oplus L^{\vee}$ and hence $[M \perp d]/(M \oplus L) \subset [M \perp d]^{\vee}/(M \oplus L) \subset A_M \oplus A_L \cong \mathbf{Z}_2^{l(M)+1}$, we have that $A_{[M \perp d]} = [M \perp d]^{\vee}/[M \perp d]$ is a vector space over $\mathbf{Z}_2$. Hence $[M \perp d]$ is 2-elementary. □

Lemma 11.4. *Let $d \in \Delta_{M^{\perp}}$. Then*

$$l([M \perp d]) = l(M^{\perp} \cap d^{\perp}) = \begin{cases} l(M^{\perp}) + 1 & \text{if } d \in \Delta'_{M^{\perp}}, \\ l(M^{\perp}) - 1 & \text{if } d \in \Delta''_{M^{\perp}}. \end{cases}$$

Proof. See [21], Proposition 3.1. □

Lemma 11.5. *Let $d \in \Delta_{M^{\perp}}$. Then*

$$g([M \perp d]) = \begin{cases} g(M) - 1 & \text{if } d \in \Delta'_{M^{\perp}}, \\ g(M) & \text{if } d \in \Delta''_{M^{\perp}}. \end{cases}$$

Proof. Since $r(M^{\perp} \cap d^{\perp}) = r(M^{\perp}) - 1$ and

$$g(M) = \{r(M^{\perp}) - l(M^{\perp})\}/2, \quad g([M \perp d]) = \{r(M^{\perp} \cap d^{\perp}) - l(M^{\perp} \cap d^{\perp})\}/2,$$

the result follows from Lemma 11.4. □

11.3. The $K3$-graph. In [21], Finashin and Kharlamov introduced the notion of the lattice graph Γ_L for an even unimodular lattice L. When $L = \mathbb{L}_{K3}$, the $K3$-graph $\Gamma_{K3} = \Gamma_{\mathbb{L}_{K3}}$ is defined as follows (cf. [21], Section 3):

The set of vertices of Γ_{K3}, denoted by V_{K3}, consists of the isometry classes of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$. For a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$, write $[M] \in V_{K3}$ for its isometry class. We identify $[M]$ with the triplet $(r(M), l(M), \delta(M))$. The vertex $[M] \in V_{K3}$ is even (resp. odd) if $\delta(M) = 0$ (resp. $\delta(M) = 1$). In [21], an even (resp. odd) vertex is said to be of type I (resp. type II). The set V_{K3} was determined by Nikulin [46], [48].

The set of oriented edges of Γ_{K3}, denoted by E_{K3}, consists of the $O(\mathbb{L}_{K3})$-orbits of the pairs $(M, [d])$, where M is a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ and $[d] \in \Delta_{M^{\perp}}/O(M^{\perp})$. The oriented edge represented by $(M, [d])$ is denoted by $[(M, [d])]$. Then $[(M, [d])]$ connects the vertices $[M]$ and $[M \perp d]$ with arrow starting from $[M]$ to $[M \perp d]$. By identifying $(M, [d])$ with the divisor $\overline{H}_d = O(M^{\perp}) \cdot H_d \subset \overline{\mathscr{D}}_{M^{\perp}}$, there is a bijection between the following sets:

(i) The edges of Γ_{K3} starting from $[M]$.

(ii) The irreducible components of $\overline{\mathscr{D}}_{M^{\perp}}$.

By the equivalence of (i) and (ii), two vertices $[M], [M'] \in \Gamma_{K3}$ are connected by an oriented edge of Γ_{K3} going from $[M]$ to $[M']$ if and only if there exist $\gamma \in O(\mathbb{L}_{K3})$ and $d \in \Delta_{M^{\perp}}$ such that $\mathscr{M}_{\gamma(M')^{\perp}}$ is an irreducible component of $\overline{\mathscr{D}}_{M^{\perp}}$.

An edge $[(M, [d])]$ with $[d] \in \Delta'_{M^{\perp}}/O(M^{\perp})$ is called an odd edge. An edge $[(M, [d])]$ with $[d] \in \Delta''_{M^{\perp}}/O(M^{\perp})$ is called an even edge. If an even edge $[(M, [d])]$ satisfies $\delta(d^{\perp} \cap M^{\perp}) = 0$, then $[(M, [d])]$ is called an even Wu edge. If $\delta(d^{\perp} \cap M^{\perp}) = 1$, $[(M, [d])]$ is called an even non-Wu edge. The set E_{K3} was determined by Finashin–Kharlamov [21]. See [21], p. 694, Figure 1, for the $K3$-graph Γ_{K3}.

Proposition 11.6. *The following hold*:

(1) $(M, [d]) = (M, [d'])$ *in* E_{K3} *if and only if* $[M \perp d] = [M \perp d']$ *in* V_{K3}.

(2) *If* $[(M, [d])] \in E_{K3}$ *is odd, then*

$$\big(r([M \perp d]), l([M \perp d]), \delta([M \perp d])\big) = \big(r(M) + 1, l(M) + 1, 1\big).$$

(3) *If* $[(M, [d])] \in E_{K3}$ *is even Wu, then*

$$\big(r([M \perp d]), l([M \perp d]), \delta([M \perp d])\big) = \big(r(M) + 1, l(M) - 1, 0\big).$$

(4) *If* $[(M,[d])] \in E_{K3}$ *is even non-Wu, then*

$$\big(r([M \perp d]), l([M \perp d]), \delta([M \perp d])\big) = \big(r(M)+1, l(M)-1, 1\big).$$

(5) Γ_{K3} *contains no multiple edges. In particular,* $\#[\Delta'_{M^\perp}/O(M^\perp)] \leqq 1$ *and* $\#[\Delta''_{M^\perp}/O(M^\perp)] \leqq 2$.

Proof. The proof can be found in [21], Section 3. For the sake of completeness, we give it here.

We get (1) by [21], Proposition 3.3. When $[(M,[d])] \in E_{K3}$ is odd, the equality $r([M \perp d]) = r(M)+1$ is trivial, the equality $l([M \perp d]) = l(M)+1$ follows from [21], Proposition 3.1, and the equality $\delta([M \perp d]) = 1$ follows from [21], proof of Proposition 3.3, because $A_{\langle 2\rangle}$ is a direct summand of $A_{[M \perp d]}$. This proves (2). When $[(M,[d])] \in E_{K3}$ is even Wu (resp. non-Wu), the equality $r([M \perp d]) = r(M)+1$ is trivial, the equality $l([M \perp d]) = l(M)-1$ follows from [21], Proposition 3.1, and the equality $\delta([M \perp d]) = 0$ (resp. $\delta([M \perp d]) = 1$) follows from the definition of a Wu (resp. non-Wu) edge. This proves (3) and (4). We get (5) by (1), (2), (3), (4). □

11.4. The irreducibility of the boundary locus: the case $r(\Lambda) \leqq 4$. The following was used in the proof of Theorem 9.1.

Proposition 11.7. *If* $r(\Lambda) \leqq 4$, *then* $B_\Lambda = \mathscr{M}_\Lambda^* \setminus \mathscr{M}_\Lambda$ *is irreducible.*

Proof. When $r(\Lambda) = 2$, we get $\Lambda \cong (\mathbb{A}_1^+)^{\oplus 2}$ and $B_\Lambda = \emptyset$. Assume $r(\Lambda) \geqq 3$. Let $I_{\max}(\Lambda)$ be the set of maximal primitive isotropic sublattices of Λ. The number of the irreducible components of B_Λ of maximal dimension is given by $\#[I_{\max}(\Lambda)/O(\Lambda)]$ (cf. [53], Section 2.1). We must prove that when $3 \leqq r(\Lambda) \leqq 4$,

$$\#[I_{\max}(\Lambda)/O(\Lambda)] \leqq 1. \tag{11.2}$$

Since $3 \leqq r(\Lambda) \leqq 4$, Λ is one of the following 7 lattices (cf. Table 1):

$$\mathbb{U}(k)^{\oplus 2} (k=1,2), \quad (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus l}\ (l=1,2), \quad \mathbb{U} \oplus \mathbb{U}(2), \quad \mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1, \quad \mathbb{U} \oplus \mathbb{A}_1^+.$$

Case 1. Assume that $\Lambda = \mathbb{U}(k) \oplus \mathbb{U}(k)$ $(k \leqq 2)$ or $\Lambda = (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus l}$ $(l=1,2)$. Since there exist an indefinite unimodular lattice Λ' and $k \in \mathbf{Z}_{>0}$ with $\Lambda = \Lambda'(k)$, we get (11.2) by [46], Proposition 1.17.1.

Case 2. Assume that $\Lambda = \mathbb{U} \oplus \mathbb{A}_1^+$. There exist isomorphisms $\Omega_\Lambda^+ \cong \mathfrak{H}$ and $O^+(\Lambda) \cong \mathrm{SL}_2(\mathbf{Z})$ such that the $O^+(\Lambda)$-action on Ω_Λ^+ is identified with the projective action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$ via these isomorphisms (cf. [17], Theorem 7.1). Hence $\mathscr{M}_\Lambda \cong \mathrm{SL}_2(\mathbf{Z}) \setminus \mathfrak{H} \cong \mathbf{C}$ and $\mathscr{M}_\Lambda^* \setminus \mathscr{M}_\Lambda = \{+i\infty\}$, which implies (11.2) in this case.

Case 3. Assume that $\Lambda = \mathbb{U} \oplus \mathbb{U}(2)$ or $\Lambda = \mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1$. Let $L \subset \Lambda$ be a primitive isotropic sublattice of rank 2. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be a basis of L. Extending this basis of L, we get a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of Λ with Gram matrix G as follows:

$$G = (\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle)_{1 \leqq i,j \leqq 4} = \begin{pmatrix} O & A \\ A & B \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B \in M_2(\mathbf{Z}), \quad {}^tB = B.$$

Set $e_3' := e_3 + \alpha e_1 + \beta e_2$ and $e_4' := e_4 + \gamma e_1 + \delta e_2$, where $\alpha, \beta, \gamma, \delta \in \mathbf{Z}$. The Gram matrix of Λ with respect to $\{e_1, e_2, e_3', e_4'\}$ is given by

$$G' = \begin{pmatrix} O & A \\ A & B + C \end{pmatrix}, \quad C = \begin{pmatrix} 2\alpha & \gamma + 2\beta \\ \gamma + 2\beta & 4\delta \end{pmatrix}.$$

Since Λ is even, we can write $\langle e_4, e_4 \rangle = 4k$ or $4k+2$. We set $\alpha := -\langle e_3, e_3 \rangle / 2$, $\beta := 0$, $\gamma := -\langle e_3, e_4 \rangle$ and $\delta := -2k$. Then we get $B + C = O$ if $\delta(\Lambda) = 0$ (i.e., $\langle e_4, e_4 \rangle \equiv 0 \bmod 4$) and $B + C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ if $\delta(\Lambda) = 1$ (i.e., $\langle e_4, e_4 \rangle \equiv 2 \bmod 4$). This proves the existence of a basis $\{e_{1,L}, e_{2,L}, e_{3,L}, e_{4,L}\}$ of Λ with $L = \mathbf{Z}e_{1,L} + \mathbf{Z}e_{2,L}$, such that the Gram matrix of Λ with respect to this basis is of the form $\begin{pmatrix} O & A \\ A & B \end{pmatrix}$. Here $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = O$ if $\delta(\Lambda) = 0$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ if $\delta(\Lambda) = 1$. If $L' \subset \Lambda$ is another primitive isotropic sublattice of rank 2, then we get an isometry of Λ sending L to L' by identifying the basis $\{e_{1,L}, e_{2,L}, e_{3,L}, e_{4,L}\}$ and $\{e_{1,L'}, e_{2,L'}, e_{3,L'}, e_{4,L'}\}$ via the map $e_{i,L} \mapsto e_{i,L'}$. This proves (11.2). $\square$

References

[1] *V. Alexeev* and *V. V. Nikulin*, Del Pezzo and $K3$ surfaces, MSJ Mem. **15**, Mathematical Society Japan, 2006.

[2] *E. Arbarello*, *M. Cornalba*, *P. Griffiths* and *J. Harris*, Geometry of algebraic curves, vol. I, Springer, Berlin 1985.

[3] *W. L. Baily* and *A. Borel*, Compactification of arithmetic quotients of bounded symmetric domains, Ann. of Math. **84** (1966), 442–528.

[4] *W. Barth*, *C. Peters*, *A. Van de Ven*, Compact complex surfaces, Springer, Berlin 1984.

[5] *J.-M. Bismut*, Equivariant immersions and Quillen metrics, J. Differential Geom. **41** (1995), 53–157.

[6] *J.-M. Bismut*, *H. Gillet* and *C. Soulé*, Analytic torsion and holomorphic determinant bundles, I, II, III, Commun. Math. Phys. **115** (1988), 49–78, 79–126, 301–351.

[7] *J.-M. Bismut* and *G. Lebeau*, Complex immersions and Quillen metrics, Publ. Math. IHES **74** (1991), 1–297.

[8] *R. E. Borcherds*, Automorphic forms on $O_{s+2,s}(\mathbf{R})$ and infinite products, Invent. Math. **120** (1995), 161–213.

[9] *R. E. Borcherds*, The moduli space of Enriques surfaces and the fake monster Lie superalgebra, Topology **35** (1996), 699–711.

[10] *R. E. Borcherds*, Automorphic forms with singularities on Grassmanians, Invent. Math. **132** (1998), 491–562.

[11] *R. E. Borcherds*, Reflection groups of Lorentzian lattices, Duke Math. J. **104** (2000), 319–366.

[12] *R. E. Borcherds*, *L. Katzarkov*, *T. Pantev* and *N. I. Shepherd-Barron*, Families of $K3$ surfaces, J. Algebraic Geom. **7** (1998), 183–193.

[13] *A. Borel*, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, J. Differential Geom. **6** (1972), 543–560.

[14] *J. H. Bruinier*, Borcherds products on $O(2,l)$ and Chern classes of Heegner divisors, Lect. Notes Math. **1780**, Springer, Berlin 2002.

[15] *D. Burns* and *M. Rapoport*, On the Torelli problems for Kählerian $K3$ surfaces, Ann. Sci. Éc. Norm. Supér. (IV) **8** (1975), 235–274.

[16] *J. H. Conway* and *N. J. A. Sloan*, Sphere packings, lattices and groups, Springer, Berlin 1999.

[17] *A. Degtyarev*, *I. Itenberg* and *V. Kharlamov*, Real Enriques surfaces, Lect. Notes Math. **1746**, Springer, Berlin 2000.

[18] *I. Dolgachev*, Mirror symmetry for lattice polarized $K3$ surfaces, J. Math. Sci. **81** (1996), 2599–2630.

[19] *H. Fang*, *Z. Lu* and *K.-I. Yoshikawa*, Analytic torsion for Calabi–Yau threefolds, J. Differential Geom. **80** (2008), 175–250.

[20] *J. D. Fay*, Theta functions on Riemann surfaces, Lect. Notes Math. **352**, Springer, Berlin 1973.

[21] *S. Finashin* and *V. Kharlamov*, Deformation classes of real four-dimensional cubic hypersurfaces, J. Algebraic Geom. **17** (2008), 677–707.

[22] *E. Freitag* and *C. F. Hermann*, Some modular varieties of low dimension, Adv. Math. **152** (2000), 203–287.

[23] *E. Freitag* and *R. Salvati Manni*, Modular forms for the even unimodular lattice of signature (2, 10), J. Algebraic Geom. **16** (2007), 753–791.
[24] *H. Gillet* and *C. Soulé*, An arithmetic Riemann–Roch theorem, Invent. Math. **110** (1992), 473–543.
[25] *P. Griffiths* and *J. Harris*, Principles of algebraic geometry, Wiley, New York 1978.
[26] *V. A. Gritsenko*, Reflective modular forms in algebraic geometry, preprint 2010, http://arxiv.org/abs/1005.3753v1.
[27] *V. A. Gritsenko* and *V. V. Nikulin*, Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras, Amer. J. Math. **119** (1997), 181–224.
[28] *V. A. Gritsenko* and *V. V. Nikulin*, Automorphic forms and Lorentzian Kac–Moody algebras, I, II, Internat. J. Math. **9** (1998), 153–199, 201–275.
[29] *R. Hartshorne*, Ample subvarieties of algebraic varieties, Lect. Notes Math. **156**, Springer, Berlin 1970.
[30] *J. Harvey* and *G. Moore*, Algebras, BPS states, and strings, Nuclear Phys. B **463** (1996), 315–368.
[31] *J.-I. Igusa*, Modular forms and projective invariants, Amer. J. Math. **89** (1967), 817–855.
[32] *K. Köhler* and *D. Roessler*, A fixed point formula of Lefschetz type in Arakelov geometry, I, Invent. Math. **145** (2001), 333–396.
[33] *S. Kondō*, The moduli space of Enriques surfaces and Borcherds products, J. Algebraic Geom. **11** (2002), 601–627.
[34] *S. Kondō*, The moduli space of 8 points on $\mathbf{P}^1$ and automorphic forms, in: Algebraic geometry, Contemporary Math. **422**, American Mathematical Society, Providence (2007), 89–106.
[35] *J. Lewittes*, Riemann surfaces and the theta function, Acta Math. **111** (1964), 37–61.
[36] *E. Looijenga*, Compactifications defined by arrangements, II, Duke Math. J. **119** (2003), 527–588.
[37] *S. Ma*, The unirationality of the moduli spaces of 2-elementary $K3$ surfaces, preprint 2010, http://arxiv.org/abs/1011.1963v1.
[38] *X. Ma*, Submersions and equivariant Quillen metrics, Ann. Inst. Fourier **50** (2000), 1539–1588.
[39] *V. Maillot* and *D. Rössler*, Formes automorphes et théorèmes de Riemann–Roch arithmétiques, in: From probability to geometry, II, Astérisque **328** (2009), 233–249.
[40] *A. Mayer*, Special divisors and the Jacobian variety, Math. Ann. **153** (1964), 163–167.
[41] *D. Mumford*, On the Kodaira dimension of the Siegel modular variety, in: Algebraic geometry—open problems, Lect. Notes Math. **997**, Springer, Berlin (1983), 348–375.
[42] *D. Mumford*, Tata lectures on theta, I, Progress in Math. **28**, Birkhäuser, Boston 1983.
[43] *D. Mumford*, Tata lectures on theta, II, Progress in Math. **43**, Birkhäuser, Boston 1984.
[44] *N. Nakayama*, Classification of log del Pezzo surfaces of index two, J. Math. Sci. Univ. Tokyo **14** (2007), 293–498.
[45] *Y. Namikawa*, Periods of Enriques surfaces, Math. Ann. **270** (1985), 201–222.
[46] *V. V. Nikulin*, Integral symmetric bilinear forms and some of their applications, Math. USSR Izv. **14** (1980), 103–167.
[47] *V. V. Nikulin*, Finite automorphism groups of Kähler $K3$ surfaces, Trans. Moscow Math. Soc. **38** (1980), 71–135.
[48] *V. V. Nikulin*, Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections, J. Soviet Math. **22** (1983), 1401–1476.
[49] *V. V. Nikulin*, A remark on discriminants for moduli of $K3$ surfaces as sets of zeros of automorphic forms, J. Math. Sci. **81** (1996), 2738–2743.
[50] *G. Pappas*, Grothendieck–Riemann–Roch and the moduli of Enriques surfaces, Math. Res. Lett. **15** (2008), 117–120.
[51] *I. Piatetskii-Shapiro* and *I. R. Shafarevich*, A Torelli theorem for algebraic surfaces of type $K3$, Math. USSR Izv. **35** (1971), 530–572.
[52] *D. B. Ray* and *I. M. Singer*, Analytic torsion for complex manifolds, Ann. of Math. **98** (1973), 154–177.
[53] *F. Scattone*, On the compactification of moduli spaces for algebraic $K3$ surfaces, Mem. Am. Math. Soc. **374** (1987).
[54] *N. Scheithauer*, The fake monster superalgebras, Adv. Math. **151** (2000), 226–269.
[55] *N. Scheithauer*, On the classification of automorphic products and generalized Kac–Moody algebras, Invent. Math. **164** (2006), 641–678.
[56] *Y.-T. Siu*, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math. **27** (1974), 53–156.
[57] *Y.-T. Siu*, Extension of meromorphic maps into Kähler manifolds, Ann. of Math. **102** (1975), 421–462.
[58] *R. Smith* and *R. Varley*, Components of the locus of singular theta divisors of genus 5, in: Algebraic geometry, Lect. Notes Math. **1124**, Springer, Berlin (1985), 338–416.
[59] *M. Yoshida*, Hypergeometric functions, my love, Vieweg, Braunschweig 1997.

[60] *K.-I. Yoshikawa*, $K3$ surfaces with involution and analytic torsion, preprint 1999, math.AG/9808129v3, unpublished version of [62].

[61] *K.-I. Yoshikawa*, Discriminant of theta divisors and Quillen metrics, J. Differential Geom. **52** (1999), 73–115.

[62] *K.-I. Yoshikawa*, $K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space, Invent. Math. **156** (2004), 53–117.

[63] *K.-I. Yoshikawa*, Discriminant of certain $K3$ surfaces, in: Representation theory and automorphic forms, Progress in Math. **255**, Birkhäuser, Boston (2007), 175–210.

[64] *K.-I. Yoshikawa*, Real $K3$ surfaces without real points, equivariant determinant of the Laplacian, and the Borcherds Φ-function, Math. Z. **258** (2008), 213–225.

[65] *K.-I. Yoshikawa*, Calabi–Yau threefolds of Borcea–Voisin, analytic torsion, and Borcherds products, in: From probability to geometry, II, Astérisque **328** (2009), 351–389.

[66] *K.-I. Yoshikawa*, $K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space, III: the case $r(M) \geqq 18$, Math. Z. DOI:10.1007/s00209-011-0928-1.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
Korea Institute for Advanced Study, Hoegiro 87, Dongdaemun-gu, Seoul 130-722, Republic of Korea
e-mail: yosikawa@math.kyoto-u.ac.jp

Eingegangen 24. August 2010, in revidierter Fassung 7. Juli 2011