QUIVER VARIETIES AND TENSOR PRODUCTS, II

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Abstract. We define a family of homomorphisms on a collection of convolution algebras associated with quiver varieties, which gives a kind of coproduct on the Yangian associated with a symmetric Kac-Moody Lie algebra. We study its property using perverse sheaves.

Contents

Introduction

In the conference the author explained his joint work with Guay on a construction of a coproduct on the Yangian $Y(\mathfrak{g})$ associated with an affine Kac-Moody Lie algebra $\mathfrak{g}$. It is a natural generalization of the coproduct on the usual Yangian $Y(\mathfrak{g})$ for a finite dimensional complex simple Lie algebra $\mathfrak{g}$ given by Drinfeld [7]. Its definition is motivated also by a recent work of Maulik and Okounkov [13] on a geometric construction of a tensor product structure on equivariant homology groups of holomorphic symplectic varieties, in particular of quiver varieties. The purpose of this paper is to explain this geometric background.

For quiver varieties of finite type, the geometric coproduct corresponding to the Drinfeld coproduct on Yangian $Y(\mathfrak{g})$, or more precisely the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, was studied in [22, 18, 23]. (And one

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corresponding to the coproduct on $\mathfrak{g}$ was studied also in [12].) But the results depend on the algebraic definition of the coproduct. As it is not known how to define a coproduct on $Y(\mathfrak{g})$ for an arbitrary Kac-Moody Lie algebra $\mathfrak{g}$, the results cannot be generalized to other types.

In this paper, we take a geometric approach and define a kind of a coproduct on convolution algebras associated with quiver varieties together with a $\mathbb{C}^*$-action preserving the holomorphic symplectic form, and study its properties using perverse sheaves.

In fact, we have an ambiguity in the definition of the coproduct, and we have a family of coproducts Δ_c, parametrized by c in a certain affine space. This ambiguity of the coproduct was already noticed in [23, Remark in §5.2]. Maulik-Okounkov theory gives a canonical choice of c for a quiver variety of an arbitrary type, and gives the formula of Δ_c on standard generators of $Y(\mathfrak{g})$. Therefore we can take the formula as a definition of the coproduct and check its compatibility with the defining relations of $Y(\mathfrak{g})$. This will be done for an affine Kac-Moody Lie algebra $\mathfrak{g}$ as we explained in the conference. (The formula is a consequence of results in [13], and hence is not explained here.)

Although there is a natural choice, the author hopes that our framework, considering also other possibilities for Δ, is suitable for a modification to other examples of convolution algebras when geometry does not give us such a canonical choice. (For example, the AGT conjecture for a general group. See [21].)

Remark also that our construction is specific for $Y(\mathfrak{g})$, and is not clear how to apply for a quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$. We need to replace cohomology groups by K groups to deal with the latter, but many of our arguments work only for cohomology groups.

Finally let us comment on a difference on the coproduct for quiver varieties of finite type and other types. A coproduct on an algebra A usually means an algebra homomorphism $\Delta\colon A \to A \otimes A$ satisfying the coassociativity. In our setting the algebra A depends on the dimension vector, or equivalently dominant weight $\mathbf{w}$. Hence Δ is supposed to be a homomorphism from the algebra $A(\mathbf{w})$ for $\mathbf{w}$ to the tensor product $A(\mathbf{w}^1) \otimes A(\mathbf{w}^2)$ with $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2$. For a quiver of type ADE, this is true, but not in general. See Remark 2.4 for the crucial point. The target of Δ is, in general, larger than $A(\mathbf{w}^1) \otimes A(\mathbf{w}^2)$. Fortunately this difference is not essential, for example, study of tensor product structures of representations of Yangians.

Notations. The definition and notation of quiver varieties related to a coproduct are as in [18], except the followings:

- Linear maps i, j are denoted by a, b here.

- A quiver possibly contains edge loops. Roots are defined as in [6, §2]. They are obtained from coordinate vectors at loop free vertices or $\pm$ elements in the fundamental region by applying some sequences of reflections at loop free vertices.
- Varieties $\mathfrak{Z}$, $\widetilde{\mathfrak{Z}}$ are denote by $\mathfrak{T}$, $\widetilde{\mathfrak{T}}$ here.

We say a quiver is of *finite type*, if its underlying graph is of type ADE. We way it is of *affine type*, if it is Jordan quiver or its underlying graph is an extended Dynkin diagram of type ADE.

For $\mathbf{v} = (v_i)$, $\mathbf{v}' = (v_i') \in \mathbb{Z}^I$, we say $\mathbf{v} \le \mathbf{v}'$ if $v_i \le v_i'$ for any $i \in I$.

For a variety X, $H_*(X)$ denote its Borel-Moore homology group. It is the dual to $H_c^*(X)$ the cohomology group with compact support.

We will use the homology group $H_*(L)$ of a closed variety L in a smooth variety M in several contexts. There is often a preferred degree in the context, which is written as 'top' below. For example, if L is lagrangian, it is $\dim_{\mathbb{C}} M$. If M has several components M_α of various dimensions, we mean $H_{\mathrm{top}}(L)$ to be the direct sum of $H_{\mathrm{top}}(L \cap M_\alpha)$, though the degree 'top' changes for each $L \cap M_\alpha$.

Let $D(X)$ denote the bounded derived category of complexes of constructible $\mathbb{C}$-sheaves on X. When X is smooth, $\mathcal{C}_X \in D(X)$ denote the constant sheaf on X shifted by $\dim X$. If X is a disjoint union of smooth varieties X_α with various dimensions, we understand $\mathcal{C}_X$ as the direct sum of $\mathcal{C}_{X_\alpha}$.

The intersection cohomology (IC for short) complex associated with a smooth locally closed subvariety $Y \subset X$ and a local system ρ on Y is denoted by $IC(Y, \rho)$ or $IC(\overline{Y}, \rho)$. If ρ is the trivial rank 1 local system, we simply denote it by $IC(Y)$ or $IC(\overline{Y})$.

1. Quiver varieties

In this section we fix the notation for quiver varieties. See [14, 15] for detail.

Suppose that a finite graph is given. Let I be the set of vertices and E the set of edges. In [14, 15] the author assumed that the graph does not contain edge loops (i.e., no edges joining a vertex with itself), but most of results (in particular definitions, natural morphisms, etc) hold without this assumption.

Let H be the set of pairs consisting of an edge together with its orientation. So we have $\#H = 2\#E$. For $h \in H$, we denote by $\mathrm{i}(h)$ (resp. $\mathrm{o}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote by $\overline{h}$ the same edge as h with the reverse orientation. Choose and fix an orientation Ω of the graph, i.e., a subset $\Omega \subset H$ such that $\overline{\Omega} \cup \Omega = H$, $\Omega \cap \overline{\Omega} = \emptyset$. The pair (I, Ω) is called a *quiver*.

Let $V = (V_i)_{i\in I}$ be a finite dimensional I-graded vector space over $\mathbb{C}$. The dimension of V is a vector

$$\dim V = (\dim V_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}.$$

If V^1 and V^2 are I-graded vector spaces, we define vector spaces by

$$\mathrm{L}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{i\in I} \operatorname{Hom}(V^1_i, V^2_i), \quad \mathrm{E}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{h\in H} \operatorname{Hom}(V^1_{\mathrm{o}(h)}, V^2_{\mathrm{i}(h)}).$$

For $B = (B_h) \in \mathrm{E}(V^1, V^2)$ and $C = (C_h) \in \mathrm{E}(V^2, V^3)$, let us define a multiplication of B and C by

$$CB \overset{\text{def.}}{=} \left(\sum_{\mathrm{i}(h)=i} C_h B_{\overline{h}} \right)_i \in \mathrm{L}(V^1, V^3).$$

Multiplications ba, Ba of $a \in \mathrm{L}(V^1, V^2)$, $b \in \mathrm{L}(V^2, V^3)$, $B \in \mathrm{E}(V^2, V^3)$ are defined in the obvious manner. If $a \in \mathrm{L}(V^1, V^1)$, its trace $\operatorname{tr}(a)$ is understood as $\sum_i \operatorname{tr}(a_i)$.

For two I-graded vector spaces V, W with $\mathbf{v} = \dim V$, $\mathbf{w} = \dim W$, we consider the vector space given by

$$\mathbf{M} \equiv \mathbf{M}(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W),$$

where we use the notation $\mathbf{M}$ when $\mathbf{v}$, $\mathbf{w}$ are clear in the context. The above three components for an element of $\mathbf{M}$ will be denoted by $B = \bigoplus B_h$, $a = \bigoplus a_i$, $b = \bigoplus b_i$ respectively.

The orientation Ω defines a function $\varepsilon\colon H \to \{\pm 1\}$ by $\varepsilon(h) = 1$ if $h \in \Omega$, $\varepsilon(h) = -1$ if $h \in \overline{\Omega}$. We consider ε as an element of $\mathrm{L}(V, V)$. Let us define a symplectic form ω on $\mathbf{M}$ by

$$\omega((B, a, b), (B', a', b')) \overset{\text{def.}}{=} \operatorname{tr}(\varepsilon B\, B') + \operatorname{tr}(ab' - a'b).$$

Let $G \equiv G_{\mathbf{v}}$ be an algebraic group defined by

$$G \equiv G_{\mathbf{v}} \overset{\text{def.}}{=} \prod_i \mathrm{GL}(V_i).$$

Its Lie algebra is the direct sum $\bigoplus_i \mathfrak{gl}(V_i)$. The group G acts on $\mathbf{M}$ by

$$(B, a, b) \mapsto g \cdot (B, a, b) \overset{\text{def.}}{=} (gBg^{-1}, ga, bg^{-1})$$

preserving the symplectic structure.

The moment map vanishing at the origin is given by

$$\mu(B, a, b) = \varepsilon B\, B + ab \in \mathrm{L}(V, V),$$

where the dual of the Lie algebra of G is identified with $\mathrm{L}(V, V)$ via the trace.

We would like to consider a 'symplectic quotient' of $\mu^{-1}(0)$ divided by G. However we cannot expect the set-theoretical quotient to have a good property. Therefore we consider the quotient using the geometric invariant theory. Then the quotient depends on an additional parameter $\zeta = (\zeta_i)_{i\in I} \in \mathbb{Z}^I$ as follows: Let us define a character of G by

$$\chi_\zeta(g) \overset{\text{def.}}{=} \prod_{i\in I} (\det g_i)^{-\zeta_i}.$$

Let $A(\mu^{-1}(0))$ be the coodinate ring of the affine variety $\mu^{-1}(0)$. Set

$$A(\mu^{-1}(0))^{G,\chi_\zeta^n} \overset{\text{def.}}{=} \{f \in A(\mu^{-1}(0)) \mid f(g\cdot(B,a,b)) = \chi_\zeta(g)^n f((B,a,b))\}.$$

The direct sum with respect to $n \in \mathbb{Z}_{\geq 0}$ is a graded algebra, hence we can define

$$\mathfrak{M}_\zeta \equiv \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \equiv \mathfrak{M}_\zeta(V,W) \overset{\text{def.}}{=} \operatorname{Proj}(\bigoplus_{n\geq 0} A(\mu^{-1}(0))^{G,\chi_\zeta^n}).$$

This is the *quiver variety* introduced in [14]. Since this space is unchanged when we replace χ by a positive power χ^N ($N > 0$), this space is well-defined for $\zeta \in \mathbb{Q}^I$. We call ζ a *stability parameter*.

We use two special stability parameters in this paper. When $\zeta = 0$, the corresponding $\mathfrak{M}_0$ is an affine algebraic variety whose coordinate ring consists of the G-invariant functions on $\mu^{-1}(0)$.

Another choice is $\zeta_i = 1$ for all i. In this case, we denote the corresponding variety simply by $\mathfrak{M}$. The corresponding stability condition is that an I-graded subspace V' of V invariant under B and contained in $\operatorname{Ker} b$ is 0 [15, Lemma 3.8]. The stability and semistability are equivalent in this case, and the action of G on the set $\mu^{-1}(0)^{\mathrm{s}}$ of stable points is free, and $\mathfrak{M}$ is the quotient $\mu^{-1}(0)^{\mathrm{s}}/G$. In particular $\mathfrak{M}$ is nonsingular.

2. Tensor product varieties

Let $W^2 \subset W$ be an I-graded subspace and $W^1 = W/W^2$ be the quotient. We fix an isomorphism $W \cong W^1 \oplus W^2$. We define a one parameter subgroup $\lambda\colon \mathbb{C}^* \to G_W$ by $\lambda(t) = \mathrm{id}_{W^1} \oplus t\,\mathrm{id}_{W^2}$. Then $\mathbb{C}^*$ acts on $\mathfrak{M}$, $\mathfrak{M}_0$ through λ.

We fix $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{w}^1 = \dim W^1$, $\mathbf{w}^2 = \dim W^2$ throughout this paper. Since we use several quiver varieties with different dimension vectors, let us use the notation $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)$, etc for those, while the notation $\mathfrak{M}$ means the original $\mathfrak{M}(\mathbf{v},\mathbf{w})$.

2(i). **Fixed points.** We consider the fixed point loci $\mathfrak{M}^{\mathbb{C}^*}$, $\mathfrak{M}_0^{\mathbb{C}^*}$. The former decomposes as

$$\mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2) \tag{2.1}$$

(see [18, Lemma 3.2]). The isomorphism is given by considering the direct sum of $[B^1,a^1,b^1] \in \mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)$ and $[B^2,a^2,b^2] \in \mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$ as a point in $\mathfrak{M}$. Since quiver varieties $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)$, $\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$ are connected, this is a decomposition of $\mathfrak{M}^{\mathbb{C}^*}$ into connected components.

Let us study the second fixed point locus $\mathfrak{M}_0^{\mathbb{C}^*}$. We have a morphism

$$\sigma\colon \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{M}_0(\mathbf{v}^1,\mathbf{w}^1) \times \mathfrak{M}_0(\mathbf{v}^2,\mathbf{w}^2) \to \mathfrak{M}_0^{\mathbb{C}^*}$$

given by the direct sum as above. This cannot be an isomorphism unless $\mathbf{v}=0$ as the inverse image of 0 consists of several points corresponding to various decomposition $\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2$. This is compensated by considering the direct limit $\mathfrak{M}_0(\mathbf{w}) = \bigcup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v},\mathbf{w})$ if the underlying graph is of type ADE. But this trick does not solve the problem yet in general. For example, if the quiver is the Jordan quiver, and $\mathbf{v}^1=\mathbf{w}^1=\mathbf{v}^2=\mathbf{w}^2=1$, we have $\mathfrak{M}_0(\mathbf{v}^1,\mathbf{w}^1)=\mathfrak{M}_0(\mathbf{v}^2,\mathbf{w}^2)=\mathbb{C}^2$, while $\mathfrak{M}_0^{\mathbb{C}^*}=S^2(\mathbb{C}^2)$. The morphism σ is the quotient map $\mathbb{C}^2\times\mathbb{C}^2 \to S^2(\mathbb{C}^2)=(\mathbb{C}^2\times\mathbb{C}^2)/S_2$. Let us study σ further.

Using the stratification [15, Lemma 3.27] we decompose $\mathfrak{M}_0=\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ as

$$\mathfrak{M}_0 = \bigsqcup_{\mathbf{v}^0} \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w}) \times \mathfrak{M}_0(\mathbf{v}-\mathbf{v}^0,0), \tag{2.2}$$

where $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w})$ is the open subvariety of $\mathfrak{M}_0(\mathbf{v}^0,\mathbf{w})$ consisting of closed free orbits, and $\mathfrak{M}_0(\mathbf{v}-\mathbf{v}^0,0)$ is the quiver variety associated with $W=0$. For quiver varieties of type ADE, the factor $\mathfrak{M}_0(\mathbf{v}-\mathbf{v}^0,0)$ is a single point 0. It is nontrivial in general. For example, if the quiver is the Jordan quiver, we have $\mathfrak{M}_0(\mathbf{v}-\mathbf{v}^0,0)=S^n(\mathbb{C}^2)$ where $n=\mathbf{v}-\mathbf{v}^0$. Then

Lemma 2.3. (1) *The above stratification induces a stratification*

$$\mathfrak{M}_0^{\mathbb{C}^*} = \bigsqcup_{\substack{\mathbf{v}^0,{}^1\mathbf{v},{}^2\mathbf{v}\\ \mathbf{v}^0={}^1\mathbf{v}+{}^2\mathbf{v}}} \mathfrak{M}_0^{\mathrm{reg}}({}^1\mathbf{v},\mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}({}^2\mathbf{v},\mathbf{w}^2) \times \mathfrak{M}_0(\mathbf{v}-\mathbf{v}^0,0).$$

(2) σ *is a surjective finite morphism.*

Thus the factor with $W=0$ appears twice in $\mathfrak{M}_0(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0(\mathbf{v}^2,\mathbf{w}^2)$ while it appears only once in $\mathfrak{M}_0^{\mathbb{C}^*}$.

Proof. (1) We consider $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ as an open subvariety in $\mathfrak{M}(\mathbf{v}^0, \mathbf{w})$ and restrict the decomposition (2.1). Then it is easy to check that $(x, y) \in \mathfrak{M}({}^1\mathbf{v}, \mathbf{w}^1) \times \mathfrak{M}({}^2\mathbf{v}, \mathbf{w}^2)$ is contained in $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ if and only if x, y are in $\mathfrak{M}_0^{\mathrm{reg}}({}^1\mathbf{v}, \mathbf{w}^1)$, $\mathfrak{M}_0^{\mathrm{reg}}({}^2\mathbf{v}, \mathbf{w}^2)$ respectively. Thus $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})^{\mathbb{C}^*} = \mathfrak{M}_0^{\mathrm{reg}}({}^1\mathbf{v}, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}({}^2\mathbf{v}, \mathbf{w}^2)$. Now the assertion is clear as $\mathbb{C}^*$ acts trivially on the factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$.

(2) The coordinate ring of $\mathfrak{M}_0$ is generated by the following two types of functions:

- $\operatorname{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} \colon V_{\mathrm{o}(h_1)} \to V_{\mathrm{i}(h_N)} = V_{\mathrm{o}(h_1)})$, where h_1, …, h_N is a cycle in our graph.
- $\chi(b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)})$, where h_1, …, h_N is a path in our graph, and χ is a linear form on $\operatorname{Hom}(W_{\mathrm{o}(h_1)}, W_{\mathrm{i}(h_N)})$.

Then the generators for $\mathfrak{M}_0^{\mathbb{C}^*}$ are the first type functions and second type functions with $\chi = (\chi_1, \chi_2) \in \operatorname{Hom}(W^1_{\mathrm{o}(h_1)}, W^1_{\mathrm{i}(h_N)}) \oplus \operatorname{Hom}(W^2_{\mathrm{o}(h_1)}, W^2_{\mathrm{i}(h_N)})$.

If we pull back these functions by σ, they become sums of the same types of functions for $\mathfrak{M}_0(\mathbf{v}^1, \mathbf{w}^1)$ and $\mathfrak{M}_0(\mathbf{v}^2, \mathbf{w}^2)$. From this observation, we can easily see that σ is a finite morphism. From (1) it is clearly surjective. □

Remark 2.4. Let $Z(\mathbf{v}^a, \mathbf{w}^a)$ be the fiber product $\mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a) \times_{\mathfrak{M}_0(\mathbf{v}^a, \mathbf{w}^a)} \mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a)$ for $a = 1, 2$. The fiber product $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$ is larger than the union of the products $Z(\mathbf{v}^1, \mathbf{w}^1) \times Z(\mathbf{v}^2, \mathbf{w}^2)$ in general. For example, consider the Jordan quiver variety with $\mathbf{v}^1 = \mathbf{v}^2 = \mathbf{w}^1 = \mathbf{w}^2 = 1$. Then $\mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a)$ is $\mathbb{C}^2$. The product $Z(\mathbf{v}^1, \mathbf{w}^1) \times Z(\mathbf{v}^2, \mathbf{w}^2)$ is consisting of points (p_1, q_1, p_2, q_2) with $p_1 = q_1$, $p_2 = q_2$. On the other hand, $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$ contains also points with $p_1 = q_2$, $p_2 = q_1$.

On the other hand, if the quiver is of type ADE, we do not have the factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$, and they are the same.

2(ii). **Review of** [18]**.** In this subsection we recall results in [18, §3], with emphasis on subvarieties in the affine quotient $\mathfrak{M}_0$.

We first define the following varieties which were implicitly introduced in [18, §3]:

$$\mathfrak{T}_0 \overset{\text{def.}}{=} \{x \in \mathfrak{M}_0 \mid \lim_{t \to 0} \lambda(t) x \text{ exists}\},$$
$$\widetilde{\mathfrak{T}}_0 \overset{\text{def.}}{=} \{x \in \mathfrak{M}_0 \mid \lim_{t \to 0} \lambda(t) x = 0\}.$$

By the proof of [18, Lemma 3.6] we have the following: $x = [B, a, b]$ is in $\mathfrak{T}_0$ (resp. $\widetilde{\mathfrak{T}}_0$) if and only if

- $b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)}$ maps $W^2_{\mathrm{o}(h_1)}$ to $W^2_{\mathrm{i}(h_N)}$ (resp. $W^2_{\mathrm{o}(h_1)}$ to 0 and the whole $W_{\mathrm{o}(h_1)}$ to $W^2_{\mathrm{i}(h_N)}$) for any path in the doubled quiver.

From this description it also follows that $\mathfrak{T}_0$, $\widetilde{\mathfrak{T}}_0$ are closed subvarieties in $\mathfrak{M}_0$.

We have the inclusion $i\colon \mathfrak{T}_0 \to \mathfrak{M}_0$ and the projection $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ defined by taking $\lim_{t\to 0}\lambda(t)x$. The latter is defined as $\mathfrak{M}_0$ is affine.

We define $\mathfrak{T} \overset{\text{def.}}{=} \pi^{-1}(\mathfrak{T}_0)$, $\widetilde{\mathfrak{T}} \overset{\text{def.}}{=} \pi^{-1}(\widetilde{\mathfrak{T}}_0)$. These definitions coincide with ones in [18, §3]. Note that we do not have an analog of $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ for $\mathfrak{T}$. Instead we have a decomposition

$$\mathfrak{T} = \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{T}(\mathbf{v}^1,\mathbf{w}^1;\mathbf{v}^2,\mathbf{w}^2) \tag{2.5}$$

into locally closed subvarieties, and the projection

$$p_{(\mathbf{v}^1,\mathbf{v}^2)}\colon \mathfrak{T}(\mathbf{v}^1,\mathbf{w}^1;\mathbf{v}^2,\mathbf{w}^2) \to \mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2), \tag{2.6}$$

which is a vector bundle. These are defined by considering the limit $\lim_{t\to 0}\lambda(t)x$. Note that they intersect in their closures, contrary to (2.1), which was the decomposition into connected components. Since pieces in (2.5) are mapped to different components, $p_{(\mathbf{v}^1,\mathbf{v}^2)}$'s do not give a morphism defined on $\mathfrak{T}$.

As a vector bundle, $\mathfrak{T}(\mathbf{v}^1,\mathbf{w}^1;\mathbf{v}^2,\mathbf{w}^2)$ is the subbundle of the normal bundle of $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$ in $\mathfrak{M}$ consisting of positive weight spaces. Its rank is half of the codimension of $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$. In fact, the restriction of the tangent space of $\mathfrak{M}$ to $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$ decomposes into weight ± 1 and 0 spaces such that

- the weight 0 subspace gives the tangent bundle of $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w}^2)$,
- the weight 1 and -1 subspaces are dual to each other with respect to the symplectic form on $\mathfrak{M}$.

We define a partial order $<$ on the set $\{(\mathbf{v}^1,\mathbf{v}^2) \mid \mathbf{v}^1+\mathbf{v}^2=\mathbf{v}\}$ defined by $(\mathbf{v}^1,\mathbf{v}^2)\le(\mathbf{v}'^1,\mathbf{v}'^2)$ if and only if $\mathbf{v}^1\le\mathbf{v}'^1$. We extend it to a total order and denote it also by $<$. Let

$$\mathfrak{T}_{\le(\mathbf{v}^1,\mathbf{v}^2)} \overset{\text{def.}}{=} \bigcup_{(\mathbf{v}'^1,\mathbf{v}'^2)\le(\mathbf{v}^1,\mathbf{v}^2)} \mathfrak{T}(\mathbf{v}^1,\mathbf{w}^1;\mathbf{v}^2,\mathbf{w}^2),$$

and let $\mathfrak{T}_{<(\mathbf{v}^1,\mathbf{v}^2)}$ be the union obtained similarly by replacing $\le$ by $<$. Then $\mathfrak{T}_{\le(\mathbf{v}^1,\mathbf{v}^2)}$, $\mathfrak{T}_{<(\mathbf{v}^1,\mathbf{v}^2)}$ are closed subvarieties in $\mathfrak{T}$.

2(iii). **The fiber product $Z_{\mathfrak{T}}$.** We introduce one more variety, following [13]. Let us consider $\mathfrak{T}_0$ as a subvariety in $\mathfrak{M}_0 \times \mathfrak{M}_0^{\mathbb{C}^*}$, where the projection to the second factor is given by p. Let $Z_{\mathfrak{T}} \subset \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ be the inverse image of $\mathfrak{T}_0$ under the restriction of the projective morphism $\pi \times \pi$. This variety is an analog of the variety $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$ introduced in [15, §7]. Note that $Z_{\mathfrak{T}}$ is also given as a fiber product $\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$, where $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$ is given by the composition of $\pi\colon \mathfrak{T} \to \mathfrak{T}_0$ and $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$. We will consider a cycle in $Z_{\mathfrak{T}}$ as a correspondence in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ later. Note that the projection $p_1\colon Z_{\mathfrak{T}} \to \mathfrak{M}$ is proper, but $p_2\colon Z_{\mathfrak{T}} \to \mathfrak{M}^{\mathbb{C}^*}$ is not.

We consider the two decompositions (2.1, 2.5). For brevity, we change the notation as

$$\mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\alpha} \mathfrak{M}_\alpha, \quad \mathfrak{T} = \bigsqcup_{\alpha} \mathfrak{T}_\alpha.$$

We also recall

$$\mathfrak{T}_{\le\alpha} = \bigsqcup_{\beta\le\alpha} \mathfrak{T}_\beta, \qquad \mathfrak{T}_{<\alpha} = \bigsqcup_{\beta<\alpha} \mathfrak{T}_\beta$$

These are closed subvarieties in $\mathfrak{T}$.

Then they induce a decomposition

$$Z_{\mathfrak{T}} = \bigsqcup_{\alpha,\beta} Z_{\mathfrak{T},\alpha,\beta},$$

with

$$Z_{\mathfrak{T},\alpha,\beta} \overset{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_\alpha \times \mathfrak{M}_\beta)\,.$$

We have the corresponding decomposition

$$Z^{\mathbb{C}^*} = \mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\alpha,\beta} Z_{\alpha,\beta}$$

induced from the the decomposition of the first and second factors.

We also define

$$Z_{\mathfrak{T},\le\alpha,\beta} \overset{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_{\le\alpha} \times \mathfrak{M}_\beta)\,, \qquad Z_{\mathfrak{T},<\alpha,\beta} \overset{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_{<\alpha} \times \mathfrak{M}_\beta)\,.$$

They are closed subvarieties in $Z_{\mathfrak{T}}$ and $Z_{\mathfrak{T},\alpha,\beta}$ is an open subvariety in $Z_{\mathfrak{T},\le\alpha,\beta}$. On the other hand, each $Z_{\alpha,\beta}$ is a closed subvariety in $\mathfrak{M}_\alpha \times \mathfrak{M}_\beta$.

Each piece $Z_{\mathfrak{T},\alpha,\beta}$ is a vector bundle over $Z_{\alpha,\beta}$, which is pull-back of $\mathfrak{T}_\alpha \to \mathfrak{M}_\alpha$. Therefore its rank is half of the codimension of $\mathfrak{M}_\alpha$ in $\mathfrak{M}$.

Proposition 2.7. (1) *Each irreducible component of $Z_{\mathfrak{T},\alpha,\beta}$ is at most half dimensional in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$, and hence the same is true for $Z_{\mathfrak{T}}$.*

(2) *Irreducible components of $Z_{\mathfrak{T}}$ of half dimension are lagrangian subvarieties in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$.*

Here $\mathfrak{M}^{\mathbb{C}^*}$ has several connected components of various dimensions, so the above more precisely meant half dimensional in each component $\mathfrak{M}\times\mathfrak{M}_\beta$.

Proof. (1) It is known that $\pi\colon \mathfrak{M}\to\mathfrak{M}_0$ is semismall, if we replace the target by the image $\pi(\mathfrak{M}_0)$. (This is a consequence of [14, 6.11] as explained in [20, 2.23].) Therefore irreducible components of $Z = \mathfrak{M}\times_{\mathfrak{M}_0}\mathfrak{M}$ are at most half dimensional in $\mathfrak{M}\times\mathfrak{M}$. As σ is a finite morphism, the same is true for $Z^{\mathbb{C}^*}$. Now the assertion for $Z_{\mathfrak{T},\alpha,\beta}$ follows as it is a vector bundle over $Z_{\alpha,\beta}$ whose rank is equal to the half of codimension of $\mathfrak{M}_\alpha$.

(2) This follows from the local description of π in [17, Theorem 3.3.2] which respects the symplectic form from its proof, together with the fact that $\pi^{-1}(0)$ is isotropic by the proof of [14, Theorem 5.8]. $\square$

3. Coproduct

In this section we define a kind of a coproduct on the convolution algebra $H_*(Z)$. The target of Δ is, in general, larger than the tensor product $H_*(Z(\mathbf{w}^1))\otimes H_*(Z(\mathbf{w}^2))$ as we mentioned in the introduction.

3(i). **Convolution algebras.** Recall the fiber product $Z=\mathfrak{M}\times_{\mathfrak{M}_0}\mathfrak{M}$. The convolution product defines an algebra structure on $H_*(Z)$:

$$a * b \overset{\text{def.}}{=} p_{13*}(p_{12}^*(a)\cap p_{23}^*(b)),\quad a,b\in H_*(Z),$$

where p_{ij} is the projection from $\mathfrak{M}\times\mathfrak{M}\times\mathfrak{M}$ to the product of the i^{th} and j^{th}-factors, and Z is viewed as a subvariety in $\mathfrak{M}\times\mathfrak{M}$ for the cap product. (See [5, §2.7] for more detail.)

As $\pi\colon\mathfrak{M}\to\pi(\mathfrak{M})$ is a semismall morphism, the top degree component $H_{\text{top}}(Z)$ is a subalgebra, where 'top' is equal to the complex dimension of $\mathfrak{M}\times\mathfrak{M}$. Moreover $H_*(Z)$ is a graded algebra, where the degree p elements are in $H_{\text{top}-p}(Z)$. (See [5, §8.9].)

Take $x\in\mathfrak{M}_0$. We consider the inverse image $\pi^{-1}(x)\subset\mathfrak{M}$ and denote it by $\mathfrak{M}_x$. (When $x=0$, this is denoted by $\mathfrak{L}$ usually.) Then the convolution gives $\bigoplus H_{\text{top}-p}(\mathfrak{M}_x)$ a structure of a module of $H_*(Z)$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}$ and the stratum containing x.

Similarly we can define a graded algebra structure on $H_*(Z^{\mathbb{C}^*}) = \bigoplus H_{\text{top}-p}(Z^{\mathbb{C}^*})$, where 'top' means the complex dimension of $\mathfrak{M}^{\mathbb{C}^*}\times\mathfrak{M}^{\mathbb{C}^*}$, possibly different on various connected components. By §2(i) it is close to

$$\bigoplus_{\mathbf{v}^1+\mathbf{v}^2=\mathbf{v}^2} H_*(Z(\mathbf{v}^1,\mathbf{w}^1))\otimes H_*(Z(\mathbf{v}^2,\mathbf{w}^2)),$$

but is different in general, as explained in Remark 2.4.

We denote by $\mathfrak{M}_x^{\mathbb{C}^*}$ the inverse image $(\pi^{\mathbb{C}^*})^{-1}(x)$ in $\mathfrak{M}^{\mathbb{C}^*}$ for $x \in \mathfrak{M}_0^{\mathbb{C}^*}$. Its homology $\bigoplus H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ is a graded module of $H_*(Z^{\mathbb{C}^*})$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}^{\mathbb{C}^*}$ (resp. $\mathfrak{M}$) and the stratum containing x.

3(ii). **Convolution by $Z_{\mathfrak{T}}$.** Take $x \in \mathfrak{M}_0^{\mathbb{C}^*}$. We consider the inverse image $(p \circ \pi_{\mathfrak{T}})^{-1}(x) \subset \mathfrak{T} \subset \mathfrak{M}$ and denote it by $\mathfrak{T}_x$. (When $x = 0$, this is denoted by $\widetilde{\mathfrak{T}}$ in §2(ii).) By the convolution product its homology $H_*(\mathfrak{T}_x) = \bigoplus H_{\mathrm{top}-p}(\mathfrak{T}_x)$ is a graded module of $H_*(Z)$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}$ and the stratum containing x.

Let $\mathfrak{T}_{\alpha,x}$, $\mathfrak{T}_{\le\alpha,x}$, $\mathfrak{T}_{<\alpha,x}$ be the intersection of $\mathfrak{T}_x$ with $\mathfrak{T}_\alpha$, $\mathfrak{T}_{\le\alpha}$, $\mathfrak{T}_{<\alpha}$ respectively. We have a short exact sequence

$$0 \to H_{\mathrm{top}-p}(\mathfrak{T}_{<\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\le\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\alpha,x}) \to 0.$$

(See [18, §3] and (3.12) below.)

Let us restrict the projection $p_\alpha\colon \mathfrak{T}_\alpha \to \mathfrak{M}_\alpha$ in (2.6) to $\mathfrak{T}_{\alpha,x}$. As $\pi^{\mathbb{C}^*} \circ p_\alpha = p \circ \pi$, it identifies $\mathfrak{T}_{\alpha,x}$ with its inverse image of $\mathfrak{M}_{\alpha,x} \overset{\text{def.}}{=} \mathfrak{M}_\alpha \cap \mathfrak{M}_x^{\mathbb{C}^*}$. Therefore we can replace the last term of the short exact sequence by $H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$ thanks to the Thom isomorphism:

$$0 \to H_{\mathrm{top}-p}(\mathfrak{T}_{<\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\le\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x}) \to 0. \tag{3.1}$$

Our convention of 'top' is compatible for $\mathfrak{T}_x$ and $\mathfrak{M}_x$ as the rank of the vector bundle is the half of codimension of $\mathfrak{M}_\alpha$ in $\mathfrak{M}$. Since $\mathfrak{T}_{\le\alpha} = \mathfrak{T}$ when α is the maximal element, we get

Lemma 3.2. *$H_{\mathrm{top}-p}(\mathfrak{T}_x)$ has a filtration whose associated graded is isomorphic to $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$.*

Choice of splittings $H_{\mathrm{top}-p}(\mathfrak{T}_{\le\alpha,x}) \leftarrow H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$ in (3.1) for all α gives an isomorphism $H_{\mathrm{top}-p}(\mathfrak{T}_x) \cong H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$. Our next goal is to understand the space of all splittings in a geometric way.

For this purpose we consider the top degree homology group $H_{\mathrm{top}}(Z_{\mathfrak{T}})$. They are spanned by lagrangian irreducible components of $Z_{\mathfrak{T}}$ by Proposition 2.7.

Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ and $p \in \mathbb{Z}$. The convolution product

$$a \mapsto c * a \overset{\text{def.}}{=} p_{1*}(c \cap p_2^*(a))$$

defines an operator

$$c*\colon H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*}) \to H_{\mathrm{top}-p}(\mathfrak{T}_x), \tag{3.3}$$

where p_1, p_2 are projections from $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ to the first and second factors. The degree shift p is preserved by the argument in [5, §8.9]. (If we choose c from $H_{\mathrm{top}-k}(Z_{\mathfrak{T}})$, the convolution maps $H_{\mathrm{top}-p}$ to $H_{\mathrm{top}-p-k}$.) Note also that the above operation is well-defined as p_1 is proper, while the operator $p_{2*}(c \cap p_1^*(-))$ is not in this setting.

An arbitrary class $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ does not give a splitting of (3.1), as it is nothing to do with the decomposition $\mathfrak{T} = \bigsqcup \mathfrak{T}_\alpha$. Let us write down a sufficient condition to give a splitting.

Since c is in the top degree, it is a linear combination of fundamental classes of lagrangian irreducible components of $Z_{\mathfrak{T}}$. From Proposition 2.7(1), half-dimensional irreducible components are closures of half-dimensional irreducible components of $Z_{\mathfrak{T},\beta,\alpha}$ for some pair α, β. Therefore we can write

$$c = \sum_{\alpha,\beta} c_{\beta,\alpha}.$$

Moreover its proof there, the latter are pull-backs of half-dimensional irreducible components of $Z_{\beta,\alpha}$ under the projection $p_\beta \times \mathrm{id}_{\mathfrak{M}_\alpha}$.

We impose the following conditions on c:

$$c_{\beta,\alpha} = 0 \text{ unless } \alpha \geq \beta, \tag{3.4a}$$

$$c_{\alpha,\alpha} = [\overline{(p_\alpha \times \mathrm{id}_{\mathfrak{M}_\alpha})^{-1}(\Delta_{\mathfrak{M}_\alpha})}]. \tag{3.4b}$$

The first condition also means that c is is in the image of $\bigoplus_\alpha H_{\mathrm{top}}(Z_{\mathfrak{T},\leq\alpha,\alpha}) \to H_{\mathrm{top}}(Z_{\mathfrak{T}})$. Note that $\bigsqcup_\alpha Z_{\mathfrak{T},\leq\alpha,\alpha}$ is a disjoint union of closed subvarieties in $Z_{\mathfrak{T}}$, and hence the push-forward homomorphism is defined.

Proposition 3.5. *Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ with the conditions* (3.4 a,b). *Then $c*$ is an isomorphism and gives a splitting of* (3.1) *for all α.*

We will show the converse in §4(ii): $c*$ gives a splitting if and only if c satisfies (3.4).

Proof. By the first condition the operator $c*$ restricts to $H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\leq\alpha,x})$. And by the second condition it gives the identity if we compose $H_{\mathrm{top}-p}(\mathfrak{T}_{\leq\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$. Thus $c*$ gives a splitting of (3.1). □

Next we construct the inverse of $c*$ also by a convolution product. We consider

$$\mathfrak{T}_0^- \overset{\text{def.}}{=} \{x \in \mathfrak{M}_0 \mid \lim_{t\to\infty} \lambda(t)x \text{ exists}\},$$

and the similarly defined variety $\mathfrak{T}^-$ also by replacing $t \to 0$ by $t \to \infty$. We have the inclusion $i^- \colon \mathfrak{T}_0^- \to \mathfrak{M}_0$ and the projection $p^- \colon \mathfrak{T}_0^- \to \mathfrak{M}_0^{\mathbb{C}^*}$. Note also that $\mathfrak{T}_0 \cap \mathfrak{T}_0^- = \mathfrak{M}_0^{\mathbb{C}^*}$.

Let us define $Z_{\mathfrak{T}^-}$ as the fiber product $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-$, and consider it as a subvariety in $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M}$. We swap the first and second factors from $Z_{\mathfrak{T}}$ as it becomes more natural when we consider a composite of correspondences.

Since p_1 is proper on $Z_{\mathfrak{T}^-} \cap p_2^{-1}(\mathfrak{T}_x)$, a class $c^- \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ defines the well-defined convolution product $c^- * a = p_{1*}(c^- \cap p_2^*(a))$ for $a \in H_{\mathrm{top}-p}(\mathfrak{T}_x)$, and defines an operator

$$c^- *\colon H_{\mathrm{top}-p}(\mathfrak{T}_x) \to H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*}). \tag{3.6}$$

By the associativity of the convolution product, the composite $c^- * (c * \bullet) \in \mathrm{End}(H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*}))$ is given by the convolution of

$$c^- * c = p_{13*}(p_{12}^*(c^-) \cap p_{23}^*(c)),$$

where p_{ij} is the projection from $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ to the product of the i^{th} and j^{th}-factors.

Proposition 3.7. *Suppose that $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ satisfies the conditions* (3.4 a,b). *Then there exists a class $c^{-1} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ such that $c^{-1} * c$ is equal to $[\Delta_{\mathfrak{M}^{\mathbb{C}^*}}]$.*

Proof. We have decomposition $\mathfrak{T}^- = \bigsqcup_\alpha \mathfrak{T}_\alpha^-$, and the projection $p_\alpha^-\colon \mathfrak{T}_\alpha^- \to \mathfrak{M}_\alpha$. The index set $\{\alpha\}$ is the same as before, as it parametrizes the connected components of $\mathfrak{M}^{\mathbb{C}^*}$.

Since the order $<$ plays the opposite role for $\mathfrak{T}^-$,

$$\mathfrak{T}^-_{\ge\alpha} \overset{\text{def.}}{=} \bigsqcup_{\beta\ge\alpha} \mathfrak{T}^-_\beta, \qquad \mathfrak{T}^-_{>\alpha} \overset{\text{def.}}{=} \bigsqcup_{\beta>\alpha} \mathfrak{T}^-_\beta,$$

are closed subvarieties in $\mathfrak{T}^-$.

We define $Z_{\mathfrak{T}^-,\gamma,\beta} \overset{\text{def.}}{=} Z_{\mathfrak{T}^-} \cap (\mathfrak{M}_\gamma \cap \mathfrak{T}^-_\beta)$ and $Z_{\mathfrak{T}^-,\gamma,\ge\beta}$ as above. We then impose the following conditions on $c^- = \sum c^-_{\gamma,\beta}$:

$$\begin{aligned} &c^-_{\gamma,\beta} = 0 \text{ unless } \gamma \le \beta, \\ &c^-_{\gamma,\gamma} = [\overline{(\mathrm{id}_{\mathfrak{M}_\gamma} \times p_\gamma^-)^{-1}(\Delta_{\mathfrak{M}_\gamma})}]. \end{aligned}$$

These conditions imply that $c^- * c$ is unipotent, more precisely is upper triangular with respect to the block decomposition $H_{\mathrm{top}}(Z^{\mathbb{C}^*}) = \bigoplus_{\gamma,\alpha} H_{\mathrm{top}}(Z_{\gamma,\alpha})$, and the $H_{\mathrm{top}}(Z_{\alpha,\alpha})$ component is $[\Delta_{\mathfrak{M}_\alpha}]$ for all α. Noticing that we can represent $(c^- * c)^{-1}$ as a class in $H_{\mathrm{top}}(Z^{\mathbb{C}^*})$ by the convolution product, we define $c^{-1} = (c^- * c)^{-1} * c^- \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ to get $c^{-1} * c = [\Delta_{Z^{\mathbb{C}^*}}]$.

□

Remark 3.9. If we consider the convolution product in the opposite order, we get

$$c * c^{-1} \in H_{\mathrm{top}}(\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-),$$

where $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$ (resp. $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$) is $p \circ \pi$ (resp. $p^- \circ \pi$). In general, there are no inclusion relations between $\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-$ and $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. Therefore the equality $c * c^{-1} = [\Delta_{\mathfrak{M}}]$ does not make sense at the first sight. However the actual thing we need is the operator $c^- *$ in (3.6). Proposition 3.7 implies that the composite $c^{-1} * (c*)$ of the operator is the identity on $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ for each x. Then we have $c * (c^{-1}*)$ is also the identity on $H_{\mathrm{top}-p}(\mathfrak{T}_x)$, as both $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ and $H_{\mathrm{top}-p}(\mathfrak{T}_x)$ are vector spaces of same dimension.

Later we will see that we do not loose any information when we consider c^{-1} as such an operator. In particular, we will see that c^{-1} is uniquely determined by c, i.e., we will prove the uniqueness of the left inverse in the proof of Theorem 3.20.

3(iii). **Coproduct by convolution.** We define a coproduct using the convolution in this subsection.

Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ be a class satisfying the conditions (3.4). We take the class $c^{-1} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ as in Proposition 3.7. We define a homomorphism $\Delta_c \colon H_*(Z) \to H_*(Z^{\mathbb{C}^*})$ by

$$\Delta_c(\bullet) = c^{-1} * \bullet * c = p_{14*}(p_{12}^*(c^{-1}) \cap p_{23}^*(\bullet) \cap p_{34}^*(c)), \tag{3.10}$$

where we consider the convolution product in $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$. This preserves the grading.

Since $c^{-1} * c = 1$, we have $\Delta_c(1) = 1$. But it is not clear at this moment that Δ_c is an algebra homomorphism since we do not know $c * c^{-1} = 1$, as we mentioned in Remark 3.9. The proof is postponed until the next subsection.

3(iv). **Sheaf-theoretic analysis.** In this subsection, we reformulate the result in the previous subsection using perverse sheaves.

By [5, §8.9] we have a natural graded algebra isomorphism

$$H_*(Z) \cong \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0)}(\pi_! \mathcal{C}_{\mathfrak{M}}, \pi_! \mathcal{C}_{\mathfrak{M}}),$$

where the multiplication on the right hand side is given by the Yoneda product and the grading is the natural one. Here the semismallness of π guarantees that the grading is preserved.

We have similarly

$$H_*(Z^{\mathbb{C}^*}) \cong \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}).$$

In this subsection we define a functor sending $\pi_! \mathbb{C}_{\mathfrak{M}}$ to $\pi_!^{\mathbb{C}^*} \mathbb{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ to give a homomorphism $H_*(Z) \to H_*(Z^{\mathbb{C}^*})$ which coincides with Δ_c.

For a later purpose, we slightly generalize the setting from the previous subsection. If $\mathbf{v}' \leq \mathbf{v}$, we have a closed embedding $\mathfrak{M}_0(\mathbf{v}', \mathbf{w}) \subset$

$\mathfrak{M}_0 = \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, given by adding the trivial representation with dimension $\mathbf{v} - \mathbf{v}'$.

We consider the push-forward $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})}$ as a complex in $D(\mathfrak{M}_0)$. By the decomposition theorem [1] it is a semisimple complex. Furthermore $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})}$ is a perverse sheaf, as $\pi\colon \mathfrak{M}(\mathbf{v}', \mathbf{w}) \to \pi(\mathfrak{M}(\mathbf{v}', \mathbf{w}))$ is semismall [3]. Let $P(\mathfrak{M}_0)$ denote the full subcategory of $D(\mathfrak{M}_0)$ consisting of all perverse sheaves that are finite direct sums of perverse sheaves L, which are isomorphic to direct summand of $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})}$ with various $\mathbf{v}'$.

Replacing $\mathfrak{M}_0$, $\mathfrak{M}(\mathbf{v}', \mathbf{w})$ by $\mathfrak{M}_0^{\mathbb{C}^*}$, $\mathfrak{M}(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*}$ respectively, we introduce the full subcategory $P(\mathfrak{M}_0^{\mathbb{C}^*})$ of $D(\mathfrak{M}_0^{\mathbb{C}^*})$ as above. Here we replace π by $\pi^{\mathbb{C}^*}\colon \mathfrak{M}(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*} \to \mathfrak{M}_0(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*}$, which is the restriction of π.

Let $i\colon \mathfrak{T}_0 \to \mathfrak{M}_0$ and $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ as in §2(ii). We consider $p_! i^*\colon D(\mathfrak{M}_0) \to D(\mathfrak{M}_0^{\mathbb{C}^*})$. This is an analog of the restriction functor in [9, §4], [10, §9.2], and was introduced in the quiver variety setting in [23, §5]. It is an example of the hyperbolic localization.

Lemma 3.11. (1) *The functor $p_! i^*$ sends $P(\mathfrak{M}_0)$ to $P(\mathfrak{M}_0^{\mathbb{C}^*})$.*

(2) *Let $\mathbf{v}' \le \mathbf{v}$. The complex $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})}$ has a canonical filtration whose associated graded is canonically identified with $\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})^{\mathbb{C}^*}}$.*

This was proved in [23, Lemma 5.1] for quiver varieties of finite type, but the proof actually gives the above statements for general types.

Let us recall how the filtration is defined. Let us assume $\mathbf{v}' = \mathbf{v}$ for brevity. Consider the diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[d, "\pi"'] & \mathfrak{T} = \bigsqcup \mathfrak{T}_\alpha \arrow[l, "i'"'] \arrow[r, "\bigsqcup p_\alpha"] \arrow[d, "\pi_{\mathfrak{T}}"'] & \bigsqcup \mathfrak{M}_\alpha = \mathfrak{M}^{\mathbb{C}^*} \arrow[d, "\pi^{\mathbb{C}^*}"] \\
\mathfrak{M}_0 & \mathfrak{T}_0 \arrow[l, "i"] \arrow[r, "p"'] & \mathfrak{M}_0^{\mathbb{C}^*}
\end{tikzcd}$$

where i' is the inclusion, $\pi_{\mathfrak{T}}$ is the restriction of π to $\mathfrak{T}$, and p_α is the projection of the vector bundle (2.6). Note that each p_α is a morphism, but the union $\bigsqcup p_\alpha$ does *not* gives a morphism $\mathfrak{T} \to \mathfrak{M}^{\mathbb{C}^*}$.

Recall the order $<$ on the set $\{\alpha\}$ of fixed point components, and closed subvarieties $\mathfrak{T}_{\le\alpha}$, $\mathfrak{T}_{<\alpha}$ in §2(ii). Let $\pi_{\le\alpha}$, $\pi_{<\alpha}$ be the restrictions of $\pi_{\mathfrak{T}}$ to $\mathfrak{T}_{\le\alpha}$, $\mathfrak{T}_{<\alpha}$ respectively. Then the main point in [23, Lemma 5.1] (based on [9, §4]) was to note that there is the canonical short exact sequence

$$0 \to \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}_\alpha} \to (p \circ \pi_{\le\alpha})_! \mathcal{C}_{\mathfrak{T}_{\le\alpha}} \to (p \circ \pi_{<\alpha})_! \mathcal{C}_{\mathfrak{T}_{<\alpha}} \to 0. \tag{3.12}$$

Since $\mathfrak{T}_{\le\alpha} = \mathfrak{T}$ for the maximal element α and we have $i^*\pi_!\mathcal{C}_{\mathfrak{M}} = \pi_{\mathfrak{T}!}i'^*\mathcal{C}_{\mathfrak{M}}$, this gives the desired filtration.

During the proof it was also shown that $(p\circ\pi_{\le\alpha})_!\mathcal{C}_{\mathfrak{T}_{\le\alpha}}$, $(p\circ\pi_{<\alpha})_!\mathcal{C}_{\mathfrak{T}_{<\alpha}}$ are semisimple. (It is not stated explicitly in [23], but comes from [9, 4.7].) Therefore the short exact sequence (3.12) splits, and hence $p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}$ and $\bigoplus_\alpha \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\alpha} = \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ is isomorphic. The choice of an isomorphism depends on the choice of splittings of the above short exact sequences for all α.

The exact sequence (3.12) is the sheaf theoretic counterpart of (3.1). More precisely it is more natural to consider the transpose of (3.1):

$$0 \to (p\circ\pi_{<\alpha})_*\mathcal{C}_{\mathfrak{T}_{<\alpha}} \to (p\circ\pi_{\le\alpha})_*\mathcal{C}_{\mathfrak{T}_{\le\alpha}} \to \pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\alpha} \to 0, \tag{3.13}$$

obtained by applying the Verdier duality.

Recall that we study $H_{\mathrm{top}}(Z_{\mathfrak{T}})$ in order to describe a splitting of (3.1) by convolution.

Lemma 3.14. *We have a natural isomorphism*

$$H_{\mathrm{top}}(Z_{\mathfrak{T}}) \cong \mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}).$$

The proof is exactly the same as [5, Lemma 8.6.1], once we use the base change $i^*\pi_!\mathcal{C}_{\mathfrak{M}} = \pi_{\mathfrak{T}!}i'^*\mathcal{C}_{\mathfrak{M}}$.

This isomorphism is compatible with the convolution operator (3.3) in the following way: Let i_x denote the inclusion $\{x\} \to \mathfrak{M}_0^{\mathbb{C}^*}$. Then an element c in $\mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) \cong \mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}})$ defines an operator

$$H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) \to H^p(i_x^!p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}}) \tag{3.15}$$

by the Yoneda product. (See [5, 8.6.13].) We have

$$H^p(i_x^!p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}}) \cong H^p(i_x^!(p\circ\pi_{\mathfrak{T}})_*i'^!\mathcal{C}_{\mathfrak{M}}) \cong H^p((p\circ\pi_{\mathfrak{T}})_*i_x'^!\mathcal{C}_{\mathfrak{M}}),$$

where i_x' is the inclusion of $\mathfrak{T}_x$ in $\mathfrak{M}$. The last one is nothing but $H_{\mathrm{top}-p}(\mathfrak{T}_x)$. Similarly $H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}})$ is naturally isomorphic to $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$. Then we have

Lemma 3.16. *Under the isomorphism in Lemma 3.14, the operator* (3.15) *given by $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is equal to one in* (3.3).

The proof is the same as in [5, §8.6].

The conditions (3.4) on $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is translated into a language for the right hand side. We have the following equivalent to the condition

(3.4):

(3.17a) c maps $(p\circ\pi_{\le\alpha})_*\mathcal{C}_{\mathfrak{T}_{\le\alpha}}$ to $\displaystyle\bigoplus_{\beta\le\alpha}\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\beta}$,

(3.17b) $c\colon (p\circ\pi_{\le\alpha})_*\mathcal{C}_{\mathfrak{T}_{\le\alpha}}/(p\circ\pi_{<\alpha})_*\mathcal{C}_{\mathfrak{T}_{<\alpha}}\to\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\alpha}$ is the identity.

Here the identity means the natural homomorphism given by (3.13).

Thus c satisfying (3.17) gives a splitting of (3.12) and hence an isomorphism $p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}\cong\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$. Therefore we have a graded algebra homomorphism

(3.18)
$$\begin{aligned}\operatorname{Ext}^\bullet_{D(\mathfrak{M}_0)}(\pi_!\mathcal{C}_{\mathfrak{M}},\pi_!\mathcal{C}_{\mathfrak{M}})&\xrightarrow{p_!i^*}\operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}},p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}})\\&\xrightarrow[\cong]{\operatorname{Ad}(c)}\operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}},\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}).\end{aligned}$$

It is compatible with (3.15), i.e.,

$$\begin{array}{ccc}H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) & \xrightarrow{c} & H^p(i_x^!p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}})\\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \operatorname{Ad}(c)p_*i^!(a)}\\ H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) & \xrightarrow{c} & H^p(i_x^!p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}})\end{array}$$

is commutative.

For $Z_{\mathfrak{T}^-}$ we have the following:

Lemma 3.19. *We have natural isomorphisms*

$$\begin{aligned}H_{\mathrm{top}}(Z_{\mathfrak{T}^-})&\cong\operatorname{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}},p_*^-i^{-!}\pi_!\mathcal{C}_{\mathfrak{M}})\\&\cong\operatorname{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}},p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}).\end{aligned}$$

The first isomorphism is one as in Lemma 3.14. We exchange the first and second factors, as we have changed the order of factors $\mathfrak{M}^{\mathbb{C}^*}$ and $\mathfrak{M}$ containing $Z_{\mathfrak{T}^-}$. The sheaves are replaced by their Verdier dual. The second isomorphism is induced by

$$p_*^-i^{-!}\pi_!\mathcal{C}_{\mathfrak{M}}\cong p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}},$$

proved by Braden [4] (see Theorem 1 and the equation (1) at the end of §3).

We now have

Theorem 3.20. *The coproduct Δ_c in* (3.10) *is equal to* (3.18). *In particular, Δ_c is an algebra homomorphism.*

Proof. The isomorphisms in Lemmas 3.14,3.19 are compatible with the product. Therefore, $c^{-1} * c = [\Delta_{\mathfrak{M}^{\mathbb{C}^*}}]$ means that the composite

$$\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}} \xrightarrow{c^{-1}} p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}} \xrightarrow{c} \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$$

is the identity. (Note that the order of c, c^{-1} is swapped as we need to consider the transpose of homomorphisms for convolution.)

As $\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ and $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}$ are semisimple, c, c^{-1} can be considered as linear maps between isotypic components. (See §4(ii) for explicit descriptions of isotypic components.) Therefore $c \circ c^{-1} = \mathrm{id}$ implies $c^{-1} \circ c = \mathrm{id}$ also. This, in particular, shows the uniqueness of c^{-1} mentioned in Remark 3.9. Moreover this c^{-1} is the inverse of c used in (3.18). Therefore Δ_c coincides with (3.18) again thanks to the compatibility between the convolution and Yoneda products. $\square$

3(v). **Coassociativity.** Since Δ_c depends on the choice of the class c, the coassociativity does not hold in general. We give a sufficient condition on c (in fact, various c's) to have the coassociativity in this subsection.

Let $W = W^1 \oplus W^2 \oplus W^3$ be a decomposition of the I-graded vector space. Let $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2 + \mathbf{w}^3$ be the corresponding dimension vectors. Setting $W^{23} = W^2 \oplus W^3$, we have a flag $W^3 \subset W^{23} \subset W$ with $W^3/W^{23} \cong W^2$, $W/W^{23} = W^1$. This gives us a preferred order among factors generalizing to $W^2 \subset W$ in the previous setting.

The two dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathfrak{M} = \mathfrak{M}(\mathbf{v}, \mathbf{w})$ through the homomorphism $\lambda\colon T \to G_W$ defined by $\lambda(t_2, t_3) = \mathrm{id}_{W^1} \oplus t_2 \,\mathrm{id}_{W^2} \oplus t_3 \,\mathrm{id}_{W^3}$.

We have two ways of putting braces for the sum $\mathbf{w} = (\mathbf{w}^1 + \mathbf{w}^2) + \mathbf{w}^3 = \mathbf{w}^1 + (\mathbf{w}^2 + \mathbf{w}^3)$ respecting the order. We have corresponding two $\mathbb{C}^*$'s in T given by $\{(1, t_3)\}$ and $\{(t_2, t_2)\}$. We denote the former by $\mathbb{C}^*_{12,3}$ and the latter by $\mathbb{C}^*_{1,23}$. We then consider fixed points varieties, tensor product varieties, and fiber products for both $\mathbb{C}^*$'s. We denote them by $\mathfrak{M}^{12,3}$, $\mathfrak{T}^{12,3}$, $Z_{\mathfrak{T}^{12,3}}$, $\mathfrak{M}^{1,23}$, $\mathfrak{T}^{1,23}$, $Z_{\mathfrak{T}^{1,23}}$, etc. They correspond to block matrices $\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$ and $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ respectively.

On these varieties, we have the action of the remaining $\mathbb{C}^* = T/\mathbb{C}^*_{12,3}$ and $T/\mathbb{C}^*_{1,23}$ respectively. Then we can consider the fixed point sets $(\mathfrak{M}^{12,3})^{\mathbb{C}^*}$, $(\mathfrak{M}^{12,3})^{\mathbb{C}^*}$. Both are nothing but the torus fixed points $\mathfrak{M}^T$. We denote it by $\mathfrak{M}^{1,2,3}$. We denote the corresponding fiber product by $Z_{1,2,3}$. In $\mathfrak{T}_0^{12,3}$, $\mathfrak{T}_0^{1,23}$, we consider subvarieties consisting of points $\lim_{t\to 0}$ exists as before. They can be described as the variety consisting of points $x = [B, a, b]$ such that $b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)}$ preserves the flag $W^3 \subset W^{23} \subset W$, i.e., $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$. In particular, the variety is the same for one defined in $\mathfrak{T}_0^{12,3}$ and in $\mathfrak{T}_0^{1,23}$. Therefore it is safe to write both by $\mathfrak{T}_0^{1,2,3}$. We have the corresponding fiber product $Z_{\mathfrak{T}^{1,2,3}} \overset{\text{def.}}{=} \mathfrak{T}^{1,2,3} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$.

We need two more classes of varieties corresponding to $\begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ and $\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ respectively. Tensor product varieties are

$$\mathfrak{T}_0^{(1,2),3} \overset{\text{def.}}{=} \mathfrak{T}_0^{1,2,3} \cap \mathfrak{M}_0^{12,3}, \quad \mathfrak{T}_0^{1,(2,3)} \overset{\text{def.}}{=} \mathfrak{T}_0^{1,2,3} \cap \mathfrak{M}_0^{1,23}$$

respectively. We define the fiber products $Z_{\mathfrak{T}^{(1,2),3}} = \mathfrak{T}^{(1,2),3} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$, $Z_{\mathfrak{T}^{1,(2,3)}} = \mathfrak{T}^{1,(2,3)} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$.

A class $c^{12,3} \in H_{\text{top}}(Z_{\mathfrak{T}^{12,3}})$ gives the coproduct

$$\Delta_{c^{12,3}} \colon H_*(Z) \to H_*(Z_{12,3}),$$

and similarly $c^{1,23} \in H_{\text{top}}(Z_{\mathfrak{T}^{12,3}})$ gives $\Delta_{c^{1,23}}$. These correspond to $\Delta \otimes 1$ and $1 \otimes \Delta$ for the usual coproduct respectively.

A class $c^{(1,2),3} \in H_{\text{top}}(Z_{\mathfrak{T}^{(1,2),3}})$ gives

$$\Delta_{c^{(1,2),3}} \colon H_*(Z_{12,3}) \to H_*(Z_{1,2,3}),$$

and similarly $c^{1,(2,3)} \in H_{\text{top}}(Z_{\mathfrak{T}^{(1,2),3}})$ gives $\Delta_{c^{1,(2,3)}}$. Thus we have two ways going from $H_*(Z)$ to $H_*(Z_{1,2,3})$:

$$\begin{array}{ccc} H_*(Z) & \xrightarrow{\Delta_{c^{12,3}}} & H_*(Z_{12,3}) \\ {\scriptstyle \Delta_{c^{1,23}}}\downarrow & & \downarrow{\scriptstyle \Delta_{c^{(1,2),3}}} \\ H_*(Z_{1,23}) & \xrightarrow[\Delta_{c^{1,(2,3)}}]{} & H_*(Z_{1,2,3}) \end{array} \tag{3.21}$$

The commutativity of this diagram means the coassociativity of our coproduct.

Proposition 3.22. *The diagram* (3.21) *is commutative if*

$$c^{12,3} * c^{(1,2),3} = c^{1,(2,3)} * c^{1,23}$$

holds in $H_{\text{top}}(Z_{\mathfrak{T}^{1,2,3}})$.

The proof is obvious.

3(vi). **Equivariant homology version.** Let $G = \prod_i \operatorname{GL}(W_i^1) \times \operatorname{GL}(W_i^2)$. The group G acts on $\mathfrak{M}$, $\mathfrak{M}^{\mathbb{C}^*}$ and various other varieties considered in the previous subsections.

We consider a $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathfrak{M}$ defined by

$$(t_1, t_2) \cdot B_h = \begin{cases} t_1 B_h & \text{if } h \in \Omega, \\ t_2 B_h & \text{if } h \in \overline{\Omega}, \end{cases} \quad (t_1, t_2) \cdot a = a, \quad (t_1, t_2) \cdot b = t_1 t_2 b.$$

Let $\mathbb{G} = \mathbb{C}^* \times \mathbb{C}^* \times G$.

Remark 3.23. When the graph does not contain a cycle, the action of a factor $\mathbb{C}^*$ of $\mathbb{C}^* \times \mathbb{C}^*$, lifted to the double cover, can be move to an action through $\mathbb{C}^* \to G$. Therefore we only have an action of $\mathbb{C}^* \times G$ essentially in this case.

The results in the previous subsections hold in the equivariant category: we replace the homology $H_*(X)$ by the equivariant homology $H_*^{\mathbb{G}}(X)$. For the derived category $D(X)$ of complexes of constructible sheaves, we use their equivariant version $D_{\mathbb{G}}(X)$, considered in [2, 11].

The following observations are obvious, but useful. Top degree components of Z give a base for both $H_{\mathrm{top}}(Z)$ and $H_{\mathrm{top}}^{\mathbb{G}}(Z)$. Therefore we have a natural isomorphism

$$H_{\mathrm{top}}(Z_{\mathfrak{T}}) \cong H_{\mathrm{top}}^{\mathbb{G}}(Z_{\mathfrak{T}}).$$

The corresponding statement for the right hand side of Lemma 3.14 is

$$\operatorname{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) \cong \operatorname{Hom}_{D_{\mathbb{G}}(\mathfrak{M}_0^{\mathbb{C}^*})}(p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}).$$

This is also true as $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}$, $\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ are $\mathbb{G}$-equivariant perverse sheaves. (See [11, 1.16(a)].)

In particular, $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ defines the coproduct Δ_c for the equivariant version $\Delta_c\colon H_*^{\mathbb{G}}(Z) \to H_*^{\mathbb{G}}(Z^{\mathbb{C}^*})$. Also to check the coassociativity of the coproduct, we only need to check the condition in Proposition 3.22 for the *non*-equivariant homology.

Remark 3.24. In a wider framework of a holomorphic symplectic manifold with torus action satisfying certain conditions, Maulik and Okounkov [13] give a 'canonical' element c. It is called the *stable envelop*. It is defined first on the analog of $Z_{\mathfrak{T}}$ for the quiver varieties with generic complex parameters (deformations of $\mathfrak{M}$, $\mathfrak{M}^{\mathbb{C}^*}$), and then as the limit when parameters go to 0. It satisfies (3.4) and the condition in Proposition 3.22. Therefore their stable envelop together with the construction in this section gives a canonical coproduct, satisfying the coassociativity.

4. Tensor product multiplicities

In this section, we give the formula of tensor product multiplicities with respect to the coproduct Δ_c in terms of IC sheaves.

4(i). **Decomposition of the direct image sheaf.** We give the decomposition of $\pi_!(\mathcal{C}_{\mathfrak{M}})$ in this subsection. For this purpose, we introduce a refinement of the stratification (2.2). We do not need to worry about the first factor $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ as it cannot be decomposed further.

On the other hand the second factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ parametrizes isomorphism classes of semisimple modules M of the preprojective algebra corresponding to the quiver. They decompose into direct sum of simple modules as

$$M = M_1^{\oplus n_1} \oplus M_2^{\oplus n_2} \oplus \cdots \oplus M_N^{\oplus n_N}.$$

Dimension vectors of all simple modules have been classified by Crawley-Boevey [6, Th. 1.2]. (In fact, he also classifies pairs $(\mathbf{v}^0, \mathbf{w})$ with $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}) \neq \emptyset$.) Let δ_1, δ_2, …, δ_N be such vectors which are $\le \mathbf{v}$. They are all positive roots satisfying certain conditions. For example, for a quiver of type ADE, they are simple roots. For a quiver of affine type ADE, they are simple roots and the positive generator δ of imaginary roots. For a Jordan quiver, it is the vector $1 \in \mathbb{Z} = \mathbb{Z}^I$.

We then have

$$\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0) = S^{n_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1, 0) \times S^{n_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2, 0) \times \cdots \times S^{n_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N, 0),$$

with $\mathbf{v}^0 + n_1\delta_1 + \cdots + n_N\delta_N = \mathbf{v}$. Here $\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ parametrizes simple modules with dimension vector δ_k, or equivalently points in $\mathfrak{M}_0(\delta_k, 0)$ whose stabilizers are nonzero scalars times the identity. Its symmetric power $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ parametrizes semisimple modules

$$M_1^{\oplus m_1} \oplus M_2^{\oplus m_2} \oplus \cdots$$

such that M_1, M_2, …are distinct simple modules with dimension δ_k and the total number of simple factors is n_k.

The symmetric power $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ decomposes further according to multiplicities m_1, m_2, …. As we may assume $m_1 \ge m_2 \ge \dots$, they define partition λ_k of n_k. Let us denote by $S_{\lambda_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ the space parametrizing semisimple modules having multiplicities λ_k.

Thus we have

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}) \times \mathfrak{M}_0(\vec{\lambda}) \tag{4.1}$$

with $\mathbf{v}^0 + |\lambda_1|\delta_1 + \cdots + |\lambda_N|\delta_N = \mathbf{v}$, where

$$\mathfrak{M}_0(\vec{\lambda}) \overset{\text{def.}}{=} S_{\lambda_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1, 0) \times S_{\lambda_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2, 0) \times \cdots \times S_{\lambda_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N, 0).$$

This is nothing but the decomposition given in [14, 6.5], [15, 3.27].

This stratification has a simple form when the quiver is of type ADE. Each δ_k is a simple root α_i, and $\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ is a one point given by the simple module S_i. The symmetric product $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ is also a one point $S_i^{\oplus n_k}$, and hence we do not need to consider the partition λ_k. Thus we can safely forget factors $S_{\lambda_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ and get

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}),$$

with $\mathbf{v}^0 \le \mathbf{v}$.

For the affine case δ_k is either simple root or δ, as we mentioned above. If δ_k is a simple root, we can forget the factor $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ as in the ADE cases. If $\delta_k = \delta$, then $\mathfrak{M}_0^{\mathrm{reg}}(\delta, 0)$ is $\mathbb{C}^2$ for the Jordan quiver or $\mathbb{C}^2 \setminus \{0\}/\Gamma$ for the affine quiver corresponding to a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ via the McKay correspondence. Therefore we have

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}) \times \left(S_\lambda \mathbb{C}^2 \text{ or } S_\lambda(\mathbb{C}^2 \setminus \{0\})/\Gamma\right). \tag{4.2}$$

Return back to a general quiver. We denote each stratum in (4.1) by $\mathfrak{M}_0(\mathbf{v}^0; \vec{\lambda})$ for brevity. Here $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$. For a simple local system ρ on this stratum, we consider the corresponding IC sheaf

$$IC(\mathfrak{M}_0(\mathbf{v}^0; \vec{\lambda}), \rho).$$

Then the decomposition theorem for a semismall projective morphism [3] implies a canonical direct sum decomposition

$$\pi_! \mathcal{C}_{\mathfrak{M}} \cong \bigoplus IC(\mathfrak{M}_0(\mathbf{v}^0; \vec{\lambda}), \rho) \otimes H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})_\rho. \tag{4.3}$$

Here $x_{\mathbf{v}^0;\vec{\lambda}}$ is a point in the stratum $\mathfrak{M}_0(\mathbf{v}^0; \vec{\lambda})$ and $\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}} = \pi^{-1}(x_{\mathbf{v}^0;\vec{\lambda}})$ as before. Then $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})_\rho$ denotes the isotypic component of ρ in the homology group $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})$ of the fiber with respect to the monodromy action.

This decomposition determines representations of the convolution algebra $H_{\mathrm{top}}(Z) = \mathrm{End}_{D(\mathfrak{M}_0)}(\pi_! \mathcal{C}_{\mathfrak{M}})$ (see [5, §8.9]):

Theorem 4.4. (1) $\{H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})_\rho\}$ *is the set of isomorphism classes of simple modules of* $H_{\mathrm{top}}(Z)$.

(2) *We have*

$$H_{\mathrm{top}}(Z) \cong \bigoplus \mathrm{End}(H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})_\rho).$$

When the quiver is of type ADE, it was proved that only trivial local systems on strata appear [17, §15] in the direct summand of $\pi_! \mathcal{C}_{\mathfrak{M}}$, and hence we have

$$\pi_!(\mathcal{C}_{\mathfrak{M}}) \cong \bigoplus IC(\mathfrak{M}_0(\mathbf{v}^0, \mathbf{w})) \otimes H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0}}),$$

where we remove the local system ρ from the notation for the IC sheaves.

For a quiver of general type, the argument used in [17, §15] implies that the simple local system ρ is trivial on the factor $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$, i.e., all simple modules M_1, M_2, …are of the form S_i. In general, the author does not know what kind of local system ρ can appear on these factors. But we can show that only trivial local system appears for an affine quiver:

Lemma 4.5. *Suppose that the quiver is of affine type. Then*

$$\pi_!(\mathcal{C}_{\mathfrak{M}}) \cong \bigoplus_{\mathbf{v}^0,\lambda} IC(\mathfrak{M}_0(\mathbf{v}^0,\mathbf{w})) \boxtimes \left(\mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2)}} \text{ or } \mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2/\Gamma)}}\right) \otimes H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\lambda}}).$$

Proof. By the argument in [17, §15], it is enough to assume $\mathbf{v}^0 = 0$ and hence $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w})$ is a single point. Then a point in the stratum $x_{\mathbf{v}^0;\lambda}$ is a point in $S_\lambda\mathbb{C}^2$ or $S_\lambda(\mathbb{C}^2\setminus\{0\})/\Gamma)$, and hence is written as $m_1x_1 + m_2x_2 + \cdots$, where x_1, x_2 are distinct points in $\mathbb{C}^2$ or $\mathbb{C}^2\setminus\{0\})/\Gamma$. Then the fiber $\mathfrak{M}_{x_{\mathbf{v}^0;\lambda}}$ is the product of punctual Quot schemes parametrizing quotients Q of the trivial rank r sheaf $\mathcal{O}_{\mathbb{C}^2}^{\oplus r}$ over $\mathbb{C}^2$ such that Q is supported at 0 and the length is m_i. Here r is given by $\langle \mathbf{w}, c\rangle$, where c is the central element of the affine Lie algebra or $\mathbf{w}$ itself for the Jordan quiver. This follows from the alternative description of quiver varieties of affine types, explained in [19]. (Remark: In [19, §4], it was written that the fiber is the product of punctual Hilbert schemes, but it is wrong.) It is known that top degree part H_{top} of a punctual Quot scheme is 1-dimensional (see [16, Ex. 5.15]). Therefore the monodromy action is trivial. Moreover $\overline{S_\lambda(\mathbb{C}^2)}$ and $\overline{S_\lambda(\mathbb{C}^2/\Gamma)}$ only have finite quotient singularities, and hence are rationally smooth. Therefore the intersection complexes are constant sheaves, shifted by dimensions. □

4(ii). **A description of $H_{\mathrm{top}}(Z_{\mathfrak{T}})$.** As in Theorem 4.4 we have a natural isomorphism

$$H_{\mathrm{top}}(Z_{\mathfrak{T}}) \cong \bigoplus_{\mathbf{v}^1,\mathbf{v}^2,\vec{\lambda},\rho} \mathrm{Hom}(H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho, H_{\mathrm{top}}(\mathfrak{T}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho) \tag{4.6}$$

from Lemma 3.14 and the above decomposition.

Thus $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is determined by its convolution action $H_{\mathrm{top}}(\mathfrak{M}_x^{\mathbb{C}^*}) \to H_{\mathrm{top}}(\mathfrak{T}_x)$ for $x = x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}$ in each stratum. Then the converse of Proposition 3.5 is clear.

4(iii). **Tensor product multiplicities in terms of IC sheaves.** As in the previous subsection, we also refine the stratification in Lemma 2.3 as

$$\mathfrak{M}_0^{\mathbb{C}^*} = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2) \times \mathfrak{M}_0(\vec{\lambda}),$$

where

$$\mathfrak{M}_0(\vec{\lambda}) = S_{\lambda_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1,0) \times S_{\lambda_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2,0) \times \cdots \times S_{\lambda_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N,0)$$

as before. For a simple local system ρ on $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times\mathfrak{M}_0(\vec{\lambda})$, we consider the corresponding IC sheaf. We then have

$$\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}} = \bigoplus IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times\mathfrak{M}_0(\vec{\lambda}),\rho) \otimes H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho,$$

where $x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}$ is a point in the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times\mathfrak{M}_0(\vec{\lambda})$. Then $H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho$ is a simple module of $H_{\mathrm{top}}(Z^{\mathbb{C}^*})$, and any simple module is isomorphic to a module of this form as before.

By Δ_c in (3.10) we consider $H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho$ as a module over $H_{\mathrm{top}}(Z)$. Since $H_{\mathrm{top}}(Z)$ is semisimple, it decomposes into a direct sum of $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}}})_{\rho'}$ with various $\mathbf{v}^0$, $\vec{\lambda}'$, ρ'. Let us define the 'tensor product multiplicity' by

$$n^{\mathbf{v}^0;\vec{\lambda}',\rho'}_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda},\rho} \overset{\text{def.}}{=} [H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda}}})_\rho : H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\vec{\lambda}'}})_{\rho'}]. \tag{4.7}$$

These multiplicity has a geometric description:

Theorem 4.8. *The multiplicity $n^{\mathbf{v}^0;\vec{\lambda}',\rho'}_{\mathbf{v}^1,\mathbf{v}^2;\vec{\lambda},\rho}$ is equal to*

$$[p_!i^*IC(\mathfrak{M}_0(\mathbf{v}^0;\vec{\lambda}'),\rho') : IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times\mathfrak{M}_0(\vec{\lambda}),\rho)].$$

Recall that $p_!i^*IC(\mathfrak{M}_0(\mathbf{v}^0;\vec{\lambda}),\rho)$ is a direct sum of $IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times\mathfrak{M}_0(\vec{\lambda}'),\rho')$ with various $\mathbf{v}^1$, $\mathbf{v}^2$, $\vec{\lambda}'$, ρ' by Lemma 3.11. The right hand side of the above formula denote the decomposition multiplicity.

This formula is a direct consequence of decompositions of $\pi_!\mathcal{C}_{\mathfrak{M}}$, $\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ and the identification of Δ_c with $\mathrm{Ad}(c)p_!i^*$ in (3.18). (See also [23, Th. 5.1].)

Remark 4.9. For a quiver of type ADE, we do not have data $\vec{\lambda}$, ρ, $\vec{\lambda}'$, ρ', and multiplicities $n^{\mathbf{v}^0}_{\mathbf{v}^1,\mathbf{v}^2}$ is nothing but the usual tensor product multiplicity of finite dimensional representations of the Lie algebra $\mathfrak{g}$ of type ADE [23, Th. 5.1].

In general, the author does not know how to understand the behavior of $IC(\mathfrak{M}_0(\mathbf{v}^0;\vec{\lambda}),\rho)$ under $p_!i^*$. For affine types, only constant sheaves $\mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2/\Gamma)}}$ appear in $\pi_!\mathcal{C}_{\mathfrak{M}}$, and local systems on $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times S_\lambda(\mathbb{C}^2\setminus\{0\}/\Gamma)$ can be determined. It should be possible to determine multiplicities from the tensor product multiplicity for the affine Lie algebra. But it is yet to be clarified.

4(iv). **Fixed point version.** Let a be a semisimple element in the Lie algebra of $\mathbb{G}$. Then it defines a homomorphism

$$\rho_a \colon H^*_{\mathbb{G}}(\mathrm{pt}) \to \mathbb{C}.$$

Let A be the smallest torus whose Lie algebra contains a. Let Z^A be the fixed point set Then we have a homomorphism

$$r_a \colon H^{\mathbb{G}}_*(Z) \otimes_{H^*_{\mathbb{G}}(\mathrm{pt})} \mathbb{C} \to H_*(Z^A)$$

as the composite of the pull back and the multiplication of $1 \otimes \rho_a(e(N))^{-1}$, where N is the normal bundle of $\mathfrak{M}^A$ in $\mathfrak{M}$, and $e(N)$ is its A-equivariant Euler class. (See [5, §5.11].) Then r_a is an algebra isomorphism. Similarly we have

$$r_a \colon H^{\mathbb{G}}_*(Z^{\mathbb{C}^*}) \otimes_{H^*_{\mathbb{G}}(\mathrm{pt})} \mathbb{C} \to H_*((Z^{\mathbb{C}^*})^A).$$

We then have a specialized coproduct

$$\Delta_c \colon H_*(Z^A) \to H_*((Z^{\mathbb{C}^*})^A).$$

Those convolution algebras can be studied in terms of perverse sheaves appearing whose shifts appear in direct summand in $\pi^A \mathbb{C}_{\mathfrak{M}^A}$, $(\pi^{\mathbb{C}^*})^A_! \mathbb{C}_{(\mathfrak{M}^{\mathbb{C}^*})^A}$, where π^A, $(\pi^{\mathbb{C}^*})^A$ are restrictions of π and $\pi^{\mathbb{C}^*}$ to A-fixed point sets $\mathfrak{M}^A$ and $(\mathfrak{M}^{\mathbb{C}^*})^A$. See [5, §8.6] for detail.

The tensor product multiplicities with respect to the specialized Δ_c are described by the functor $p^A_!(i^A)^*$, where p^A, i^A are restrictions of p and i to A-fixed point sets. Since the result is almost the same as Theorem 5.10, we omit the detail. The difference is that the algebra is not semisimple in general, and multiplicities are considered in the Grothendieck group of the category of modules of convolution algebras. In geometric side, perverse sheaves are *not* preserved by the functor $p^A_!(i^A)^*$. They are sent to direct sums of *shifts* of perverse sheaves in general.

As we mentioned in the introduction, the target of Δ_c in (3.10) is $H_*(Z^{\mathbb{C}^*})$, which is larger than the tensor product of the corresponding algebra for $\mathbf{w}^1$, $\mathbf{w}^2$ in general. This is because of the existence of the third factor in Lemma 2.3(1). To avoid this, we assume that generators $\mathrm{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} \colon V_{\mathrm{o}(h_1)} \to V_{\mathrm{i}(h_N)} = V_{\mathrm{o}(h_1)})$ have nontrivial weights with respect to A. Then the A-fixed point set in the third factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ is automatically trivial, and hence we have

$$(Z^{\mathbb{C}^*})^A = \bigsqcup_{\mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}} Z(\mathbf{v}^1, \mathbf{w}^1)^A \times Z(\mathbf{v}^2, \mathbf{w}^2)^A.$$

This assumption is rather mild and satisfied for example if the compositions of $A \to \mathbb{G}$ with the projections $\mathbb{G} \to \mathbb{C}^*$ to the first and second factor of $\mathbb{G}$ both have positive weights. This condition occurs when we study modules of $Y(\mathfrak{g})$ for example, as both are identities in that case.

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