J. ALGEBRAIC GEOMETRY
00 (XXXX) 000–000
S 1056-3911(XX)0000-0

DISCRETENESS OF LOG DISCREPANCIES OVER LOG CANONICAL TRIPLES ON A FIXED PAIR

MASAYUKI KAWAKITA

Abstract

For a fixed pair and fixed exponents, we prove the discreteness of log discrepancies over all log canonical triples formed by attaching a product of ideals with given exponents.

1. Introduction

The log minimal model program (LMMP) is a program to find a good representative in each birational equivalence class of varieties by comparing the log canonical divisors. The log discrepancy, appearing in the relative log canonical divisor, is hence a fundamental invariant in the LMMP. For a triple $(X, \Delta, \mathfrak{a})$, the *log discrepancy* $a_E(X, \Delta, \mathfrak{a})$ is attached to each divisor E over X. The minimum of those $a_E(X, \Delta, \mathfrak{a})$ with E mapped onto a subset Z of X is called the *minimal log discrepancy* at η_Z and denoted by $\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$. Refer to Section 2 for the precise definitions.

Shokurov conjectures the ascending chain condition (ACC) [10], [11, Conjecture 4.2] of the set of the minimal log discrepancies of all pairs with given coefficients in fixed dimension. Its importance is recognised in his reduction [12] of the termination of flips (in the relatively projective case) to this ACC and the lower semi-continuity of minimal log discrepancies. The main theorem of this paper is the *discreteness* of log discrepancies over log canonical triples on a fixed pair. Let $\mathcal{D}_X$ denote the set of divisors over X.

Theorem 1.1. *Let (X, Δ) be a pair and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \Delta, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid \mathfrak{a}_j \subset \mathcal{O}_X,\ E \in \mathcal{D}_X,\ (X, \Delta, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \text{ lc at } \eta_{c_X(E)}\}$$

is discrete in $\mathbb{R}$, where $c_X(E)$ is the centre of E on X.

Note that it is trivial in the case of rational boundary and exponents. The condition of log canonicity is necessary, see Remark 5.1.

Partial support was provided by Grant-in-Aid for Young Scientists (A) 24684003.

Theorem 1.1 asserts a special case of Shokurov's ACC conjecture.

Theorem 1.2. *Let (X, Δ) be a pair and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\mathrm{mld}_{\eta_Z}(X, \Delta, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid \mathfrak{a}_j \subset \mathcal{O}_X,\ Z \subset X\}$$

is finite.

Theorem 1.2 follows from Theorem 1.1 immediately since the minimal log discrepancies in Theorem 1.2 are bounded from above by the maximum of $\mathrm{mld}_x(X, \Delta)$ for all $x \in X$.

We prove Theorem 1.1 in Section 4 by developing the study [3], [4], [5], [9] of the ACC for *log canonical thresholds* due to de Fernex, Ein, Mustaţă and Kollár. We use their construction of a *generic limit* $(W, \mathfrak{a})$, reviewed in Section 3, from a collection of varieties W_i with bounded singularities and ideals $\mathfrak{a}_i$. W is the spectrum of a complete local ring over an extension of the ground field. They showed the equivalence of the log canonicity of $(W, \mathfrak{a})$ and general $(W_i, \mathfrak{a}_i)$ in order to obtain the ACC for log canonical thresholds on varieties with bounded singularities. We apply this equivalence to small perturbations of the exponents in $\mathfrak{a}$. It implies the boundedness of the orders appearing in the expression of $a_{F_i}(W_i, \mathfrak{a}_i)$, leading to Theorem 1.1.

Several extensions of Theorems 1.1, 1.2 and relevant remarks are given in Section 5. For example, applying Theorem 1.1 and inversion of adjunction [6] for locally complete intersection (lci) singularities, we obtain the following in Corollary 5.4.

Theorem 1.3. *Fix an integer d and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\mathrm{mld}_{\eta_Z}(X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid X \text{ lci},\ \dim X \leq d,\ \mathfrak{a}_j \subset \mathcal{O}_X,\ Z \subset X\}$$

is finite.

We work over an algebraically closed field k of characteristic zero.

2. Log discrepancies

A *pair* (X, Δ) consists of a normal variety X and a *boundary* Δ, that is, an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We treat a *triple* $(X, \Delta, \mathfrak{a})$ by attaching a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ with real exponents $r_j \in \mathbb{R}_{\geq 0}$. An *extraction* of X is a normal variety X' with a proper birational morphism $\varphi\colon X' \to X$. A prime divisor E on such an extraction X' is called a divisor *over* X, and the image $\varphi(E)$ on X is called the *centre* of E on X and denoted by $c_X(E)$. We denote

by $\mathcal{D}_X$ the set of divisors over X. We define the *log discrepancy* of E with respect to the triple $(X, \Delta, \mathfrak{a})$ as

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \operatorname{ord}_E(K_{X'} - \varphi^*(K_X + \Delta)) - \operatorname{ord}_E \mathfrak{a},$$

where $\operatorname{ord}_E \mathfrak{a} := \sum_j r_j \operatorname{ord}_E \mathfrak{a}_j$ for $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$. The triple $(X, \Delta, \mathfrak{a})$ is said to be *log canonical* (*lc*), *Kawamata log terminal* (*klt*) if $a_E(X, \Delta, \mathfrak{a}) \geq 0$, > 0 respectively for all $E \in \mathcal{D}_X$, and said to be *canonical*, *terminal* if $a_E(X, \Delta, \mathfrak{a}) \geq 1$, > 1 respectively for all exceptional $E \in \mathcal{D}_X$. Let Z be an irreducible closed subset of X and η_Z its generic point. The *minimal log discrepancy* $\operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ at η_Z is the infimum of $a_E(X, \Delta, \mathfrak{a})$ for all $E \in \mathcal{D}_X$ with centre Z. It is either a non-negative real number or $-\infty$. The log canonicity of $(X, \Delta, \mathfrak{a})$ about η_Z is equivalent to $\operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) \geq 0$. We say that $E \in \mathcal{D}_X$ *computes* $\operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ if $c_X(E) = Z$ and $a_E(X, \Delta, \mathfrak{a}) = \operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ (or negative when $\operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = -\infty$). We are often reduced to the case when Z is a closed point since $\operatorname{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = \operatorname{mld}_z(X, \Delta, \mathfrak{a}) - \dim Z$ for general $z \in Z$, see [1, Proposition 2.1].

3. Generic limits

The generic limit is a limit of ideals in a fixed local ring. It was constructed first by de Fernex and Mustaţă in [5] using ultraproducts, and the construction was then simplified by Kollár in [9]. It is clearly exposed in [3, Section 4], [4, Section 3].

Set $R = k[[x_1, \ldots, x_N]]$ with maximal ideal $\tilde{\mathfrak{m}}$. We fix integers m and e. For every l, let $\mathcal{H}_l$ be the Hilbert scheme parametrising ideals in R containing $\tilde{\mathfrak{m}}^l$. Let $\mathcal{G}$ be the parameter space for ideals in R generated by polynomials in $\tilde{\mathfrak{m}}$ of degree $\leq m$. Set $\mathcal{Z}_l = \mathcal{G} \times (\mathcal{H}_l)^e$. We have a natural surjective map $t_l \colon \mathcal{Z}_l \to \mathcal{Z}_{l-1}$, and by generic flatness, there exists a stratification of $\mathcal{Z}_l$ such that the restriction of t_l on each stratum is a morphism.

We define a generic limit of the collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \ldots, \tilde{\mathfrak{a}}_{ie})\}_{i \in I}$ of $(e+1)$-tuples of ideals in R indexed by an infinite set I, where $\tilde{\mathfrak{p}}_i$ are generated by polynomials in $\tilde{\mathfrak{m}}$ of degree $\leq m$. One can construct locally closed irreducible subsets $Z_l^\circ \subset \mathcal{Z}_l$ such that

(i) t_l induces a dominant morphism $Z_l^\circ \to Z_{l-1}^\circ$,
(ii) $I_l := \{i \in I \mid (\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1} + \tilde{\mathfrak{m}}^l, \ldots, \tilde{\mathfrak{a}}_{ie} + \tilde{\mathfrak{m}}^l) \in Z_l^\circ\}$ is infinite,
(iii) the set of points in $\mathcal{Z}_l$ indexed by I_l is dense in Z_l°.

We take the union $K = \bigcup_l k(Z_l^\circ)$ of the function fields by the inclusions $k(Z_{l-1}^\circ) \subset k(Z_l^\circ)$. For each l, the morphism $\operatorname{Spec} K \to Z_l^\circ \subset \mathcal{Z}_l$ corresponds to an $(e+1)$-tuple $(\tilde{\mathfrak{p}}(l); \tilde{\mathfrak{a}}_1(l), \ldots, \tilde{\mathfrak{a}}_e(l))$ of ideals in $R_K = K[[x_1, \ldots, x_N]]$.

Then there exists an $(e+1)$-tuple $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_e)$ of ideals in R_K such that $\tilde{\mathfrak{p}}(l) = \tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{a}}_j(l) = \tilde{\mathfrak{a}}_j + \tilde{\mathfrak{m}}_K^l$, where $\tilde{\mathfrak{m}}_K = \tilde{\mathfrak{m}} R_K$. This $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_e)$ is a *generic limit* of our collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \dots, \tilde{\mathfrak{a}}_{ie})\}_{i \in I}$.

We set $W_i = \operatorname{Spec} R/\tilde{\mathfrak{p}}_i$, $W = \operatorname{Spec} R_K/\tilde{\mathfrak{p}}$ with closed points $o_i \in W_i$, $o \in W$, and $\mathfrak{m}_i = \tilde{\mathfrak{m}}/\tilde{\mathfrak{p}}_i$, $\mathfrak{m} = \tilde{\mathfrak{m}}_K/\tilde{\mathfrak{p}}$. Note that $\tilde{\mathfrak{p}}$ is generated by polynomials in $\tilde{\mathfrak{m}}_K$ of degree $\leq m$, so W is the spectrum of a completion of a finitely generated K-algebra. We suppose $\tilde{\mathfrak{p}}_i \subset \tilde{\mathfrak{a}}_{ij}$, then $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{a}}_j$ and write $\mathfrak{a}_{ij} = \tilde{\mathfrak{a}}_{ij}/\tilde{\mathfrak{p}}_i$, $\mathfrak{a}_j = \tilde{\mathfrak{a}}_j/\tilde{\mathfrak{p}}$.

We compare log discrepancies over W and W_i. The notions in Section 2 are extended to the spectra of our complete local rings by the existence of their log resolutions due to Temkin in [13] after Hironaka. This extension is discussed in [4], [5] by de Fernex, Ein and Mustaţă. The following proposition associates the minimal log discrepancy of the generic limit to those of the $\mathfrak{m}_i$-adic approximations of the original pairs. Proposition 3.2 is a consequence of the basic fact that for a family of pairs, the minimal log discrepancy is constant on an open subfamily. The corresponding statement for log canonical thresholds is [3, Proposition 4.4] or [4, Proposition 3.3]. Our (iii) is stronger, but it just needs the extra condition $l_E > \operatorname{ord}_E \mathfrak{a}_j$ for any j.

Definition 3.1. A subset J_l of I_l is said to be *dense* if J_l is infinite and the set of points in $\mathcal{Z}_l$ indexed by J_l is dense in Z_l°.

Proposition 3.2. *With the above notation, suppose that each W_i is klt. Then*

(i) *W is klt.*

(ii) *For each l, there exists a dense subset I_l° of I_l such that*

$$\operatorname{mld}_o(W, \prod_j (\mathfrak{a}_j + \mathfrak{m}^l)^{r_j}) = \operatorname{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j})$$

for all $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$ and all $i \in I_l^\circ$.

(iii) *Fix $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$ and $E \in \mathcal{D}_W$ computing $\operatorname{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j})$. Then there exist an integer l_E, a dense subset I_l^E of I_l° for each $l \geq l_E$, and $E_i \in \mathcal{D}_{W_i}$ computing $\operatorname{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j})$ for each $i \in I_l^E$, such that*

$$\operatorname{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = \operatorname{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j}),$$

$$\operatorname{ord}_E \mathfrak{a}_j = \operatorname{ord}_E(\mathfrak{a}_j + \mathfrak{m}^l) = \operatorname{ord}_{E_i}(\mathfrak{a}_{ij} + \mathfrak{m}_i^l) = \operatorname{ord}_{E_i} \mathfrak{a}_{ij} < l,$$

for all $i \in I_l^E$ with $l \geq l_E$.

Remark 3.3. One can choose $l_{E(s)}$ and $I_l^{E(s)}$ in Proposition 3.2(iii) commonly for a finite collection $\{(r_1(s), \dots, r_e(s); E(s))\}_s$.

We shall use the effective ideal-adic semi-continuity of log canonicity due to Kollár, and de Fernex, Ein and Mustaţă. They applied it to the ACC for log canonical thresholds on varieties with bounded singularities.

Theorem 3.4 ([9, Theorem 32], [3, Theorem 1.4])**.** *Let $W = \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ with closed point o for some log canonical singularity $x \in X$, and $\mathfrak{a}_1, \dots, \mathfrak{a}_e$, $\mathfrak{b}_1, \dots, \mathfrak{b}_e \subset \mathcal{O}_W$, $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Suppose $\operatorname{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$ and it is computed by $E \in \mathcal{D}_W$. If $\mathfrak{a}_j + \mathfrak{p}_j = \mathfrak{b}_j + \mathfrak{p}_j$ for every j, where $\mathfrak{p}_j = \{f \in \mathcal{O}_W \mid \operatorname{ord}_E f > \operatorname{ord}_E \mathfrak{a}_j\}$, then $\operatorname{mld}_o(W, \prod_j \mathfrak{b}_j^{r_j}) = 0$.*

Corollary 3.5. *If $\operatorname{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$ in Proposition 3.2(iii), then $\operatorname{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j}) = 0$ for $i \in I_l^E$.*

4. Discreteness

The purpose of this section is to prove Theorem 1.1. The theorem is reduced to the case when X has $\mathbb{Q}$-factorial terminal singularities by the existence of a $\mathbb{Q}$-factorial terminal extraction $\varphi\colon X' \to X$ with $\Delta' \geq 0$ for $K_{X'} + \Delta' = \varphi^*(K_X + \Delta)$, thanks to [2]. Then we may assume $\Delta = 0$ by forcing $\prod_j \mathfrak{a}_{ij}^{r_j}$ to absorb Δ. Moreover, by the equality $\operatorname{mld}_{\eta_Z}(X, \mathfrak{a}) = \operatorname{mld}_z(X, \mathfrak{a}) - \dim Z$ given in Section 2, we may consider only the log discrepancies of divisors whose centres are closed points. Hence it suffices to prove the following theorem.

Theorem 4.1. *Let X be a variety with klt singularities and $a, r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Let I be an infinite set indexing $a_i = a_{E_i}(X, \prod_j \mathfrak{a}_{ij}^{r_j}) \leq a$ with $\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ie} \subset \mathcal{O}_X$, $E_i \in \mathcal{D}_X$ such that $x_i = c_X(E_i)$ is a closed point at which $(X, \prod_j \mathfrak{a}_{ij}^{r_j})$ is log canonical. Then there exists an infinite subset $I^\circ \subset I$ such that a_i is constant for $i \in I^\circ$.*

Since X is covered by finitely many affine open subvarieties, we can fix integers N and m so that for each $i \in I$ there exists an ideal $\tilde{\mathfrak{p}}_i$ in $R = k[[x_1, \dots, x_N]]$ generated by polynomials of degree $\leq m$ satisfying $\widehat{\mathcal{O}}_{X,x_i} \simeq R/\tilde{\mathfrak{p}}_i$. We apply the generic limit construction in Section 3 to the collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \dots, \tilde{\mathfrak{a}}_{ie})\}_{i \in I}$; here we let $\mathfrak{a}_{ij}$ denote also the image in $R/\tilde{\mathfrak{p}}_i$ of $\mathfrak{a}_{ij}\widehat{\mathcal{O}}_{X,x_i}$ by abuse of notation, and define $\tilde{\mathfrak{a}}_{ij}$ as the inverse image in R of $\mathfrak{a}_{ij}$. The generic limit $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_e)$ is defined in some $R_K = K[[x_1, \dots, x_N]]$. We follow the notation in Section 3. We have $a_i = a_{F_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j})$ for $F_i = E_i \times_X W_i$, and $(W_i, \prod_j \mathfrak{a}_{ij}^{r_j})$ is log canonical. By Proposition 3.2(ii), $(W, \prod_j \mathfrak{a}_j^{r_j})$ is also log canonical.

We shall find perturbations of the exponents r_j preserving the log canonicity. Set $r_0 = 1$. By permutation, we may assume that $r_0, \dots, r_{e'}$ for some $0 \leq e' \leq e$ form a basis of the $\mathbb{Q}$-vector space spanned by $r_0, \dots, r_e$. We write $r_j = \sum_{j'=0}^{e'} q_{jj'} r_{j'}$ with $q_{jj'} \in \mathbb{Q}$, then $\prod_j \mathfrak{a}_j^{r_j} = \prod_{j'=0}^{e'} (\prod_j \mathfrak{a}_j^{q_{jj'}})^{r_{j'}}$ formally.

We put $\mathfrak{b}_{j'} := \prod_j \mathfrak{a}_j^{q_{jj'}}$ and $\mathfrak{b}_{ij'} := \prod_j \mathfrak{a}_{ij}^{q_{jj'}}$. Setting $s_0 = 1$, for $\varepsilon > 0$ we define the finite set

$$S_\varepsilon := \big\{(s_1, \dots, s_e) \bigm| |s_{j'} - r_{j'}| = \varepsilon \ \forall 1 \le j' \le e',\ s_j = \sum_{j'=0}^{e'} q_{jj'} s_{j'} \ \forall j\big\}.$$

Lemma 4.2. *There exist ε, l and a dense subset $J_l \subset I_l$ such that all $s_j \ge 0$ and $(W_i, \prod_j \mathfrak{a}_{ij}^{s_j})$ is log canonical for any $(s_j)_{1\le j\le e} \in S_\varepsilon$ and $i \in J_l$.*

Proof. First we find ε such that all $s_j \ge 0$ and $(W, \prod_j \mathfrak{a}_j^{s_j})$ is log canonical for any $(s_j) \in S_\varepsilon$. Indeed, if $E \in \mathcal{D}_W$ has $a_E(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$, then $a_E(W) = \sum_{j'=0}^{e'} r_{j'} \operatorname{ord}_E \mathfrak{b}_{j'}$, and thus $\operatorname{ord}_E \mathfrak{b}_{j'} = 0$ for $1 \le j' \le e'$ by the $\mathbb{Q}$-linear independence of $r_{j'}$. This means that the log discrepancy of E remains zero when we perturb the exponents $r_1, \dots, r_{e'}$ in $\prod_{j'} \mathfrak{b}_{j'}^{r_{j'}} (= \prod_j \mathfrak{a}_j^{r_j})$. Hence one sees that on a fixed log resolution of $(W, \prod_j \mathfrak{a}_j^{r_j})$, every divisor has non-negative log discrepancy for any sufficiently small perturbation of the exponents $r_1, \dots, r_{e'}$. This guarantees the existence of the required ε.

For each $s = (s_j) \in S_\varepsilon$, we fix $t_s \ge 0$ such that $\operatorname{mld}_o(W, \prod_j \mathfrak{a}_j^{s_j} \mathfrak{m}^{t_s}) = 0$. By Corollary 3.5 and Remark 3.3, we obtain l and $J_l \subset I_l$ such that $\operatorname{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{s_j} \mathfrak{m}_i^{t_s}) = 0$ for any $s \in S_\varepsilon$ and $i \in J_l$, implying the log canonicity of $(W_i, \prod_j \mathfrak{a}_{ij}^{s_j})$. q.e.d.

Lemma 4.3. $\sum_{j'=1}^{e'} |\operatorname{ord}_{F_i} \mathfrak{b}_{ij'}| \le \varepsilon^{-1} a_i$ *for $i \in J_l$.*

Proof. We choose $s_{ij'} = r_{j'} \pm \varepsilon$ for $1 \le j' \le e'$ so that $\operatorname{ord}_{F_i} \mathfrak{b}_{ij'}/(s_{ij'} - r_{j'}) \ge 0$, and extend $(s_{ij'})$ to the e-tuple $(s_{ij}) \in S_\varepsilon$ by $s_{ij} = \sum_{j'=0}^{e'} q_{jj'} s_{ij'}$. Then by Lemma 4.2, $0 \le a_{F_i}(W_i, \prod_j \mathfrak{a}_{ij}^{s_{ij}}) = a_i - \varepsilon \sum_{j'=1}^{e'} |\operatorname{ord}_{F_i} \mathfrak{b}_{ij'}|$. q.e.d.

Fix a positive integer n such that nK_X is a Cartier divisor and $nq_{jj'} \in \mathbb{Z}$ for all j, j'. We define the finite set

$$A := [0, a] \cap \Big(\frac{1}{n}\mathbb{Z} + \big\{\sum_{j'=1}^{e'} r_{j'} m_{j'} \bigm| \sum_{j'=1}^{e'} |m_{j'}| \le \varepsilon^{-1} a,\ m_{j'} \in \frac{1}{n}\mathbb{Z} \ \forall j'\big\}\Big).$$

Then by Lemma 4.3, we have $a_i = a_{F_i}(W_i, \prod_{j'=0}^{e'} \mathfrak{b}_{ij'}^{r_{j'}}) = a_{F_i}(W_i, \mathfrak{b}_{i0}) - \sum_{j'=1}^{e'} r_{j'} \operatorname{ord}_{F_i} \mathfrak{b}_{ij'} \in A$ for $i \in J_l$. Theorem 4.1, and Theorem 1.1, are therefore completed.

5. Extensions

First we remark the need of log canonicity in Theorem 1.1.

Remark 5.1. Consider a non-lc pair $(\mathbb{A}^2, (1+r)l)$ where $r > 0$ is irrational and l is a line. Let E_1 be the exceptional divisor of the blow-up of $\mathbb{A}^2$ at a

point on l, and define E_p inductively as the exceptional divisor of the blow-up at the intersection of E_{p-1} and the strict transform of l. Let $E_{p,0} = E_p$ and define $E_{p,q}$ inductively as the exceptional divisor of the blow-up at a general point on $E_{p,q-1}$. Then $a_{E_{p,q}}(\mathbb{A}^2, (1+r)l) = q - pr$. The set of these log discrepancies is dense in $\mathbb{R}$.

The generic limit construction is applicable to bounded singularities in the sense [4] of de Fernex, Ein and Mustaţă. We say that a collection $\{x_i \in X_i\}_i$ of singularities is *bounded* if there exist m and N such that for each i there exists an ideal $\tilde{\mathfrak{p}}_i$ in $R = k[[x_1, \dots, x_N]]$ generated by polynomials of degree $\leq m$ satisfying $\widehat{\mathcal{O}}_{X_i,x_i} \simeq R/\tilde{\mathfrak{p}}_i$. Theorem 1.1 can be formulated for such a collection.

Theorem 5.2. *Let $\mathcal{X}$ be a collection of varieties with bounded klt singularities, and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid X \in \mathcal{X},\ \mathfrak{a}_j \subset \mathcal{O}_X,\ E \in \mathcal{D}_X,\ (X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j})\ lc\ at\ \eta_{c_X(E)}\}$$

is discrete.

We apply it to the minimal log discrepancies of Gorenstein singularities.

Corollary 5.3. *Let $\mathcal{X}$ be a collection of varieties with bounded normal Gorenstein singularities, and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\mathrm{mld}_{\eta_Z}(X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid X \in \mathcal{X},\ \mathfrak{a}_j \subset \mathcal{O}_X,\ Z \subset X\}$$

is finite.

This follows from the boundedness [7, Theorem 2.2] of the minimal log discrepancies of Gorenstein singularities with bounded embedding dimensions. Note that even for Gorenstein log canonical singularities, [7, Theorem 2.2] holds by its proof, and Proposition 3.2(ii), (iii) hold since the klt assumption on W_i in Proposition 3.2 is needed only to bound the Gorenstein index by the results in [4, Appendix B].

We have a further application to lci singularities.

Corollary 5.4. *Fix an integer d and $r_1, \dots, r_e \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j}) \mid X\ lci,\ \dim X \leq d,\ \mathfrak{a}_j \subset \mathcal{O}_X,\ E \in \mathcal{D}_X,$$
$$(X, \prod_{j=1}^{e} \mathfrak{a}_j^{r_j})\ lc\ at\ \eta_{c_X(E)}\}$$

is discrete.

Corollary 5.4, and Theorem 1.3, follow from the consequence of inversion of adjunction [6] that an lci log canonical singularity of dimension d has embedding dimension $\leq 2d$, see [3, Proposition 6.3]. Indeed, it suffices to treat $a_E(X, \prod_j \mathfrak{a}_j^{r_j})$ such that $c_X(E)$ is a closed point x at which X is lci and lc of dimension d. Then $x \in X$ is locally embedded into a smooth ambient space A of dimension $2d$ as the complete intersection of some hypersurfaces $H_1, \dots, H_d$ on A. We have $a_E(X, \prod_j \mathfrak{a}_j^{r_j}) = a_{\tilde{E}}(A, \sum_i H_i, \prod_j \tilde{\mathfrak{a}}_j^{r_j})$ for $\tilde{\mathfrak{a}}_j \subset \mathcal{O}_A$ with $\tilde{\mathfrak{a}}_j \mathcal{O}_X = \mathfrak{a}_j$ and a divisor $\tilde{E} \in \mathcal{D}_A$ on a log resolution of $(A, \sum_i H_i, \prod_j \tilde{\mathfrak{a}}_j)$ whose restriction to the strict transform of X defines E as a valuation.

On the other hand, Shokurov's ACC conjecture is generalised to the case when the exponents vary in a fixed set satisfying the descending chain condition (DCC). Mustaţă observed that an effective ideal-adic semi-continuity for minimal log discrepancies implies the generalised ACC on a fixed pair (see [8, Remark 1.5.1]). This semi-continuity is known in the klt case [8, Theorem 1.6], and for example, we can prove the following.

Proposition 5.5. *Let (X, Δ) be a pair with rational Δ, and R a subset of $\mathbb{R}_{\geq 0}$ satisfying the DCC. Suppose that any accumulation point of R is irrational. Then the set*

$$\{\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}^r) \mid \mathfrak{a} \subset \mathcal{O}_X,\ r \in R,\ Z \subset X\}$$

satisfies the ACC.

Proof. As in the beginning of Section 4, we are reduced to the case of terminal X, and we want the stability of any non-decreasing sequence of $a_i = \mathrm{mld}_{x_i}(X, \Delta, \mathfrak{a}_i^{r_i}) \geq 0$ with $r_i \in R$, x_i closed point, where $i \in \mathbb{N}$. By passing to a subsequence, we may assume that $\mathfrak{a}_i$ is non-trivial at x_i. Then r_i are bounded by the maximum b of $\mathrm{mld}_x(X, \Delta)$ for all $x \in X$, hence we may further assume that $\{r_i\}_i$ is a non-decreasing sequence which has a limit r. If $r \in \mathbb{Q}$, then $r_i = r$ for large i by the assumption on R, and the stability is trivial. Henceforth we assume $r \notin \mathbb{Q}$. As in Section 4, we construct a generic limit $o \in W$, $\mathfrak{d}, \mathfrak{a}$ of $o_i \in W_i$, $\mathfrak{d}_i, \mathfrak{a}_i$ with $a_i = \mathrm{mld}_{o_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r_i})$, where $\mathfrak{d}_i$ is an ideal with fixed rational exponent, corresponding to Δ. We fix $F_i \in \mathcal{D}_{W_i}$ computing $\mathrm{mld}_{o_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r_i})$, that is,

$$a_i = a_{F_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^r) + (r - r_i) \operatorname{ord}_{F_i} \mathfrak{a}_i. \tag{1}$$

By Proposition 3.2 or [4, Corollary 3.4], $(W, \mathfrak{d}\mathfrak{a}^r)$ is log canonical. If $E \in \mathcal{D}_W$ has $a_E(W, \mathfrak{d}\mathfrak{a}^r) = 0$, then $\operatorname{ord}_E \mathfrak{a} = 0$ by $r \notin \mathbb{Q}$. Thus we can find $t > 0$ such that $(W, \mathfrak{d}\mathfrak{a}^{r+t})$ is log canonical as in the proof of Lemma 4.2. We take $t' \geq 0$ such that $\mathrm{mld}_o(W, \mathfrak{d}\mathfrak{a}^{r+t}\mathfrak{m}^{t'}) = 0$. Then Corollary 3.5 shows $\mathrm{mld}_{o_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r+t} \mathfrak{m}_i^{t'}) = 0$ for infinitely many i. In particular,

$$t \operatorname{ord}_{F_i} \mathfrak{a}_i \leq a_{F_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^r) \leq a_i \leq b. \tag{2}$$

Theorem 1.1 and (2) imply the finiteness of possible choices for $a_{F_i}(W_i, \mathfrak{d}_i\mathfrak{a}_i^r)$ and $\mathrm{ord}_{F_i}\mathfrak{a}_i$ for such i. Hence they are constant for infinitely many i. Now (1) provides the constancy of a_i and r_i for large i. q.e.d.

Remark 5.6. When $\mathfrak{d}_i = \mathcal{O}_{W_i}$ and $r \notin \mathbb{Q}$ in the proof, one can further prove $a_i = a := \mathrm{mld}_o(W, \mathfrak{a}^r)$ for large i. Indeed, we may assume $r_i = r$, and $a = \mathrm{mld}_{o_i}(W_i, (\mathfrak{a}_i + \mathfrak{m}_i^l)^r) \geq a_i$ for some $l > t^{-1}b$ by Proposition 3.2(iii). Hence with (2), we have $a \geq a_i = a_{F_i}(W_i, \mathfrak{a}_i^r) = a_{F_i}(W_i, (\mathfrak{a}_i + \mathfrak{m}_i^l)^r) \geq a$, meaning $a_i = a$.

Remark 5.6 is a special case of the following conjecture. The corresponding statement for log canonical thresholds is [4, Corollary 3.4].

Conjecture 5.7. *With the notation in Section 3, we suppose that each W_i is klt and fix $r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}$. Let*

$$J := \{i \in I \mid \mathrm{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j}) = \mathrm{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j})\}.$$

Then $I_l \cap J$ is dense for each l.

Acknowledgements. This paper has its origin in Mustaţă's problem on the ideal-adic semi-continuity for minimal log discrepancies. I am grateful to him for valuable discussions.

References

1. F. Ambro, On minimal log discrepancies, Math. Res. Lett. **6** (1999), 573-580
2. C. Birkar, P. Cascini, C. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, J. Am. Math. Soc. **23** (2010), 405-468
3. T. de Fernex, L. Ein and M. Mustaţă, Shokurov's ACC conjecture for log canonical thresholds on smooth varieties, Duke Math. J. **152** (2010), 93-114
4. T. de Fernex, L. Ein and M. Mustaţă, Log canonical thresholds on varieties with bounded singularities, *Classification of algebraic varieties*, EMS Series of Congress Reports (2011), 221-257
5. T. de Fernex and M. Mustaţă, Limits of log canonical thresholds, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 491-515
6. L. Ein and M. Mustaţă, Inversion of adjunction for local complete intersection varieties, Am. J. Math. **126** (2004), 1355-1365
7. M. Kawakita, Towards boundedness of minimal log discrepancies by Riemann–Roch theorem, Am. J. Math. **133** (2011), 1299-1311
8. M. Kawakita, Ideal-adic semi-continuity problem for minimal log discrepancies, arXiv:1012.0395
9. J. Kollár, Which powers of holomorphic functions are integrable?, arXiv:0805.0756
10. V. Shokurov, Problems about Fano varieties, *Birational geometry of algebraic varieties, Open problems*, Katata 1988, 30-32
11. V. Shokurov, 3-fold log models, J. Math. Sci. **81** (1996), 2667-2699
12. V. Shokurov, Letters of a bi-rationalist V. Minimal log discrepancies and termination of log flips, Tr. Mat. Inst. Steklova **246** (2004), 328-351, translation in Proc. Steklov Inst. Math. **246** (2004), 315-336

13. M. Temkin, Desingularization of quasi-excellent schemes in characteristic zero, Adv. Math. **219** (2008), 488-522

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: masayuki@kurims.kyoto-u.ac.jp