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AUTHOR(S):
Kawakita, Masayuki

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DISCRETENESS OF LOG DISCREPANCIES OVER LOG CANONICAL Triples ON A FIXED PAIR

MASAYUKI KAWAKITA

Abstract

For a fixed pair and fixed exponents, we prove the discreteness of log discrepancies over all log canonical triples formed by attaching a product of ideals with given exponents.

1. Introduction

The log minimal model program (LMMP) is a program to find a good representative in each birational equivalence class of varieties by comparing the log canonical divisors. The log discrepancy, appearing in the relative log canonical divisor, is hence a fundamental invariant in the LMMP. For a triple $(X, \Delta, a)$, the log discrepancy $a_E(X, \Delta, a)$ is attached to each divisor $E$ over $X$. The minimum of those $a_E(X, \Delta, a)$ with $E$ mapped onto a subset $Z$ of $X$ is called the minimal log discrepancy at $\eta_Z$ and denoted by $\operatorname{mld}_{\eta_Z}(X, \Delta, a)$. Refer to Section 2 for the precise definitions.

Shokurov conjectures the ascending chain condition (ACC) [10], [11, Conjecture 4.2] of the set of the minimal log discrepancies of all pairs with given coefficients in fixed dimension. Its importance is recognised in his reduction [12] of the termination of flips (in the relatively projective case) to this ACC and the lower semi-continuity of minimal log discrepancies. The main theorem of this paper is the discreteness of log discrepancies over log canonical triples on a fixed pair. Let $\mathcal{D}_X$ denote the set of divisors over $X$.

Theorem 1.1. Let $(X, \Delta)$ be a pair and $r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}$. Then the set

$$\{a_E(X, \Delta, \prod_{j=1}^e a_j^{r_j}) \mid a_j \subset O_X, E \in \mathcal{D}_X, (X, \Delta, \prod_{j=1}^e a_j^{r_j}) \text{ lc at } \eta_{c_X(E)}\}$$

is discrete in $\mathbb{R}$, where $c_X(E)$ is the centre of $E$ on $X$.

Note that it is trivial in the case of rational boundary and exponents. The condition of log canonicity is necessary, see Remark 5.1.

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Theorem 1.1 asserts a special case of Shokurov’s ACC conjecture.

**Theorem 1.2.** Let \((X, \Delta)\) be a pair and \(r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}\). Then the set

\[
\{ \operatorname{mld}_{\eta_Z}(X, \Delta, \prod_{j=1}^e a_j^{r_j}) \mid a_j \subset O_X, \ Z \subset X \}
\]

is finite.

Theorem 1.2 follows from Theorem 1.1 immediately since the minimal log discrepancies in Theorem 1.2 are bounded from above by the maximum of \(\operatorname{mld}_x(X, \Delta)\) for all \(x \in X\).

We prove Theorem 1.1 in Section 4 by developing the study [3], [4], [5], [9] of the ACC for log canonical thresholds due to de Fernex, Ein, Mustaţă and Kollár. We use their construction of a generic limit \((W, a)\), reviewed in Section 3, from a collection of varieties \(W_i\) with bounded singularities and ideals \(a_i\). \(W\) is the spectrum of a complete local ring over an extension of the ground field. They showed the equivalence of the log canonicity of \((W, a)\) and general \((W_i, a_i)\) in order to obtain the ACC for log canonical thresholds on varieties with bounded singularities. We apply this equivalence to small perturbations of the exponents in \(a\). It implies the boundedness of the orders appearing in the expression of \(a_{F_i}(W_i, a_i)\), leading to Theorem 1.1.

Several extensions of Theorems 1.1, 1.2 and relevant remarks are given in Section 5. For example, applying Theorem 1.1 and inversion of adjunction [6] for locally complete intersection (lci) singularities, we obtain the following in Corollary 5.4.

**Theorem 1.3.** Fix an integer \(d\) and \(r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}\). Then the set

\[
\{ \operatorname{mld}_{\eta_Z}(X, \prod_{j=1}^e a_j^{r_j}) \mid X \text{ lci}, \ \dim X \leq d, \ a_j \subset O_X, \ Z \subset X \}
\]

is finite.

We work over an algebraically closed field \(k\) of characteristic zero.

2. Log discrepancies

A pair \((X, \Delta)\) consists of a normal variety \(X\) and a boundary \(\Delta\), that is, an effective \(\mathbb{R}\)-divisor such that \(K_X + \Delta\) is an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor. We treat a triple \((X, \Delta, a)\) by attaching a formal product \(a = \prod_{j} a_j^{r_j}\) of finitely many coherent ideal sheaves \(a_j\) with real exponents \(r_j \in \mathbb{R}_{\geq 0}\). An extraction of \(X\) is a normal variety \(X'\) with a proper birational morphism \(\varphi: X' \to X\). A prime divisor \(E\) on such an extraction \(X'\) is called a divisor over \(X\), and the image \(\varphi(E)\) on \(X\) is called the centre of \(E\) on \(X\) and denoted by \(c_X(E)\). We denote
by $D_X$ the set of divisors over $X$. We define the log discrepancy of $E$ with respect to the triple $(X, \Delta, a)$ as

$$ a_E(X, \Delta, a) := 1 + \text{ord}_E(K_X^F - \varphi^*(K_X + \Delta)) - \text{ord}_E a, $$

where $\text{ord}_E a := \sum_j r_j \text{ord}_E a_j$ for $a = \prod_j a_j^{r_j}$. The triple $(X, \Delta, a)$ is said to be log canonical (lc), Kawamata log terminal (klt) if $a_E(X, \Delta, a) \geq 0$, $> 0$ respectively for all $E \in D_X$, and said to be canonical, terminal if $a_E(X, \Delta, a) \geq 1, > 1$ respectively for all exceptional $E \in D_X$. Let $Z$ be an irreducible closed subset of $X$ and $\eta_Z$ its generic point. The minimal log discrepancy $\text{mld}_{\eta_Z}(X, \Delta, a)$ at $\eta_Z$ is the infimum of $a_E(X, \Delta, a)$ for all $E \in D_X$ with centre $Z$. It is either a non-negative real number or $-\infty$. The log canonicity of $(X, \Delta, a)$ about $\eta_Z$ is equivalent to $\text{mld}_{\eta_Z}(X, \Delta, a) \geq 0$. We say that $E \in D_X$ computes $\text{mld}_{\eta_Z}(X, \Delta, a)$ if $c_X(E) = Z$ and $a_E(X, \Delta, a) = \text{mld}_{\eta_Z}(X, \Delta, a)$ (or negative when $\text{mld}_{\eta_Z}(X, \Delta, a) = -\infty$). We are often reduced to the case when $Z$ is a closed point since $\text{mld}_{\eta_Z}(X, \Delta, a) = \text{mld}_z(X, \Delta, a) - \dim Z$ for general $z \in Z$, see [1, Proposition 2.1].

3. Generic limits

The generic limit is a limit of ideals in a fixed local ring. It was constructed first by de Fernex and Mustaţă in [5] using ultraproducts, and the construction was then simplified by Kollár in [9]. It is clearly exposed in [3, Section 4], [4, Section 3].

Set $R = k[[x_1, \ldots, x_N]]$ with maximal ideal $\mathfrak{m}$. We fix integers $m$ and $e$. For every $l$, let $H_l$ be the Hilbert scheme parametrising ideals in $R$ containing $\mathfrak{m}^l$. Let $\mathcal{G}$ be the parameter space for ideals in $R$ generated by polynomials in $\mathfrak{m}$ of degree $\leq m$. Set $Z_l = \mathcal{G} \times (H_l)^e$. We have a natural surjective map $t_l: Z_l \to Z_{l-1}$, and by generic flatness, there exists a stratification of $Z_l$ such that the restriction of $t_l$ on each stratum is a morphism.

We define a generic limit of the collection $\{(\hat{p}; \hat{a}_1, \ldots, \hat{a}_e)\}_{i \in I}$ of $(e + 1)$-tuples of ideals in $R$ indexed by an infinite set $I$, where $\hat{p}_i$ are generated by polynomials in $\mathfrak{m}$ of degree $\leq m$. One can construct locally closed irreducible subsets $Z_i^\circ \subset Z_l$ such that

(i) $t_l$ induces a dominant morphism $Z_i^\circ \to Z_{l-1}^\circ$,

(ii) $I_i := \{ i \in I \mid (\hat{p}_i; \hat{a}_i + \hat{m}_i, \ldots, \hat{a}_i, \hat{m}_i) \in Z_l^\circ \}$ is infinite,

(iii) the set of points in $Z_l$ indexed by $I_l$ is dense in $Z_l^\circ$.

We take the union $K = \bigcup_l k(Z_l^\circ)$ of the function fields by the inclusions $k(Z_{l-1}^\circ) \subset k(Z_l^\circ)$. For each $l$, the morphism $\text{Spec} K \to Z_l^\circ \subset Z_l$ corresponds to an $(e + 1)$-tuple $((\hat{p}(l); \hat{a}_1(l), \ldots, \hat{a}_e(l))$ of ideals in $R_K = K[[x_1, \ldots, x_N]]$. 
Then there exists an \((e + 1)\)-tuple \((\ilde{p}; \tilde{a}_1, \ldots, \tilde{a}_r)\) of ideals in \(R_K\) such that \(\tilde{p}(l) = \tilde{p}\) and \(\tilde{a}_j(l) = \tilde{a}_j + \tilde{m}_j^l\), where \(\tilde{m}_j = \tilde{m}R_K\). This \((\tilde{p}; \tilde{a}_1, \ldots, \tilde{a}_r)\) is a generic limit of our collection \(\{(\tilde{p}_i; \tilde{a}_{i1}, \ldots, \tilde{a}_{ir})\}_{i \in I}\).

We set \(W_i = \text{Spec } R/\tilde{p}_i\), \(W = \text{Spec } \tilde{R}/\tilde{p}\) with closed points \(a_i \in W_i\), \(o \in W\), and \(m_i = \tilde{m}/\tilde{p}_i\), \(m = \tilde{m}_K/\tilde{p}\). Note that \(\tilde{p}\) is generated by polynomials in \(\tilde{m}_K\) of degree \(\leq m\), so \(W\) is the spectrum of a completion of a finitely generated \(K\)-algebra. We suppose \(\tilde{p}_i \subset \tilde{a}_j\), then \(\tilde{p} \subset \tilde{a}_j\) and write \(a_{ij} = a_{ij}/\tilde{p}_i\), \(a_j = \tilde{a}_j/\tilde{p}\).

We compare log discrepancies over \(W\) and \(W_i\). The notions in Section 2 are extended to the spectra of our complete local rings by the existence of their log resolutions due to Temkin in [13] after Hironaka. This extension is discussed in [4], [5] by de Fernex, Ein and Mustaţă. The following proposition associates the minimal log discrepancy of the generic limit to those of the \(m_j\)-adic approximations of the original pairs. Proposition 3.2 is a consequence of the basic fact that for a family of pairs, the minimal log discrepancy is constant on an open subfamily. The corresponding statement for log canonical thresholds is [3, Proposition 4.4] or [4, Proposition 3.3]. Our (iii) is stronger, but it just needs the extra condition \(l_E > \text{ord}_E a_j\) for any \(j\).

**Definition 3.1.** A subset \(J_i\) of \(I_i\) is said to be dense if \(J_i\) is infinite and the set of points in \(Z_i\) indexed by \(J_i\) is dense in \(Z_i\).

**Proposition 3.2.** With the above notation, suppose that each \(W_i\) is klt. Then

(i) \(W\) is klt.

(ii) For each \(l\), there exists a dense subset \(I_i^l\) of \(I_i\) such that

\[
\text{mld}_o(W, \prod_j (a_j + m_j^l)^{r_j}) = \text{mld}_o(W_i, \prod_j (a_{ij} + m_j^l)^{r_j})
\]

for all \(r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}\) and all \(i \in I_i^l\).

(iii) Fix \(r_1, \ldots, r_e \in \mathbb{R}_{\geq 0}\) and \(E \in D_W\) computing \(\text{mld}_o(W, \prod_j (a_j^r)^{r_j})\). Then there exist an integer \(l_E\), a dense subset \(I_i^{l_E}\) of \(I_i^l\) for each \(l \geq l_E\), and \(E_i \subset D_{W_i}\) computing \(\text{mld}_o(W_i, \prod_j (a_{ij} + m_i^l)^{r_j})\) for each \(i \in I_i^{l_E}\), such that

\[
\text{mld}_o(W, \prod_j (a_j^r)^{r_j}) = \text{mld}_o(W_i, \prod_j (a_{ij} + m_i^l)^{r_j}),
\]

\[
\text{ord}_{E_i} a_j = \text{ord}_{E} (a_j + m_i^l) = \text{ord}_{E_i} (a_{ij} + m_i^l) = \text{ord}_{E_i} a_{ij} < l,
\]

for all \(i \in I_i^{l_E}\) with \(l \geq l_E\).

**Remark 3.3.** One can choose \(l_{E(s)}\) and \(l_i^{E(s)}\) in Proposition 3.2(iii) commonly for a finite collection \(\{(r_1(s), \ldots, r_e(s); E(s))\}_s\).
We shall use the effective ideal-adic semi-continuity of log canonicity due to Kollár, and de Fernex, Ein and Mustaţă. They applied it to the ACC for log canonical thresholds on varieties with bounded singularities.

**Theorem 3.4** ([9, Theorem 32], [3, Theorem 1.4]). \( W = \text{Spec} \mathcal{O}_{X,e} \) with closed point \( o \) for some log canonical singularity \( x \in X \), and \( a_1, \ldots, a_e, b_1, \ldots, b_e \subset \mathcal{O}_W \), \( r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Suppose \( \text{mld}_o(W, \prod_j a_j^{r_j}) = 0 \) and it is computed by \( E \in D_W \). If \( a_i + p_j = b_i + p_j \) for every \( j \), where \( p_j = \{ f \in \mathcal{O}_W \mid \text{ord}_f f > \text{ord}_E f \} \), then \( \text{mld}_o(W, \prod_j b_j^{p_j}) = 0 \).

**Corollary 3.5.** If \( \text{mld}_o(W, \prod_j a_j^{r_j}) = 0 \) in Proposition 3.2(ii), then \( \text{mld}_o(W, \prod_j a_j^{r_j}) = 0 \) for \( i \in I^F \).

4. Discreteness

The purpose of this section is to prove Theorem 1.1. The theorem is reduced to the case when \( X \) has \( \mathbb{Q} \)-factorial terminal singularities by the existence of a \( \mathbb{Q} \)-factorial terminal extraction \( \varphi: X' \to X \) with \( \Delta' \geq 0 \) for \( K_{X'} + \Delta' = \varphi^*(K_X + \Delta) \), thanks to [2]. Then we may assume \( \Delta = 0 \) by forcing \( \prod_j a_j^{r_j} \) to absorb \( \Delta \). Moreover, by the equality \( \text{mld}_a(X, a) = \text{mld}_s(X, a) = \dim Z \) given in Section 2, we may consider only the log discrepancies of divisors whose centres are closed points. Hence it suffices to prove the following theorem.

**Theorem 4.1.** Let \( X \) be a variety with klt singularities and \( a_1, r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Let \( I \) be an infinite set indexing \( a_i = a_{E_i} = \prod_j a_j^{r_j} \leq a \) with \( a_1, \ldots, a_e \subset \mathcal{O}_X \), \( E_i \in D_X \) such that \( x_i = c_X(E_i) \) is a closed point at which \( X, \prod_j a_j^{r_j} \) is log canonical. Then there exists an infinite subset \( I_0 \subset I \) such that \( a_i \) is constant for \( i \in I_0 \).

Since \( X \) is covered by finitely many affine open subvarieties, we can fix integers \( N \) and \( m \) so that for each \( i \in I \) there exists an ideal \( \mathfrak{p}_i \) in \( R = k[[x_1, \ldots, x_N]] \) generated by polynomials of degree \( \leq m \) satisfying \( \mathfrak{O}_{X,x_i} \cong R/\mathfrak{p}_i \). We apply the generic limit construction in Section 3 to the collection \( \{(\mathfrak{p}_i; \mathfrak{a}_1, \ldots, \mathfrak{a}_e)\}_{i \in I} \); here we let \( \mathfrak{a}_{ij} \) denote also the image in \( R/\mathfrak{p}_i \) of \( \mathfrak{a}_{ij} \mathcal{O}_{X,x_i} \) by abuse of notation, and define \( \mathfrak{a}_{ij} \) as the inverse image in \( R \) of \( \mathfrak{a}_{ij} \). The generic limit \( (\hat{\mathfrak{p}}; \hat{\mathfrak{a}}_1, \ldots, \hat{\mathfrak{a}}_e) \) is defined in some \( R_K = K[[x_1, \ldots, x_N]] \).

We shall find perturbations of the exponents \( r_j \) preserving the log canonicity. Set \( r_0 = 1 \). By permutation, we may assume that \( r_0, \ldots, r_{e'} \) for some \( 0 \leq e' \leq e \) form a basis of the \( \mathbb{Q} \)-vector space spanned by \( r_0, \ldots, r_e \). We write \( r_j' = \sum_{j'=0}^{e'} q_{jj'} r_{j'} \) with \( q_{jj'} \in \mathbb{Q} \), then \( \prod_j a_j^{r_{jj'}} = \prod_j a_j^{q_{jj'} r_{j'}} \) formally.
We put $b_{j'} := \prod_j a_{ij}^{q_{ij}'}$ and $b_i := \prod_j a_{ij}^{q_{ij}}$. Setting $s_0 = 1$, for $\varepsilon > 0$ we define the finite set

$$S_{\varepsilon} := \{ (s_1, \ldots, s_e) \mid |s_j - r_{j'}| = \varepsilon \forall 1 \leq j' \leq e', \ s_j = \sum_{j'=0}^{e'} q_{ij}^{'j} s_{j'} \forall j \}.$$  

**Lemma 4.2.** There exist $\varepsilon$, $l$ and a dense subset $J_1 \subset J_1$ such that all $s_j \geq 0$ and $(W_i, \prod_j a_{ij}^s)$ is log canonical for any $(s_j)_{1 \leq j \leq e} \in S_{\varepsilon}$ and $i \in J_1$.

**Proof.** First we find $\varepsilon$ such that all $s_j \geq 0$ and $(W_i, \prod_j a_{ij}^s)$ is log canonical for any $(s_j) \in S_{\varepsilon}$. Indeed, if $E \in D(W)$ has $a_E(W, \prod_j a_{ij}^s) = 0$, then $a_E(W) = \sum_{j'=0}^{e'} r_{j'} \ord_E b_{j'}$, and thus $\ord_E b_{j'} = 0$ for $1 \leq j' \leq e'$ by the $\mathbb{Q}$-linear independence of $r_{j'}$. This means that the log discrepancy of $E$ remains zero when we perturb the exponents $r_1, \ldots, r_{e'}$ in $\prod_{j'} b_{j'}^{r_{j'}} (= \prod_j a_{ij}^s)$. Hence one sees that on a fixed log resolution of $(W_i, \prod_j a_{ij}^s)$, every divisor has non-negative log discrepancy for any sufficiently small perturbation of the exponents $r_1, \ldots, r_{e'}$. This guarantees the existence of the required $\varepsilon$.

For each $s = (s_j) \in S_{\varepsilon}$, we fix $s'(s) \geq 0$ such that $\mld_n(W_i, \prod_j a_{ij}^{s'} m^{t_s}) = 0$. By Corollary 3.5 and Remark 3.3, we obtain $l$ and $J_1 \subset I_1$ such that $\mld_n(W_i, \prod_j a_{ij}^{s'} m^{t_s}) = 0$ for any $s \in S_{\varepsilon}$ and $i \in J_1$, implying the log canonicity of $(W_i, \prod_j a_{ij}^s)$.

**Lemma 4.3.** $\sum_{j'=1}^{e'} |\ord_{F_i} b_{j'}| \leq \varepsilon^{-1} a_i$ for $i \in J_1$.

**Proof.** Choose $s_{j'} = r_{j'} \pm \varepsilon$ for $1 \leq j' \leq e'$ so that $\ord_{F_i} b_{j'}/(s_{j'} - r_{j'}) \geq 0$, and extend $(s_{j'})$ to the $e$-tuple $(s_{ij}) \in S_{\varepsilon}$ by $s_{ij} = \sum_{j'=0}^{e'} q_{ij}^{'j} s_{j'}$. Then by Lemma 4.2, $0 \leq a_{F_i}(W_i, \prod_j a_{ij}^{s_{ij}}) = a_i - \varepsilon \sum_{j'=1}^{e'} |\ord_{F_i} b_{j'}|$. q.e.d.

Fix a positive integer $n$ such that $nK_X$ is a Cartier divisor and $nq_{ij'} \in \mathbb{Z}$ for all $j, j'$. We define the finite set

$$A := [0, a] \cap \left( \frac{1}{n} \mathbb{Z} + \left\{ \sum_{j=1}^{e'} r_{j'} m_{j'}, \sum_{j'=1}^{e'} m_{j'} \leq \varepsilon^{-1} a, m_{j'} \in \frac{1}{n} \mathbb{Z} \forall j' \right\} \right).$$

Then by Lemma 4.3, we have

$$a_i = a_{F_i}(W_i, \prod_{j'=0}^{e'} b_{j'}^{r_{j'}}) = a_{F_i}(W_i, b_0) - \sum_{j'=1}^{e'} r_{j'} \ord_{F_i} b_{j'} \in A \quad \text{for } i \in J_1.$$  

Theorem 4.1, and Theorem 1.1, are therefore completed.

5. Extensions

First we remark the need of log canonicity in Theorem 1.1.

**Remark 5.1.** Consider a non-lc pair $(\mathbb{A}^2, (1+r)l)$ where $r > 0$ is irrational and $l$ is a line. Let $E_1$ be the exceptional divisor of the blow-up of $\mathbb{A}^2$ at a
point on \( l \), and define \( E_p \) inductively as the exceptional divisor of the blow-up at the intersection of \( E_{p-1} \) and the strict transform of \( l \). Let \( E_{p,0} = E_p \) and define \( E_{p,q} \) inductively as the exceptional divisor of the blow-up at a general point on \( E_{p,q-1} \). Then \( a_{E_{p,q}}(k^2, (1 + r)l) = q - pr \). The set of these log discrepancies is dense in \( \mathbb{R} \).

The generic limit construction is applicable to bounded singularities in the sense [4] of de Fernex, Ein and Mustaţă. We say that a collection \( \{ x_i \in X_i \}_{i} \) of singularities is bounded if there exist \( m \) and \( N \) such that for each \( i \) there exists an ideal \( \tilde{p}_i \) in \( R = k[[x_1, \ldots, x_N]] \) generated by polynomials of degree \( \leq m \) satisfying \( \mathcal{O}_{X_i,x_i} \simeq R/\tilde{p}_i \). Theorem 1.1 can be formulated for such a collection.

**Theorem 5.2.** Let \( \mathcal{X} \) be a collection of varieties with bounded klt singularities, and \( r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Then the set

\[
\{ a_E(X, \prod_{j=1}^{e} a_j^r) \mid X \in \mathcal{X}, \; a_j \subset \mathcal{O}_X, \; E \in \mathcal{D}_X, \; (X, \prod_{j=1}^{e} a_j^r) \text{ lc at } \eta_{e_X}(E) \}
\]

is discrete.

We apply it to the minimal log discrepancies of Gorenstein singularities.

**Corollary 5.3.** Let \( \mathcal{X} \) be a collection of varieties with bounded normal Gorenstein singularities, and \( r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Then the set

\[
\{ \text{mld}\eta_Z(X, \prod_{j=1}^{e} a_j^r) \mid X \in \mathcal{X}, \; a_j \subset \mathcal{O}_X, \; Z \subset X \}
\]

is finite.

This follows from the boundedness [7, Theorem 2.2] of the minimal log discrepancies of Gorenstein singularities with bounded embedding dimensions. Note that even for Gorenstein log canonical singularities, [7, Theorem 2.2] holds by its proof, and Proposition 3.2(ii), (iii) hold since the klt assumption on \( W_i \) in Proposition 3.2 is needed only to bound the Gorenstein index by the results in [4, Appendix B].

We have a further application to lci singularities.

**Corollary 5.4.** Fix an integer \( d \) and \( r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Then the set

\[
\{ a_E(X, \prod_{j=1}^{e} a_j^r) \mid X \text{ lci, dim } X \leq d, \; a_j \subset \mathcal{O}_X, \; E \in \mathcal{D}_X, \}
\]

\[
(X, \prod_{j=1}^{e} a_j^r) \text{ lc at } \eta_{e_X}(E) \}
\]

is discrete.
Corollary 5.4, and Theorem 1.3, follow from the consequence of inversion of adjunction [6] that an lc log canonical singularity of dimension \(d\) has embedding dimension \(\leq 2d\), see [3, Proposition 6.3]. Indeed, it suffices to treat \(a_E(X, \prod_j a_j)\) such that \(c_X(E)\) is a closed point \(x\) at which \(X\) is lc and of dimension \(d\). Then \(x \in X\) is locally embedded into a smooth ambient space \(A\) of dimension \(2d\) as the complete intersection of some hypersurfaces \(H_1, \ldots, H_q\) on \(A\). We have \(a_E(X, \prod_j a_j) = a_E(A, \sum_i H_i, \prod_j \tilde{a}_j)\) for \(\tilde{a}_j \subset O_A\) with \(\tilde{a}_j O_X = a_j\) and a divisor \(\tilde{E} \in D_A\) on a log resolution of \((A, \sum_i H_i, \prod_j \tilde{a}_j)\) whose restriction to the strict transform of \(X\) defines \(E\) as a valuation.

On the other hand, Shokurov’s ACC conjecture is generalised to the case when the exponents vary in a fixed set satisfying the descending chain condition (DCC). Mustaţă observed that an effective ideal-adic semi-continuity for minimal log discrepancies implies the generalised ACC on a fixed pair (see [8, Remark 1.5.1]). This semi-continuity is known in the klt case [8, Theorem 1.6], and for example, we can prove the following.

**Proposition 5.5.** Let \((X, \Delta)\) be a pair with rational \(\Delta\), and \(R\) a subset of \(\mathbb{R}_{\geq 0}\) satisfying the DCC. Suppose that any accumulation point of \(R\) is irrational. Then the set

\[
\{ \text{mld}_{nR}(X, \Delta, a^r) \mid a \subset O_X, \; r \in R, \; Z \subset X \}
\]

satisfies the ACC.

**Proof.** As in the beginning of Section 4, we are reduced to the case of terminal \(X\), and we want the stability of any non-decreasing sequence of \(a_i = \text{mld}_{n_i}(X, \Delta, a_i^r)\) \(\geq 0\) with \(r_i \in R\), \(x_i\) closed point, where \(i \in \mathbb{N}\). By passing to a subsequence, we may assume that \(a_i\) is non-trivial at \(x_i\). Then \(r_i\) are bounded by the maximum \(b\) of \(\text{mld}_x(X, \Delta)\) for all \(x \in X\), hence we may further assume that \(\{r_i\}\) is a non-decreasing sequence which has a limit \(r\). If \(r \in \mathbb{Q}\), then \(r_i = r\) for large \(i\) by the assumption on \(R\), and the stability is trivial. Henceforth we assume \(r \notin \mathbb{Q}\). As in Section 4, we construct a generic limit \(o \in W_i, o_0, a_0\) of \(o_i \in W_i, o_i, a_i\) with \(a_i = \text{mld}_{n_i}(W_i, o_i, a_i^r)\), where \(o_i\) is an ideal with fixed rational exponent, corresponding to \(\Delta\). We fix \(F_i \in D_{W_i}\), computing \(\text{mld}_{n_i}(W_i, o_i, a_i^r)\), that is,

\[
a_i = a_{F_i}(W_i, o_i, a_i^r) + (r - r_i) \text{ord}_{F_i} a_i.
\]

(1)

By Proposition 3.2 or [4, Corollary 3.4], \((W, o a^r)\) is log canonical. If \(E \in D_{W_i}\) has \(a_E(W, o a^r) = 0\), then \(\text{ord}_E a = 0\) by \(r \notin \mathbb{Q}\). Thus we can find \(t > 0\) such that \((W, o a^{r+t})\) is log canonical as in the proof of Lemma 4.2. We take \(t' \geq 0\) such that \(\text{mld}_{n_i}(W, o a^{r+t'm'}) = 0\). Then Corollary 3.5 shows \(\text{mld}_{n_i}(W_i, o_i, a_i^{r+t'm'}) = 0\) for infinitely many \(i\). In particular,

\[
t \text{ord}_{F_i} a_i \leq a_{F_i}(W_i, o_i, a_i^r) \leq a_i \leq b.
\]

(2)
Theorem 1.1 and (2) imply the finiteness of possible choices for \( a_i \) and \( r_i \) for such \( i \). Hence they are constant for infinitely many \( i \). Now (1) provides the constancy of \( a_i \) and \( r_i \) for large \( i \).

**Remark 5.6.** When \( d_i = O(W_i) \) and \( r \not\in \mathbb{Q} \) in the proof, one can further prove \( a_i = a := \text{mld}_i(W, a^r_i) \) for large \( i \). Indeed, we may assume \( r_i = r \), and \( a = \text{mld}_i(W_i, (a_i + ml_i)^r) \geq a_i \) for some \( l > t^{-1}b \) by Proposition 3.2(iii).

Hence with (2), we have \( a \geq a_i = a_F(W_i, a^r_i) = a_F(W_i, (a_i + ml_i)^r) \geq a \), meaning \( a_i = a \).

Remark 5.6 is a special case of the following conjecture. The corresponding statement for log canonical thresholds is [4, Corollary 3.4].

**Conjecture 5.7.** With the notation in Section 3, we suppose that each \( W_i \) is klt and fix \( r_1, \ldots, r_e \in \mathbb{R}_{\geq 0} \). Let

\[ J := \{ i \in I \mid \text{mld}_i(W_i, \prod_j a_{ij}^r) = \text{mld}_i(W, \prod_j a_{ij}^r) \} \]

Then \( I_l \cap J \) is dense for each \( l \).

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: masayuki@kurims.kyoto-u.ac.jp