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Entanglement entropy of disjoint regions in excited states: an operator method

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Abstract: We develop the computational method of entanglement entropy based on the idea that $\mathrm{Tr}\rho_\Omega^n$ is written as the expectation value of the local operator, where ρ_Ω is a density matrix of the subsystem Ω. We apply it to consider the mutual Rényi information $I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states defined by acting the local operators at A and B on the vacuum of a $(d+1)$-dimensional field theory, in the limit when the separation r between A and B is much greater than their sizes $R_{A,B}$. For the general QFT which has a mass gap, we compute $I^{(n)}(A,B)$ explicitly and find that this result is interpreted in terms of an entangled state in quantum mechanics. For a free massless scalar field, we show that for some classes of excited states, $I^{(n)}(A,B) - I^{(n)}(A,B)|_{r\to\infty} = C_{AB}^{(n)}/r^{\alpha(d-1)}$ where $\alpha = 1$ or 2 which is determined by the property of the local operators under the transformation $\phi \to -\phi$ and $\alpha = 2$ for the vacuum state. We give a method to compute $C_{AB}^{(2)}$ systematically.

Keywords: Field Theories in Higher Dimensions, Lattice Quantum Field Theory

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Contents

1 Introduction

The entanglement entropy in the quantum field theory (QFT) plays important roles in many fields of physics such as the string theory, condensed matter physics, and the physics of the black hole. The entanglement entropy is a useful quantity which characterize quantum properties of given states. For example, the entanglement entropy of ground states follows the area law [1–4] if we consider a local quantum field theory with a UV fixed point, while non-local field theories [5, 6] or QFTs with fermi surfaces [7, 8] at UV cut off scale can violate the area law.

For a given density matrix ρ of the total system, the entanglement entropy of the subsystem Ω is defined as

$$S_\Omega = -\mathrm{Tr}\rho_\Omega \ln \rho_\Omega, \tag{1.1}$$

where $\rho_\Omega = \mathrm{Tr}_{\Omega^c}\rho$ is the reduced density matrix of the subsystem Ω and Ω^c is the complement of Ω. The Rényi entropy $S_\Omega^{(n)}$ is defined as

$$S_\Omega^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\rho_\Omega^n. \tag{1.2}$$

The limit $n \to 1$ coincides with the entanglement entropy $\lim_{n=1} S_\Omega^{(n)} = S_\Omega$.

In this paper we develop the computational method of Rényi entanglement entropy based on the idea that $\mathrm{Tr}\rho_\Omega^n$ is written as the expectation value of the local operator at Ω. We apply this method to consider the mutual Rényi information $I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states defined by acting the local operators at A and B on the vacuum of a $(d+1)$-dimensional field theory, in the limit when the separation r between A and B is much greater than their sizes $R_{A,B}$.

Our method is based on the idea that $\mathrm{Tr}\rho_\Omega^n$ is written as the expectation value of the local operator at Ω. This idea was originally used to compute $I^{(n)}(A,B)$ in the vacuum state by Cardy [9], Calabrese et al. [10] and Headrick [11]. We generalize this idea to an arbitrary density matrix ρ and construct explicitly the local operator. The density matrix of the total system ρ can be a mixed state and an excited state. We consider the general scalar field and do not specify its Hamiltonian. (Our method is applicable to QFT with interaction.) We summarize our method. We consider n copies of the scalar fields and the j-th copy of the scalar field is denoted by $\{\phi^{(j)}\}$. Thus the total Hilbert space, $H^{(n)}$, is the tensor product of the n copies of the Hilbert space, $H^{(n)} = H \otimes H \cdots \otimes H$ where H is the Hilbert space of one scalar field. We define the density matrix $\rho^{(n)}$ in $H^{(n)}$ as

$$\rho^{(n)} \equiv \rho \otimes \rho \otimes \cdots \otimes \rho \tag{1.3}$$

where ρ is an arbitrary density matrix in H. We can express $\mathrm{Tr}\rho_\Omega^n$ as

$$\mathrm{Tr}\rho_\Omega^n = \mathrm{Tr}(\rho^{(n)} E_\Omega), \tag{1.4}$$

where

$$\begin{aligned} E_\Omega = \int \prod_{j=1}^{n} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp\left[i \int d^d x \sum_{l=1}^{n} J^{(l+1)}(x)\phi^{(l)}(x)\right] \\ \times \exp\left[i \int d^d x \sum_{l=1}^{n} K^{(l)}(x)\pi^{(l)}(x)\right] \times \exp\left[-i \int d^d x \sum_{l=1}^{n} J^{(l)}\phi^{(l)}\right], \end{aligned} \tag{1.5}$$

where $\pi(x)$ is a conjugate momenta of $\phi(x)$, $[\phi(x), \pi(y)] = i\delta^d(x-y)$, and $J^{(j)}(x)$ and $K^{(j)}(x)$ exist only in Ω and $J^{(n+1)} = J^{(1)}$ and we normalize the measure of the functional integral as $\int DJ^{(j)} \exp[i \int d^d x J^{(j)}(x) f(x)] = \prod_{x\in\Omega} \delta(f(x))$ where $f(x)$ is an arbitrary function. Notice that ϕ and π in (1.5) are operators and the ordering is important. We call this operator E_Ω as *the glueing operator*. When ρ is a pure state, $\rho = |\Psi\rangle\langle\Psi|$, the equation (1.4) becomes

$$\mathrm{Tr}\rho_\Omega^n = \langle\Psi^{(n)}| E_\Omega |\Psi^{(n)}\rangle \tag{1.6}$$

where

$$|\Psi^{(n)}\rangle = |\Psi\rangle |\Psi\rangle \ldots |\Psi\rangle . \tag{1.7}$$

For a free scalar field, we can rewrite E_Ω in (1.5) using the normal ordering. In the case $n = 2$, we obtain a simple expression of E_Ω and reproduce the result that $I^{(2)}(A,B)$ in the vacuum state is proportional to the product of the electrostatic capacitance of each regions

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obtained by Cardy [9]. Furthermore the simple expression of E_Ω is useful for numerical calculation.

The advantages of this method are that we can use ordinary technique in QFT such as OPE and the cluster decomposition property and that we can use the general properties and the explicit expression of the glueing operator to compute systematically the Rényi entropy for an arbitrary state.

We apply this method to the mutual Rényi information $I^{(n)}(A,B)$ in the locally excited states. We consider the following locally excited state,

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})\,|0\rangle\,, \tag{1.8}$$

where N is a real normalization constant and O_{iA} and $O_{i'A}$ (O_{jB} and $O_{j'B}$) are operators on A (B) and i and i' (j and j') label a kind of operators. For the general QFT which has a mass gap, we compute $I^{(n)}(A,B)$ explicitly and find that this result is interpreted in terms of an entangled state in a quantum mechanical system which has finite degrees of freedom. For a free massless scalar field, we show that for some classes of excited states, $I^{(n)}(A,B) - I^{(n)}(A,B)|_{r\to\infty} = C^{(n)}_{AB}/r^{\alpha(d-1)}$ where $\alpha = 1$ or 2 which is determined by the property of the local operators under the transformation $\phi \to -\phi$ and $\alpha = 2$ for the vacuum state.

The mutual information for a free scalar field in higher dimensions has been studied in only a few papers [9, 12–16, 26]. In these papers, the authors considered only the mutual information for the vacuum state. The relation between the mutual information and the physics of the black hole was considered in [14]. Recently, it is proposed [17] that the mutual information is obtained by the quantum correction to the holographic entanglement entropy formula [18, 19]. It would be interesting to use our results to check this proposition.

The entanglement entropy for an excited state defined by acting the local operator on the vacuum was considered in [20–23]. In [20], the subsystem is a half of the total space and the local operator exists at the complement of the subsystem and the time evolution of the entanglement entropy was considered. It was found that the entanglement entropy at late time is interpreted in terms of an entangled state in quantum mechanics [20]. This result is analogous to our result in the general QFT which has a mass gap.

The present paper is organized as follows. In section 2.1 we develop the computational method of the entanglement entropy. We derive the basic formula (1.4) and construct explicitly the glueing operator E_Ω. In section 2.2 we investigate the general properties of E_Ω. In section 3 we consider the mutual Rényi information $I^{(n)}(A,B)$ in the locally excited states in the general QFT which has a mass gap. We compute $I^{(n)}(A,B)$ explicitly and find that this result is interpreted in terms of an entangled state in quantum mechanics. In section 4 we consider free scalar fields. In section 4.1 we rewrite E_Ω in (1.5) using the normal ordering. In the case $n = 2$, we obtain a simple expression of E_Ω and reproduce the result that $I^{(2)}(A,B)$ in the vacuum state is proportional to the product of the electrostatic capacitance of each regions obtained by Cardy. In section 4.2 we consider a free massless scalar field. We show that for some classes of excited states $I^{(n)}(A,B) - I^{(n)}(A,B)|_{r\to\infty} = C^{(n)}_{AB}/r^{\alpha(d-1)}$ where $\alpha = 1$ or 2 which is determined by the property of the local operators

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under the transformation $\phi \to -\phi$ and $\alpha = 2$ for the vacuum state. In section 5 we summarize our conclusion.

2 Operator formalism

2.1 Operator representation of $\mathrm{Tr}\rho_\Omega^n$

We represent the trace of the nth power of the reduced density matrix as the expectation value of the local operator. As a model amenable to unambiguous calculation we deal with the scalar field as a collection of coupled oscillators on a lattice of space points, labeled by capital Latin indices, the displacement at each point giving the value of the scalar field there. The local Hermitian variables $\hat{q}_A$ and $\hat{p}_B$ (coordinates and the conjugate momentum) obey the canonical commutation relations

$$[\hat{q}_A, \hat{p}_B] = i\delta_{AB}, \qquad [\hat{q}_A, \hat{q}_B] = [\hat{p}_A, \hat{p}_B] = 0. \tag{2.1}$$

The density matrix ρ of the total system in coordinate representation is

$$\rho(q_A; q'_B) = \langle \{q_A\}|\, \rho \,|\{q'_B\}\rangle \tag{2.2}$$

where $\{q_A\}$ denotes the collection of all q_A's. We consider the arbitrary density matrix ρ.

Now consider a subsystem (or subregion) Ω in the space. The oscillators in this region will be specified by lowercase Latin letters, and those in its complement Ω^c will be specified by Greek letters. We can obtain a reduced density matrix ρ_Ω for Ω by integrating out over $q^\alpha \in \mathbb{R}$ for each of the oscillators in Ω^c, and then we have

$$\rho_\Omega(q_a; q'_b) = \int \prod_\alpha dq_\alpha \rho(q_a, q_\alpha; q'_b, q_\alpha). \tag{2.3}$$

We obtain the trace of the nth power of the reduced density matrix ρ_Ω as

$$\begin{aligned}\mathrm{Tr}\rho_\Omega^n &= \int \prod_{j=1}^n \prod_a dq_a^{(j)} \rho_\Omega(q_a^{(1)}; q_a^{(2)})\rho_\Omega(q_a^{(2)}; q_a^{(3)}) \ldots \rho_\Omega(q_a^{(n)}; q_a^{(1)}) \\ &= \int \prod_{j=1}^n \prod_a dq_a^{(j)} \prod_\alpha dq_\alpha^{(j)} \rho(q_a^{(1)}, q_\alpha^{(1)}; q_a^{(2)}, q_\alpha^{(1)})\rho(q_a^{(2)}, q_\alpha^{(2)}; q_a^{(3)}, q_\alpha^{(2)}) \ldots \\ &\quad \times \rho(q_a^{(n)}, q_\alpha^{(n)}; q_a^{(1)}, q_\alpha^{(n)}).\end{aligned} \tag{2.4}$$

We consider n copies of the oscillators and the j-th copy of the oscillators is denoted by $\{q_A^{(j)}\}$. Thus the total Hilbert space, $H^{(n)}$, is the tensor product of the n copies of the Hilbert space, $H^{(n)} = H \otimes H \cdots \otimes H$. We define the following density matrix,

$$\rho^{(n)}(q_A^{(1)}, \ldots, q_A^{(n)}; q_B^{(1)'}, \ldots, q_B^{(n)'}) \equiv \rho(q_A^{(1)}; q_B^{(1)'})\rho(q_A^{(2)}; q_B^{(2)'}) \ldots \rho(q_A^{(n)}; q_B^{(n)'}), \tag{2.5}$$

i.e.

$$\rho^{(n)} \equiv \rho \otimes \rho \otimes \cdots \otimes \rho. \tag{2.6}$$

Then we can rewrite $\mathrm{Tr}\rho_\Omega^n$ as

$$\begin{aligned}
\mathrm{Tr}\rho_\Omega^n =& \int \prod_{j=1}^n \prod_a dq_a^{(j)} \prod_\alpha dq_\alpha^{(j)} \prod_b dq_b^{(j)'} \prod_\beta dq_\beta^{(j)'} \\
&\times \rho(q_a^{(1)}, q_\alpha^{(1)}; q_b^{(1)'}, q_\beta^{(1)'})\rho(q_a^{(2)}, q_\alpha^{(2)}; q_b^{(2)'}, q_\beta^{(2)'}) \dots \rho(q_a^{(n)}, q_\alpha^{(n)}; q_b^{(n)'}, q_\beta^{(n)'}) \\
&\times \delta(q_\alpha^{(1)} - q_\beta^{(1)'})\delta(q_\alpha^{(2)} - q_\beta^{(2)'}) \dots \delta(q_\alpha^{(n)} - q_\beta^{(n)'}) \\
&\times \delta(q_b^{(1)'} - q_a^{(2)})\delta(q_b^{(2)'} - q_a^{(3)}) \dots \delta(q_b^{(n)'} - q_a^{(1)}) \\
=& \int \prod_{j=1}^n \prod_A dq_A^{(j)} \prod_B dq_B^{(j)'} \rho^{(n)}(q_A^{(1)}, \dots, q_A^{(n)}; q_B^{(1)'}, \dots, q_B^{(n)'}) \\
&\times E_\Omega(q_B^{(1)'}, \dots, q_B^{(n)'}; q_A^{(1)}, \dots, q_A^{(n)}) \\
=& \mathrm{Tr}(\rho^{(n)} E_\Omega),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
E_\Omega(q_B^{(1)'}, \dots, q_B^{(n)'}; q_A^{(1)}, \dots, q_A^{(n)}) &\equiv \langle \{q_B^{(1)'}\}, \dots, \{q_B^{(n)'}\}| E_\Omega |\{q_A^{(1)}\}, \dots, \{q_A^{(n)}\}\rangle \\
&\equiv \prod_a \prod_\alpha \delta(q_\alpha^{(1)} - q_\beta^{(1)'})\delta(q_\alpha^{(2)} - q_\beta^{(2)'}) \dots \delta(q_\alpha^{(n)} - q_\beta^{(n)'}) \\
&\quad \times \delta(q_b^{(1)'} - q_a^{(2)})\delta(q_b^{(2)'} - q_a^{(3)}) \dots \delta(q_b^{(n)'} - q_a^{(1)}).
\end{aligned} \tag{2.8}$$

We call E_Ω as *the glueing operator*. When ρ is a pure state, $\rho = |\Psi\rangle \langle\Psi|$, the equation (2.7) becomes

$$\mathrm{Tr}\rho_\Omega^n = \langle \Psi^{(n)}| E_\Omega |\Psi^{(n)}\rangle \tag{2.9}$$

where

$$|\Psi^{(n)}\rangle = |\Psi\rangle |\Psi\rangle \dots |\Psi\rangle \,. \tag{2.10}$$

We represent E_Ω as a function of $\hat{q}$ and $\hat{p}$. We can rewrite E_Ω as

$$\begin{aligned}
E_\Omega =& \int \prod_{j=1}^n \prod_a dq_a^{(j)} \prod_b dq_b^{(j)'} |\{q_b^{(1)'}\}, \dots, \{q_b^{(n)'}\}\rangle \langle \{q_a^{(1)}\}, \dots, \{q_a^{(n)}\}| \\
&\times \delta(q_b^{(1)'} - q_a^{(2)})\delta(q_b^{(2)'} - q_a^{(3)}) \dots \delta(q_b^{(n)'} - q_a^{(1)}) \\
=& \int \prod_{j=1}^n \prod_a \frac{dJ_a^{(j)}}{2\pi} \exp[i(J_a^{(2)}\hat{q}_a^{(1)} + J_a^{(3)}\hat{q}_a^{(2)} + \dots + J_a^{(n)}\hat{q}_a^{(n-1)} + J_a^{(1)}\hat{q}_a^{(n)})] \\
&\times \int \prod_{j=1}^n \prod_a dq_a^{(j)} \prod_b dq_b^{(j)'} |\{q_b^{(1)'}\}, \dots, \{q_b^{(n)'}\}\rangle \langle \{q_a^{(1)}\}, \dots, \{q_a^{(n)}\}| \\
&\times \exp[-i(J_a^{(1)}\hat{q}_a^{(1)} + J_a^{(2)}\hat{q}_a^{(2)} + \dots + J_a^{(n)}\hat{q}_a^{(n)})],
\end{aligned} \tag{2.11}$$

where we have written the delta functions as the Fourier integrals. Note that $\hat{q}_a^{(l)}$ is a operator and not a integral variable. The middle term in (2.11) is the tensor product of the following operator,

$$\int dq' \int dq \, |q'\rangle \langle q| = \int dK \exp[iK\hat{p}], \tag{2.12}$$

where we have omitted the subscripts for simplicity. We can check easily that (2.12) is correct just by taking the matrix elements of its both sides. Thus we can rewrite the middle term in (2.11) as

$$\begin{aligned}\int \prod_{j=1}^{n} \prod_{a} dq_a^{(j)} \prod_{b} dq_b^{(j)\prime} \, |\{q_b^{(1)\prime}\}, \dots, \{q_b^{(n)\prime}\}\rangle \langle \{q_a^{(1)}\}, \dots, \{q_a^{(n)}\}| = \\ \int \prod_{j=1}^{n} \prod_{a} dK_a^{(j)} \exp[i(K_a^{(1)} \hat{p}_a^{(1)} + K_a^{(2)} \hat{p}_a^{(2)} + \dots + K_a^{(n)} \hat{p}_a^{(n)})].\end{aligned} \tag{2.13}$$

We substitute (2.13) into (2.11) and obtain

$$\begin{aligned}E_\Omega = \int \prod_{j=1}^{n} \prod_{a \in \Omega} \frac{dJ_a^{(j)}}{2\pi} dK_a^{(j)} \exp[i(J_a^{(2)} \hat{q}_a^{(1)} + J_a^{(3)} \hat{q}_a^{(2)} + \dots + J_a^{(n)} \hat{q}_a^{(n-1)} + J_a^{(1)} \hat{q}_a^{(n)})] \\ \times \exp[i(K_a^{(1)} \hat{p}_a^{(1)} + K_a^{(2)} \hat{p}_a^{(2)} + \dots + K_a^{(n)} \hat{p}_a^{(n)})] \\ \times \exp[-i(J_a^{(1)} \hat{q}_a^{(1)} + J_a^{(2)} \hat{q}_a^{(2)} + \dots + J_a^{(n)} \hat{q}_a^{(n)})].\end{aligned} \tag{2.14}$$

From (2.14), for the $(d+1)$ dimensional scalar field theory, we obtain

$$\begin{aligned}E_\Omega = \int \prod_{j=1}^{n} \prod_{x \in \Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp\left[i \int d^d x \sum_{l=1}^{n} J^{(l+1)}(x) \phi^{(l)}(x)\right] \\ \times \exp\left[i \int d^d x \sum_{l=1}^{n} K^{(l)}(x) \pi^{(l)}(x)\right] \times \exp\left[-i \int d^d x \sum_{l=1}^{n} J^{(l)} \phi^{(l)}\right].\end{aligned} \tag{2.15}$$

where $\pi(x)$ is a conjugate momenta of $\phi(x)$, $[\phi(x), \pi(y)] = i\delta^d(x-y)$, and $J^{(j)}(x)$ and $K^{(j)}(x)$ exist only in Ω and $J^{(n+1)} = J^{(1)}$ and we normalize the measure of the functional integral as $\int DJ^{(j)} \exp[i \int d^d x J^{(j)}(x) f(x)] = \prod_{x \in \Omega} \delta(f(x))$ where $f(x)$ is an arbitrary function.

2.2 General properties of the glueing operator E_Ω

We investigate some general properties of E_Ω.

(1) Symmetry: from (2.15), E_Ω is invariant under the sign changing transformation $\phi, \pi \to -\phi, -\pi$, i.e.

$$E_\Omega(\phi^{(1)}, \dots, \phi^{(n)}, \pi^{(1)}, \dots, \pi^{(n)}) = E_\Omega(-\phi^{(1)}, \dots, -\phi^{(n)}, -\pi^{(1)}, \dots, -\pi^{(n)}). \tag{2.16}$$

(2) Locality: when $\Omega = A \cup B$ and $A \cap B = \emptyset$,

$$E_{A \cup B} = E_A E_B. \tag{2.17}$$

(3) For n arbitrary operators F_j $(j = 1, 2, \dots, n)$ on H,

$$\mathrm{Tr}(F_1 \otimes F_2 \otimes \dots \otimes F_n \cdot E_\Omega) = \mathrm{Tr}(F_{1\Omega} F_{2\Omega} \dots F_{n\Omega}), \tag{2.18}$$

where $F_{j\Omega} \equiv \mathrm{Tr}_{\Omega^c} F_j$. This property is the generalization of (2.7). We can prove easily (2.18) by using the matrix elements of E_Ω in (2.8).

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(4) The cyclic property: from the cyclic property of the trace in the right hand side in (2.18), we obtain

$$\mathrm{Tr}(F_1 \otimes F_2 \otimes \cdots \otimes F_n \cdot E_\Omega) = \mathrm{Tr}(F_2 \otimes F_3 \otimes \cdots \otimes F_n \otimes F_1 \cdot E_\Omega). \tag{2.19}$$

(5) The relation between E_Ω and E_{Ω^c} for pure states: we consider two pure states $|\phi_1\rangle |\phi_2\rangle \dots |\phi_n\rangle$ and $|\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle$ in $H^{(n)}$ where $|\phi_j\rangle$ and $|\psi_j\rangle$ $(j = 1, 2, \dots, n)$ are arbitrary pure states. We can prove the following equation:

$$\langle\psi_1| \langle\psi_2| \dots \langle\psi_n| E_\Omega |\phi_1\rangle |\phi_2\rangle \dots |\phi_n\rangle = [\langle\phi_2| \langle\phi_3| \dots \langle\phi_n| \langle\phi_1| E_{\Omega^c} |\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle]^*. \tag{2.20}$$

We prove (2.20) as follows:

From (2.8) we obtain

$$\begin{aligned}
&\langle\psi_1| \langle\psi_2| \dots \langle\psi_n| E_\Omega |\phi_1\rangle |\phi_2\rangle \dots |\phi_n\rangle \\
&= \int \prod_{j=1}^{n} \prod_a dq_a^{(j)} \prod_\alpha dq_\alpha^{(j)} \\
&\quad \times \psi_1(q_a^{(1)}, q_\alpha^{(1)})^* \psi_2(q_a^{(2)}, q_\alpha^{(2)})^* \dots \psi_n(q_a^{(n)}, q_\alpha^{(n)})^* \phi_1(q_a^{(n)}, q_\alpha^{(1)}) \\
&\quad \times \phi_2(q_a^{(1)}, q_\alpha^{(2)}) \dots \phi_n(q_a^{(n-1)}, q_\alpha^{(n)}) \\
&= \Bigg[\int \prod_{j=1}^{n} \prod_a dq_a^{(j)} \prod_\alpha dq_\alpha^{(j)} \phi_2(q_a^{(1)}, q_\alpha^{(2)})^* \phi_3(q_a^{(2)}, q_\alpha^{(3)})^* \dots \phi_n(q_a^{(n-1)}, q_\alpha^{(n)})^* \\
&\quad \times \phi_1(q_a^{(n)}, q_\alpha^{(1)})^* \psi_1(q_a^{(1)}, q_\alpha^{(1)}) \psi_2(q_a^{(2)}, q_\alpha^{(2)}) \dots \psi_n(q_a^{(n)}, q_\alpha^{(n)}) \Bigg]^* \\
&= [\langle\phi_2| \langle\phi_3| \dots \langle\phi_n| \langle\phi_1| E_{\Omega^c} |\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle]^*.
\end{aligned} \tag{2.21}$$

By using (2.20), we can obtain the following basic property (see e.g. [24]) for a pure state $\rho = |\Psi\rangle \langle\Psi|$;

$$\mathrm{Tr}\rho_\Omega^n = \langle\Psi^{(n)}| E_\Omega |\Psi^{(n)}\rangle = [\langle\Psi^{(n)}| E_{\Omega^c} |\Psi^{(n)}\rangle]^* = \mathrm{Tr}\rho_{\Omega^c}^n. \tag{2.22}$$

where we have used the fact that $\mathrm{Tr}\rho_\Omega^n$ is real. So (2.20) is the generalization of $\mathrm{Tr}\rho_\Omega^n = \mathrm{Tr}\rho_{\Omega^c}^n$ for a pure state $\rho = |\Psi\rangle \langle\Psi|$.

3 Locally excited states in the general QFT which has a mass gap

We consider the mutual Rényi information $I^{(n)}(A, B) \equiv S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states of the general QFT which has the mass gap m in the limit when the separation r between A and B is much greater than their sizes $R_{A,B}$ and $1/m$ $(r \gg R_{A,B}, 1/m)$. We consider the following locally excited state,

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B}) |0\rangle\,, \tag{3.1}$$

where N is a real normalization constant and O_{iA} and $O_{i'A}$ (O_{jB} and $O_{j'B}$) are operators on A (B) and i and i' (j and j') label a kind of operators. We assume the distance between the positions of O_A (O_B) and the boundary of A (B) is much greater than $1/m$ ($R_{A,B} \gg 1/m$) (we have omitted the labels is for simplicity.) We impose following orthogonal conditions for simplicity,

$$\langle 0| O^{\dagger}_{iA} O_{i'A} |0\rangle = \langle 0| O^{\dagger}_{jB} O_{j'B} |0\rangle = 0. \tag{3.2}$$

This state is similar to the EPR state. We compute the mutual Rényi information of this state. In this calculation, the general properties (2), (5) and the cluster decomposition property in the QFT play important roles.

From the normalization condition $\langle \Psi|\Psi\rangle = 1$ we obtain

$$N^{-2} \simeq \langle 0| O^{\dagger}_{iA} O_{iA} |0\rangle \langle 0| O^{\dagger}_{jB} O_{jB} |0\rangle + \langle 0| O^{\dagger}_{i'A} O_{i'A} |0\rangle \langle 0| O^{\dagger}_{j'B} O_{j'B} |0\rangle , \tag{3.3}$$

where we have used the cluster decomposition property $\langle 0| O^{\dagger}_{A} O_A O^{\dagger}_{B} O_B |0\rangle \simeq \langle 0| O^{\dagger}_{A} O_A |0\rangle$ $\langle 0| O^{\dagger}_{B} O_B |0\rangle$ and the orthogonal conditions (3.2). By using (2.9), we obtain $\mathrm{Tr}\rho^n_{\Omega}$ for $\rho = |\Psi\rangle\langle\Psi|$ as

$$\begin{aligned}\mathrm{Tr}\rho^n_{\Omega} &= \langle \Psi^{(n)}| E_{\Omega} |\Psi^{(n)}\rangle \\ &= N^{2n} \langle 0^{(n)}| (O^{\dagger}_{iA} O^{\dagger}_{jB} + O^{\dagger}_{i'A} O^{\dagger}_{j'B})^{(1)} \dots (O^{\dagger}_{iA} O^{\dagger}_{jB} + O^{\dagger}_{i'A} O^{\dagger}_{j'B})^{(n)} \\ &\quad \times E_{\Omega} (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(1)} \dots (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(n)} |0^{(n)}\rangle ,\end{aligned} \tag{3.4}$$

where $O^{(l)}$ are operators in the Hilbert space of the l-th copy and the Ω is A, B or $A \cup B$.

First, we consider $\mathrm{Tr}\rho^n_{A\cup B}$. From (2.20) and (3.4) we obtain

$$\begin{aligned}\mathrm{Tr}\rho^n_{A\cup B} &= N^{2n} [\langle 0^{(n)}| (O^{\dagger}_{iA} O^{\dagger}_{jB} + O^{\dagger}_{i'A} O^{\dagger}_{j'B})^{(1)} \dots (O^{\dagger}_{iA} O^{\dagger}_{jB} + O^{\dagger}_{i'A} O^{\dagger}_{j'B})^{(n)} \\ &\quad \times E_{(A\cup B)^c} (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(1)} \dots (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(n)} |0^{(n)}\rangle]^* \\ &\simeq \langle \Psi|\Psi\rangle [\langle 0^{(n)}| E_{(A\cup B)^c} |0^{(n)}\rangle]^* = \langle 0^{(n)}| E_{A\cup B} |0^{(n)}\rangle = \langle 0^{(n)}| E_A E_B |0^{(n)}\rangle \\ &\simeq \langle 0^{(n)}| E_A |0^{(n)}\rangle \langle 0^{(n)}| E_B |0^{(n)}\rangle = \mathrm{Tr}\rho^n_{0A} \mathrm{Tr}\rho^n_{0B},\end{aligned} \tag{3.5}$$

where we have used the cluster decomposition property and the conditions $r, R_A, R_B \gg 1/m$ and $\rho_{0A(B)}$ is the reduced density matrix of the vacuum state.

Next we consider $\mathrm{Tr}\rho^n_A$. We expand the product in (3.4). The terms in the expansion in (3.4) have the following form,

$$\begin{aligned}&\langle 0^{(n)}| O^{(1)\dagger}_{i_1 A} \dots O^{(n)\dagger}_{i_n A} E_A O^{(1)}_{i_{n+1} A} \dots O^{(n)}_{i_{2n} A} \cdot O^{(1)\dagger}_{j_1 B} \dots O^{(n)\dagger}_{j_n B} O^{(1)}_{j_{n+1} B} \dots O^{(n)}_{j_{2n} B} |0^{(n)}\rangle \\ &\simeq \langle 0^{(n)}| O^{(1)\dagger}_{i_1 A} \dots O^{(n)\dagger}_{i_n A} E_A O^{(1)}_{i_{n+1} A} \dots O^{(n)}_{i_{2n} A} |0^{(n)}\rangle \langle 0^{(n)}| O^{(1)\dagger}_{j_1 B} \dots O^{(n)\dagger}_{j_n B} O^{(1)}_{j_{n+1} B} \dots O^{(n)}_{j_{2n} B} |0^{(n)}\rangle \\ &= \langle 0^{(n)}| O^{(1)\dagger}_{i_1 A} \dots O^{(n)\dagger}_{i_n A} E_A O^{(1)}_{i_{n+1} A} \dots O^{(n)}_{i_{2n} A} |0^{(n)}\rangle \prod_{l=1}^{n} \langle 0| O^{\dagger}_{j_l B} O_{j_{l+n} B} |0\rangle \\ &= \langle 0^{(n)}| O^{(1)\dagger}_{i_1 A} \dots O^{(n)\dagger}_{i_n A} E_A O^{(1)}_{i_{n+1} A} \dots O^{(n)}_{i_{2n} A} |0^{(n)}\rangle \prod_{l=1}^{n} \delta_{j_l j_{l+n}} \langle 0| O^{\dagger}_{j_l B} O_{j_l B} |0\rangle\end{aligned} \tag{3.6}$$

where (i_l, j_l) is (i,j) or (i',j') $(l = 1, 2, \ldots, 2n)$ and we have used the cluster decomposition property and the condition $r \gg 1/m$ in the second line and used the orthogonal conditions in (3.2) in the last line. From (3.6), the number of the nonzero terms in the the expansion in (3.4) is 2^n. From (3.4) and (3.6), we obtain

$$\begin{aligned}
\mathrm{Tr}\rho_A^n = N^{2n}[&\langle 0| O_{jB}^\dagger O_{jB} |0\rangle^n \langle 0^{(n)}| O_{iA}^{(1)\dagger} \ldots O_{iA}^{(n)\dagger} E_A O_{iA}^{(1)} \ldots O_{iA}^{(n)} |0^{(n)}\rangle \\
&+ \langle 0| O_{jB}^\dagger O_{jB} |0\rangle^{n-1} \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle \\
&\times (\langle 0^{(n)}| O_{i'A}^{(1)\dagger} O_{iA}^{(2)\dagger} \ldots O_{iA}^{(n)\dagger} E_A O_{i'A}^{(1)} O_{iA}^{(2)} \ldots O_{iA}^{(n)} |0^{(n)}\rangle + \ldots) \\
&+ \langle 0| O_{jB}^\dagger O_{jB} |0\rangle^{n-2} \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle^2 \\
&\times (\langle 0^{(n)}| O_{i'A}^{(1)\dagger} O_{i'A}^{(2)\dagger} O_{iA}^{(3)\dagger} \ldots O_{iA}^{(n)\dagger} E_A O_{i'A}^{(1)} O_{i'A}^{(2)} O_{iA}^{(3)} \ldots O_{iA}^{(n)} |0^{(n)}\rangle + \ldots) \\
&+ \cdots + \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle^n \langle 0^{(n)}| O_{i'A}^{(1)\dagger} \ldots O_{i'A}^{(n)\dagger} E_A O_{i'A}^{(1)} \ldots O_{i'A}^{(n)} |0^{(n)}\rangle].
\end{aligned} \tag{3.7}$$

The number of the terms which are proportional to $\langle 0| O_{jB}^\dagger O_{jB} |0\rangle^{n-l} \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle^l$ in (3.7) is ${}_n\mathrm{C}_l$. By using (2.20), we obtain

$$\begin{aligned}
&\langle 0^{(n)}| O_{i_1A}^{(1)\dagger} \ldots O_{i_nA}^{(n)\dagger} E_A O_{i_1A}^{(1)} \ldots O_{i_nA}^{(n)} |0^{(n)}\rangle \\
&\quad = [\langle 0^{(n)}| O_{i_2A}^{(1)\dagger} \ldots O_{i_nA}^{(n-1)\dagger} O_{i_1A}^{(n)\dagger} E_{A^c} O_{i_1A}^{(1)} \ldots O_{i_nA}^{(n)} |0^{(n)}\rangle]^* \\
&\quad \simeq [\langle 0^{(n)}| E_{A^c} |0^{(n)}\rangle \langle 0^{(n)}| O_{i_2A}^{(1)\dagger} \ldots O_{i_nA}^{(n-1)\dagger} O_{i_1A}^{(n)\dagger} O_{i_1A}^{(1)} \ldots O_{i_nA}^{(n)} |0^{(n)}\rangle]^* \\
&\quad = \langle 0^{(n)}| E_A |0^{(n)}\rangle \prod_{l=1}^{n} \langle 0| O_{i_lA}^\dagger O_{i_{l+1}A} |0\rangle \\
&\quad = \begin{cases} \mathrm{Tr}\rho_{0A}^n \langle 0| O_{iA}^\dagger O_{iA} |0\rangle^n & \text{for} \quad i = i_1 = i_2 = \cdots = i_n \\ \mathrm{Tr}\rho_{0A}^n \langle 0| O_{i'A}^\dagger O_{i'A} |0\rangle^n & \text{for} \quad i' = i_1 = i_2 = \cdots = i_n \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{3.8}$$

where $i_{n+1} = i_1$ and we have used the cluster decomposition property and the condition $R_A \gg 1/m$ in the third line and used the orthogonal conditions in (3.2) in the last line. From (3.8) the nonzero terms in (3.7) are only the first term and the last term and we obtain

$$\mathrm{Tr}\rho_A^n = \mathrm{Tr}\rho_{0A}^n \cdot N^{2n}[\langle 0| O_{iA}^\dagger O_{iA} |0\rangle^n \langle 0| O_{jB}^\dagger O_{jB} |0\rangle^n + \langle 0| O_{i'A}^\dagger O_{i'A} |0\rangle^n \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle^n]. \tag{3.9}$$

By the same way, we obtain $\mathrm{Tr}\rho_B^n$ as

$$\mathrm{Tr}\rho_B^n = \mathrm{Tr}\rho_{0B}^n \cdot N^{2n}[\langle 0| O_{iA}^\dagger O_{iA} |0\rangle^n \langle 0| O_{jB}^\dagger O_{jB} |0\rangle^n + \langle 0| O_{i'A}^\dagger O_{i'A} |0\rangle^n \langle 0| O_{j'B}^\dagger O_{j'B} |0\rangle^n]. \tag{3.10}$$

From (3.3), (3.5), (3.9) and (3.10), we obtain the mutual Rényi information $I^{(n)}(A,B) = (n-1)^{-1} \ln \frac{\mathrm{Tr}\rho_{A\cup B}^n}{\mathrm{Tr}\rho_A^n \mathrm{Tr}\rho_B^n}$ as

$$I^{(n)}(A,B) = \frac{2}{n-1} \ln \frac{(x+y)^n}{x^n + y^n} \tag{3.11}$$

where

$$x \equiv \langle 0| O^\dagger_{iA} O_{iA} |0\rangle \langle 0| O^\dagger_{jB} O_{jB} |0\rangle \,, \qquad y \equiv \langle 0| O^\dagger_{i'A} O_{i'A} |0\rangle \langle 0| O^\dagger_{j'B} O_{j'B} |0\rangle \,. \tag{3.12}$$

Taking the limit $n \to 1$ leads to the the mutual information

$$I(A,B) = 2\left[\ln(x+y) - \frac{1}{x+y}(x\ln x + y\ln y)\right]. \tag{3.13}$$

We can reproduce these results from the quantum mechanics. Let us consider the following state,

$$|\Psi\rangle_{qm} = N(|i\rangle_A |j\rangle_B + |i'\rangle_A |j'\rangle_B), \tag{3.14}$$

where N is the real normalization constant, and $|i(i')\rangle_A$ and $|j(j')\rangle_B$ are the pure state of the subsystem A and B and

$$\langle i|i'\rangle_A = \langle j|j'\rangle_B = 0. \tag{3.15}$$

From (3.14), (3.15) and the normalization condition $\langle\Psi|\Psi\rangle_{qm} = 1$, we obtain

$$\mathrm{Tr}\rho^n_{qmA} = \mathrm{Tr}\rho^n_{qmB} = N^{2n}[\langle i|i\rangle^n_A \langle j|j\rangle^n_B + \langle i'|i'\rangle^n_A \langle j'|j'\rangle^n_B] \tag{3.16}$$

where

$$N^{-2} = \langle i|i\rangle_A \langle j|j\rangle_B + \langle i'|i'\rangle_A \langle j'|j'\rangle_B \,. \tag{3.17}$$

Because the total system is a pure state, $\mathrm{Tr}\rho^n_{qmA\cup B} = 1$. From (3.16) and (3.17) we obtain the mutual Rényi information as

$$I^{(n)}_{qm}(A,B) = \frac{2}{n-1}\ln\frac{(x_{qm}+y_{qm})^n}{x^n_{qm}+y^n_{qm}}, \tag{3.18}$$

where

$$x_{qm} \equiv \langle i|i\rangle_A \langle j|j\rangle_B \,, \qquad y_{qm} \equiv \langle i'|i'\rangle_A \langle j'|j'\rangle_B \,. \tag{3.19}$$

(3.11) is the same as (3.18) when we replace the states as follows:

$$O_{i(i')A}|0\rangle \to |i(i')\rangle_A \,, \qquad O_{j(j')B}|0\rangle \to |j(j')\rangle_B \,. \tag{3.20}$$

Interestingly, the mutual information in the QFT measures only the quantum entanglement in the limit $r \to \infty$ although the mutual information measures generally the total of the quantum entanglement and the classical one [24]. So, in this limit, the mutual information is a good measure of quantum entanglement in this sense.

4 Free scalar fields

4.1 Explicit calculation of the glueing operator E_Ω

We consider $(d+1)$ dimensional free scalar field theory. For free scalar fields, it is useful to represent the glueing operator E_Ω in (2.15) as the normal ordered operator. We decompose ϕ and π into the creation and annihilation parts,

$$\phi(x) = \phi^+(x) + \phi^-(x), \qquad \pi(x) = \pi^+(x) + \pi^-(x), \tag{4.1}$$

where

$$\phi^+(x) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} a_p e^{ipx}, \qquad \phi^-(x) = (\phi^+(x))^\dagger,$$
$$\pi^+(x) = \int \frac{d^dp}{(2\pi)^d} (-i) \sqrt{\frac{E_p}{2}} a_p e^{ipx}, \qquad \pi^-(x) = (\pi^+(x))^\dagger, \tag{4.2}$$

here E_p is the energy and $[a_p, a^\dagger_{p'}] = (2\pi)^d \delta^d(p-p')$. The commutators of these operators are

$$[\phi^+(x), \phi^-(y)] = \langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^dp}{(2\pi)^d} \frac{1}{2E_p} e^{ip(x-y)} \equiv \frac{1}{2} W^{-1}(x-y),$$
$$[\pi^+(x), \pi^-(y)] = \langle 0| \pi(x)\pi(y) |0\rangle = \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{2} e^{ip(x-y)} \equiv \frac{1}{2} W(x-y), \tag{4.3}$$
$$[\pi^+(x), \phi^-(y)] = [\pi^-(x), \phi^+(y)] = -\frac{i}{2} \delta^d(x-y),$$

where we have defined the matrix W which has continuous indices x, y in (4.3) and W^{-1} is the inverse of W. By using (4.3) and the Baker-Campbell-Hausdorff (BCH) formula $e^X e^Y = e^{[X,Y]} e^Y e^X$, $e^{X+Y} = e^{-\frac{1}{2}[X,Y]} e^X e^Y$, for $[[X,Y],X] = [[X,Y],Y] = 0$, we obtain

$$\begin{aligned} &\exp\left[i \int d^dx J'\phi\right] \exp\left[i \int d^dx K\pi\right] \exp\left[-i \int d^dx J\phi\right] \\ &= : \exp\left[i \int d^dx (K\pi + (J'-J)\phi)\right] : \\ &\quad \times \exp\left[\int d^dx d^dy \left(-\frac{1}{4} K(x) W(x-y) K(y) - \frac{1}{4}(J'-J)(x) W^{-1}(x-y)(J'-J)(y)\right)\right. \\ &\quad \left. - \int d^dx \frac{i}{2} K(x)(J'+J)(x)\right], \end{aligned} \tag{4.4}$$

where $:O:$ means the normal ordered operator of O. From (4.4) we can rewrite E_Ω in (2.15) as the normal ordered operator,

$$E_\Omega = \int \prod_{j=1}^{n} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) : \exp\left[i \sum_{l=1}^{n} \int d^dx ((J^{(l+1)} - J^{(l)})\phi^{(l)} + K^{(l)}\pi^{(l)})\right] : \exp[-\tilde{S}], \tag{4.5}$$

where $J^{(n+1)} = J^{(1)}$ and

$$\begin{aligned} \tilde{S} \equiv \sum_{l=1}^{n} &\left[\int d^dx d^dy \left[\frac{1}{4} K^{(l)}(x) W(x-y) K^{(l)}(y) + \frac{1}{4}(J^{(l+1)} - J^{(l)})(x) W^{-1}(x-y)(J^{(l+1)} - J^{(l)})(y)\right]\right. \\ &\left. + \frac{i}{2} \int d^dx K^{(l)}(x)(J^{(l+1)} + J^{(l)})(x)\right]. \end{aligned} \tag{4.6}$$

For the vacuum state $\rho_0 = |0\rangle\langle 0|$, from (4.5) we obtain

$$\mathrm{Tr}\rho^n_{0\Omega} = \langle 0^{(n)}| E_\Omega |0^{(n)}\rangle = \int \prod_{j=1}^{n} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp[-\tilde{S}]. \tag{4.7}$$

We can show that (4.7) reproduces the same result as that of the real time approach [1, 2, 25] which is based on the wave functional calculation (see appendix A). By expanding the exponential in the normal ordered product in (4.5) and performing the Gauss integral of J and K, we can rewrite the E_Ω as a series of operators. Note that the odd powers of the expansion of (4.5) vanish from the symmetric property (2.16).

4.1.1 The case $n = 2$

In the case $n = 2$, it is useful to define the following linear combinations,

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi^{(1)} \pm \phi^{(2)}), \;\; \pi_\pm = \frac{1}{\sqrt{2}}(\pi^{(1)} \pm \pi^{(2)}), \;\; J_\pm = \frac{1}{\sqrt{2}}(J^{(1)} \pm J^{(2)}), \;\; K_\pm = \frac{1}{\sqrt{2}}(K^{(1)} \pm K^{(2)}). \tag{4.8}$$

From (4.8) we rewrite (4.5) as

$$\begin{aligned} E_\Omega = & \int DJ_+ DJ_- DK_+ DK_- : \exp\left[i\int d^dx(-2J_-\phi_- + K_+\pi_+ + K_-\pi_-)\right] : \\ & \times \exp\left[\int d^dx d^dy \left(-\frac{1}{4}(K_+(x)W(x-y)K_+(y) + K_-(x)W(x-y)K_-(y))\right.\right. \\ & \left.\left. - J_-(x)W^{-1}(x-y)J_-(y)\right) - i\int d^dx K_+(x)J_+(x)\right] \\ = & \int DJ_- DK_- : \exp\left[i\int d^dx(-2J_-\phi_- + K_-\pi_-)\right] : \\ & \times \exp\left[\int d^dx d^dy \left(-\frac{1}{4}K_-(x)W(x-y)K_-(y) - J_-(x)W^{-1}(x-y)J_-(y)\right)\right], \end{aligned} \tag{4.9}$$

where we have performed J_+ and K_+ integrals. In order to represent the Gauss integrals of K_- and J_-, we will use the following matrix notation,

$$W(x-y) = \begin{pmatrix} W(x_\Omega - y_\Omega) & W(x_\Omega - y_{\Omega^c}) \\ W(x_{\Omega^c} - y_\Omega) & W(x_{\Omega^c} - y_{\Omega^c}) \end{pmatrix} \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{4.10}$$

$$W^{-1}(x-y) = \begin{pmatrix} W^{-1}(x_\Omega - y_\Omega) & W^{-1}(x_\Omega - y_{\Omega^c}) \\ W^{-1}(x_{\Omega^c} - y_\Omega) & W^{-1}(x_{\Omega^c} - y_{\Omega^c}) \end{pmatrix} \equiv \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} \tag{4.11}$$

where $x_{\Omega(\Omega^c)}$ and $y_{\Omega(\Omega^c)}$ are the coordinates in $\Omega(\Omega^c)$. Thus the propagators of J_- and K_- are

$$\langle J_-(x)J_-(y)\rangle \equiv \frac{\int DJ_- J_-(x)J_-(y)e^{-\int d^dx d^dy J_-(x)W^{-1}(x-y)J_-(y)}}{\int DJ_- e^{-\int d^dx d^dy J_-(x)W^{-1}(x-y)J_-(y)}} = \frac{1}{2}D^{-1}(x-y) \tag{4.12}$$

and

$$\langle K_-(x)K_-(y)\rangle \equiv \frac{\int DK_- K_-(x)K_-(y)e^{-\int d^dx d^dy \frac{1}{4}K_-(x)W(x-y)K_-(y)}}{\int DK_- e^{-\int d^dx d^dy \frac{1}{4}K_-(x)W(x-y)K_-(y)}} = 2A^{-1}(x-y). \tag{4.13}$$

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Then we can expand the E_Ω for $n=2$ as

$$E_\Omega = \mathrm{Tr}\rho_{0\Omega}^2 \left[1 - 2\int d^dx d^dy \langle _-(x) J_-(y)\rangle : \phi_-(x)\phi_-(y) : \right. \\ \left. -\frac{1}{2}\int d^dx d^dy \langle K_-(x) K_-(y)\rangle : \pi_-(x)\pi_-(y) : + \dots \right]. \tag{4.14}$$

Next let us apply above results to the mutual Rényi information $I^{(2)}(A,B)$ of disjoint compact spatial regions A and B in the vacuum states of the massless free scalar field. We express E_Ω as a sum of the local operators at some conventionally chosen points (r_A, r_B) inside A and B. From (4.14), we have

$$E_{A(B)} = \mathrm{Tr}\rho_{0A(B)}^2 [1 - 2 : \phi_-^2(r_{A(B)}) : C_{A(B)} + \dots] \tag{4.15}$$

where

$$C_{A(B)} = \int d^dx d^dy \langle J_-(x) J_-(y)\rangle. \tag{4.16}$$

From the local property (2.17), $E_{A\cup B} = E_A E_B$, and (4.15) we obtain

$$\frac{\mathrm{Tr}\rho_{0A\cup B}^2}{\mathrm{Tr}\rho_{0A}^2 \mathrm{Tr}\rho_{0B}^2} = \frac{\langle 0^{(2)}|\, E_A E_B \,|0^{(2)}\rangle}{\langle 0^{(2)}|\, E_A \,|0^{(2)}\rangle \langle 0^{(2)}|\, E_B \,|0^{(2)}\rangle} \simeq 1 + \frac{1}{2} C_A C_B (W^{-1}(r))^2, \tag{4.17}$$

where we have used $\langle 0^{(2)}| : \phi_-^2(x) :: \phi_-^2(y) : |0^{(2)}\rangle = 2(\langle 0|\, \phi(x)\phi(y)\, |0\rangle)^2 = \frac{1}{2}(W^{-1}(x-y))^2$. We use the notation $E_p = |p|$ and

$$W^{-1}(x-y) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{|p|} a_p e^{ip(x-y)} = \frac{B_d}{|x-y|^{d-1}}, \tag{4.18}$$

where B_d is a numerical constant. Thus the the mutual Rényi information $I^{(2)}(A,B)$ is

$$I^{(2)}(A,B) \simeq \frac{1}{2} C_A C_B \frac{B_d^2}{r^{2d-2}}. \tag{4.19}$$

We can compute $C_{A(B)}$ numerically at least. The expression (4.16) of $C_{A(B)}$ is useful for numerical computation. Furthermore, we can obtain the alternative expression of $C_{A(B)}$ as follows. Let us consider the following generating function,

$$G(\sigma) = \int DJ_- \exp\left[-\int d^dx d^dy J_-(x) W^{-1}(x-y) J_-(y) + i\sigma \int d^dx J_-(x) \right], \tag{4.20}$$

where σ is a c number. We can obtain the coefficients of $: \phi_-^{2m}(r_A) :$ in the expansion of E_A from $G(\sigma)$. For example, we have

$$C_A = \frac{-1}{G(0)} \left.\frac{\partial^2 G}{\partial \sigma^2}\right|_{\sigma=0}. \tag{4.21}$$

We can rewrite $G(\sigma)$ by using the $(d+1)$ dim Euclidean path integral as

$$G(\sigma) = \frac{1}{Z(0)} \int DJ_- Z(J_-) \exp\left[i\sigma \int d^dx J_-(x) \right], \tag{4.22}$$

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where

$$Z(J_-) = \int D\varphi \exp\left[-\frac{1}{2}\int d^{d+1}x(\partial_\mu\varphi)^2 - 2i\int d^dx J_-\varphi|_{\tau=0}\right], \tag{4.23}$$

where τ is the Euclidean time coordinate and $\varphi(\tau, x)$ is a scalar field in $(d+1)$ dimensional Euclidean space. First, we perform the J_- integral and obtain

$$G(\sigma) = \frac{1}{Z(0)}\int D\varphi \exp\left[-\frac{1}{2}\int d^{d+1}x(\partial_\mu\varphi)^2\right]\prod_{x\in\Omega}\delta(\sigma - 2\varphi(\tau=0,x)). \tag{4.24}$$

We decompose φ into the quantum part φ_q and the classical part $\varphi_{\rm cl}$. The $\varphi_{\rm cl}$ is the solution of $\partial^2\varphi = 0$ and satisfy the boundary condition,

$$\varphi_{\rm cl}(\tau=0,x) = \frac{\sigma}{2} \quad (x\in A), \qquad \text{and} \qquad \varphi_{\rm cl}(\tau,x) = 0 \quad (|\tau^2 + x^2| \to \infty). \tag{4.25}$$

Thus the region $(\tau = 0, x \in A)$ acts like a conductor where electrostatic potential is $\sigma/2$ and $\varphi_{\rm cl}(\tau, x)$ is electrostatic potential at (τ, x). We perform the $\varphi_{\rm q}$ integral and obtain

$$G(\sigma) = G(0)\exp\left[-\frac{1}{2}\int d^{d+1}x(\partial_\mu\varphi_{\rm cl})^2\right] = G(0)\exp\left[-\frac{\sigma^2}{8}\mathbf{C}_A\right], \tag{4.26}$$

where $\frac{1}{2}\int d^{d+1}x(\partial_\mu\varphi_{\rm cl})^2$ is the electrostatic energy and we have rewritten it by using $\mathbf{C}_A$ which is electrostatic capacitance of the conductor $(\tau = 0, x \in A)$. From (4.21) and (4.26) we have

$$C_A = \frac{1}{4}\mathbf{C}_A. \tag{4.27}$$

Thus, from (4.19) and (4.27), we reproduced the result that $I^{(2)}(A, B)$ in the vacuum state is proportional to the product of the electrostatic capacitance of each regions obtained by Cardy [9].

4.2 The mutual information of locally excited states

We consider the mutual Rényi information $I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states of the (d+1) dimensional free massless scalar field theory in the limit when the separation r between A and B is much greater than their sizes $R_{A,B}$.

We consider the following locally excited state,

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle, \tag{4.28}$$

where N is a real normalization constant and O_{iA} and $O_{i'A}$ (O_{jB} and $O_{j'B}$) are operators on A (B) and i and i' (j and j') label a kind of operators. We impose following orthogonal conditions,

$$\langle 0| O_{iA}^\dagger O_{i'A}|0\rangle = \langle 0| O_{jB}^\dagger O_{j'B}|0\rangle = 0. \tag{4.29}$$

This state is similar to the EPR state. We compute the mutual Rényi information of this state. When $r \to \infty$, the mutual information $I^{(n)}(A, B)$ is the same as that of the general

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QFT which have a mass gap in (3.11). In the mass gap case there is a correction which is $O(e^{-mR_{A(B)}})$ for (3.11). In the free massless case there is a correction which is $O(1/R_{A(B)})$.

We consider the leading term of $I^{(n)}(A,B)$ which depends on r for $r \gg R_A, R_B$. Because the symmetric property (2.16) is important to determine the r dependence of $I^{(n)}(A,B)$ as we will show it later, from now on, we impose the condition that under the sign changing transformation $(\phi,\pi) \to (-\phi,-\pi)$ the operators O in (4.28) is transformed as

$$O \to (-1)^{|O|} O, \tag{4.30}$$

where $|O| = 0$ or 1. From the normalization condition $\langle\Psi|\Psi\rangle = 1$ we have

$$N^{-2} = \langle 0| \left[O^\dagger_{iA}O_{iA}O^\dagger_{jB}O_{jB} + O^\dagger_{iA}O_{i'A}O^\dagger_{jB}O_{j'B} + O^\dagger_{i'A}O_{iA}O^\dagger_{j'B}O_{jB} + O^\dagger_{i'A}O_{i'A}O^\dagger_{j'B}O_{j'B}\right] |0\rangle\,. \tag{4.31}$$

By using (2.9), we obtain $\mathrm{Tr}\rho^n_\Omega$ for $\rho = |\Psi\rangle\langle\Psi|$ as

$$\begin{aligned}\mathrm{Tr}\rho^n_\Omega &= \langle\Psi^{(n)}| E_\Omega |\Psi^{(n)}\rangle \\ &= N^{2n} \langle 0^{(n)}| (O^\dagger_{iA}O^\dagger_{jB} + O^\dagger_{i'A}O^\dagger_{j'B})^{(1)} \ldots (O^\dagger_{iA}O^\dagger_{jB} + O^\dagger_{i'A}O^\dagger_{j'B})^{(n)} \\ &\quad\times E_\Omega (O_{iA}O_{jB} + O_{i'A}O_{j'B})^{(1)} \ldots (O_{iA}O_{jB} + O_{i'A}O_{j'B})^{(n)} |0^{(n)}\rangle\,,\end{aligned} \tag{4.32}$$

where $O^{(l)}$ are operators in the Hilbert space of the l-th copy and the Ω is A, B or $A\cup B$.

4.2.1 The case $O_{i'A} = O_{j'B} = 0$

First we consider the case $O_{i'A} = O_{j'B} = 0$ and O_{iA} and O_{jB} are nonzero

$$|\Psi_1\rangle = N_1 O_{iA}O_{jB}|0\rangle\,. \tag{4.33}$$

From the normalization condition we have

$$N_1^{-2} = \langle 0| O^\dagger_{iA}O_{iA}O^\dagger_{jB}O_{jB}|0\rangle\,. \tag{4.34}$$

From the condition (4.30), $|O^\dagger_{iA}O_{iA}| = 0$ and the OPE of $O^\dagger_{iA}O_{iA}$ has the form

$$O^\dagger_{iA}O_{iA} = \langle 0| O^\dagger_{iA}O_{iA}|0\rangle \left[1 + C^{:\phi^2:}_{iA} :\phi^2: (r_A) + \ldots\right], \tag{4.35}$$

where (r_A, r_B) are some conventionally chosen points inside A and B. The key point is that there is not ϕ in the OPE (4.35) because of $|O^\dagger_{iA}O_{iA}| = 0$. The OPE of $O^\dagger_{jB}O_{jB}$ has the same form as that of $O^\dagger_{iA}O_{iA}$ because of $|O^\dagger_{jB}O_{jB}| = 0$. Thus we have

$$N_1^{-2} = \langle 0| O^\dagger_{iA}O_{iA}|0\rangle \langle 0| O^\dagger_{jB}O_{jB}|0\rangle \left[1 + O(1/r^{2d-2})\right], \tag{4.36}$$

where we have used $\langle 0| :\phi^2(r_A)::\phi^2(r_B): |0\rangle \propto 1/r^{2d-2}$.

We consider $\mathrm{Tr}\rho^n_A$. From (4.32) we have

$$\begin{aligned}\mathrm{Tr}\rho^n_A &= \langle\Psi^{(n)}_1| E_\Omega |\Psi^{(n)}_1\rangle \\ &= N_1^{2n} \langle 0^{(n)}| (O^\dagger_{iA}O^\dagger_{jB})^{(1)} \ldots (O^\dagger_{iA}O^\dagger_{jB})^{(n)} E_A (O_{iA}O_{jB})^{(1)} \ldots (O_{iA}O_{jB})^{(n)} |0^{(n)}\rangle \\ &= N_1^{2n} \langle 0^{(n)}| O^{(1)\dagger}_{iA} \ldots O^{(n)\dagger}_{iA} E_A O^{(1)}_{iA} \ldots O^{(n)}_{iA} \cdot O^{(1)\dagger}_{jB} \ldots O^{(n)\dagger}_{jB} O^{(1)}_{jB} \ldots O^{(n)}_{jB} |0^{(n)}\rangle\,.\end{aligned} \tag{4.37}$$

From (2.16) and (4.30), $O_{iA}^{(1)\dagger}\dots O_{iA}^{(n)\dagger}E_A O_{iA}^{(1)}\dots O_{iA}^{(n)}$ and $O_{jB}^{(1)\dagger}\dots O_{jB}^{(n)\dagger}O_{jB}^{(1)}\dots O_{jB}^{(n)}$ are even under the sign changing transformation

$$(\phi^{(1)},\dots,\phi^{(n)},\pi^{(1)},\dots,\pi^{(n)}) \to (-\phi^{(1)},\dots,-\phi^{(n)},-\pi^{(1)},\dots,-\pi^{(n)}). \tag{4.38}$$

So the OPEs of these operators are

$$\begin{aligned} &O_{iA}^{(1)\dagger}\dots O_{iA}^{(n)\dagger}E_A O_{iA}^{(1)}\dots O_{iA}^{(n)} = \\ &\quad \langle 0|\, O_{iA}^{(1)\dagger}\dots O_{iA}^{(n)\dagger}E_A O_{iA}^{(1)}\dots O_{iA}^{(n)}\,|0\rangle \left[1+\sum_{l,m}\tilde{C}_{iA}^{:\phi^{(l)}\phi^{(m)}:} :\phi^{(l)}\phi^{(m)}:(r_A)+\dots\right] \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} &O_{jB}^{(1)\dagger}\dots O_{jB}^{(n)\dagger}O_{jB}^{(1)}\dots O_{jB}^{(n)} = \\ &\quad \langle 0|\, O_{jB}^{(1)\dagger}\dots O_{jB}^{(n)\dagger}O_{jB}^{(1)}\dots O_{jB}^{(n)}\,|0\rangle \left[1+\sum_{l}\tilde{D}_{jB}^{:\phi^{(l)}\phi^{(l)}:} :\phi^{(l)}\phi^{(l)}:(r_B)+\dots\right]. \end{aligned} \tag{4.40}$$

Thus, we substitute (4.36), (4.39) and (4.40) into (4.37) and obtain the power of r of the subleading term of $\mathrm{Tr}\rho_A^n$

$$\mathrm{Tr}\rho_A^n = \mathrm{Tr}\rho_A^n|_{r\to\infty}[1+O(1/r^{2d-2})]. \tag{4.41}$$

In the same way, we can obtain the power of r of the subleading terms of $\mathrm{Tr}\rho_B^n$ and $\mathrm{Tr}\rho_{A\cup B}^n$

$$\mathrm{Tr}\rho_B^n = \mathrm{Tr}\rho_B^n|_{r\to\infty}[1+O(1/r^{2d-2})], \qquad \mathrm{Tr}\rho_{A\cup B}^n = \mathrm{Tr}\rho_{A\cup B}^n|_{r\to\infty}[1+O(1/r^{2d-2})]. \tag{4.42}$$

Thus the mutual Rényi information is

$$I^{(n)}(A,B) = O(1/r^{2d-2}), \tag{4.43}$$

where we have used $I^{(n)}(A,B)|_{r\to\infty}=0$ for $O_{i'A}=O_{j'B}=0$ from (3.11).

4.2.2 The case $O_{iA}, O_{jB}, O_{i'A}$ and $O_{j'B}$ are nonzero

There are two cases which are different powers of r of the subleading term of $I^{(n)}(A,B)$.

(i) The case $|O_{iA}| = |O_{i'A}|$.

In this case the operators at A in the expansions of N^{-2}, $\mathrm{Tr}\rho_\Omega^n$ $(\Omega = A, B, A\cup B)$ in (4.31) and (4.32) are even under the sign changing transformation (4.38). So the subleading terms of them come from the operators $:\phi^{(l)}\phi^{(m)}:(r_A)$. Thus the mutual Rényi information is

$$I^{(n)}(A,B) = I^{(n)}(A,B)|_{r\to\infty} + O(1/r^{2d-2}), \tag{4.44}$$

where $I^{(n)}(A,B)|_{r\to\infty}$ is the same form as (3.11).

(ii) The case $|O_{iA}| \neq |O_{i'A}|$ and $|O_{jB}| \neq |O_{j'B}|$.

There are some terms which are odd under the sign changing transformation (4.38). As an example, in the expansion of $\mathrm{Tr}\rho^n_{A\cup B}$ in (4.32), we consider the following term,

$$\langle 0^{(n)}|\, O^{(1)\dagger}_{i'A} O^{(2)\dagger}_{iA} \dots O^{(n)\dagger}_{iA} E_A O^{(1)}_{iA} O^{(2)}_{iA} \dots O^{(n)}_{iA} \cdot O^{(1)\dagger}_{j'B} O^{(2)\dagger}_{jB} \dots O^{(n)\dagger}_{jB} E_B O^{(1)}_{jB} O^{(2)}_{jB} \dots O^{(n)}_{jB}\, |0^{(n)}\rangle\,. \tag{4.45}$$

In (4.45), the operators at A and at B are both odd under the sign changing transformation (4.38). So the leading r dependent term of it come from the operators $\phi^{(l)}$. Thus the mutual Rényi information is

$$I^{(n)}(A,B) = I^{(n)}(A,B)|_{r\to\infty} + O(1/r^{d-1}), \tag{4.46}$$

where $I^{(n)}(A,B)|_{r\to\infty}$ is the same form as (3.11).

5 Conclusion and discussions

We developed the computational method of Rényi entanglement entropy based on the idea that $\mathrm{Tr}\rho^n_\Omega$ is written as the expectation value of the local operator at Ω. We expressed $\mathrm{Tr}\rho^n_\Omega$ as the expectation value of the glueing operator E_Ω, $\mathrm{Tr}\rho^n_\Omega = \mathrm{Tr}(\rho^{(n)} E_\Omega)$. We constructed explicitly E_Ω and investigated its general properties. For a free scalar field, we rewrote E_Ω in (2.15) using the normal ordering. In the case $n = 2$, we obtained a simple expression of E_Ω and reproduced the result that $I^{(2)}(A,B)$ in the vacuum state is proportional to the product of the electrostatic capacitance of each regions obtained by Cardy [9]. The coefficients of the expansion of E_Ω is obtained by the propagators of J_- and K_- in (4.12) and (4.13). We can compute these propagators numerically at least and the expression (4.9) is useful for numerical calculation.

The advantages of this methods are that we can use ordinary technique in QFT such as OPE and the cluster decomposition property and that we can use the general properties and the explicit expression of the glueing operator to compute systematically the Rényi entropy for an arbitrary state.

We applied this method to consider the mutual Rényi information $I^{(n)}(A,B)$ of disjoint compact spatial regions A and B in the locally excited states defined by acting the local operators at A and B on the vacuum of a $(d+1)$-dimensional field theory, in the limit when the separation r between A and B is much greater than their sizes $R_{A,B}$. For the general QFT which has a mass gap, we computed $I^{(n)}(A,B)$ explicitly and find that this result is interpreted in terms of an entangled state in quantum mechanics. Interestingly, the mutual information in the QFT measures only the quantum entanglement in the limit $r \to \infty$ although the mutual information measures generally the total of the quantum entanglement and the classical one [24]. So, in this limit, the mutual information is a good measure of quantum entanglement in this sense. For a free massless scalar field, we showed that for some classes of excited states $I^{(n)}(A,B) - I^{(n)}(A,B)|_{r\to\infty} = C^{(n)}_{AB}/r^{\alpha(d-1)}$ where $\alpha = 1$ or 2 which is determined by the property of the local operators under the transformation $\phi \to -\phi$ and $\alpha = 2$ for the vacuum state.

Finally we discuss the generalization of our method. Although we considered only the locally excited states, we can apply our method to more general excited states, for example, many particle states and thermal states. We might be able to generalize our method to fermionic fields. We could apply our method to perturbative calculation in an interacting field theory.

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A Rényi entropy for the vacuum state in free scalar field theory

In this appendix we show that (4.7) reproduces the same result as that of the real time approach [1, 2, 25] which is based on the wave functional calculation.

From (4.6) and (4.7), we perform the K integral in (4.7) and obtain

$$\mathrm{Tr}\rho_{0\Omega}^n = \left(Det\left(\frac{A}{4\pi}\right)\right)^{-n/2} \int \prod_{j=1}^{n}\prod_{x\in\Omega} DJ^{(j)}(x) \times \exp\left[-\int d^dx d^dy (J^{(1)}(x),\cdots,J^{(n)}(x)) M_n(x,y) \begin{pmatrix} J^{(1)}(y) \\ \vdots \\ J^{(n)}(y)\end{pmatrix}\right], \tag{A.1}$$

where

$$M_n = \begin{pmatrix} X & Y & 0 & \cdots & 0 & Y \\ Y & X & Y & \cdots & 0 & 0 \\ 0 & Y & X & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X & Y \\ Y & 0 & 0 & \cdots & Y & X \end{pmatrix}, \tag{A.2}$$

here

$$X = \frac{1}{2}(A^{-1}+D), \qquad Y = \frac{1}{4}(A^{-1}-D) \tag{A.3}$$

and we have used the matrix notation in (4.10) and (4.11).

The following calculation is analogous to that in [25]. From (A.1) and (A.2) we obtain

$$\mathrm{Tr}\rho_{0\Omega}^n = \left(Det\left(\frac{A}{4\pi}\right)\right)^{-n/2} (Det(4\pi M_n))^{-1/2}, \tag{A.4}$$

where we have used the normalization condition of the J integral $\int DJ^{(j)} \exp[-\int d^dx d^dy \cdot J^{(j)}(x) M(x,y) J^{(j)}(y)] = (Det(4\pi M))^{-1/2}$.

We rewrite M_n as

$$M_n = \frac{X}{2}\tilde{M_n}, \tag{A.5}$$

where

$$\tilde{M_n} = \begin{pmatrix} 2 & -Z & 0 & \cdots & 0 & -Z \\ -Z & 2 & -Z & \cdots & 0 & 0 \\ 0 & -Z & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -Z \\ -Z & 0 & 0 & \cdots & -Z & 2 \end{pmatrix}, \tag{A.6}$$

here

$$Z = -2X^{-1}Y = (1 + AD)^{-1}(AD - 1). \tag{A.7}$$

We diagonalize Z and denote the eigenvalues of Z as z_i. And we can diagonalize $\tilde{M_n}$ by Fourier transformation and obtain

$$Det\tilde{M_n} = \prod_i \prod_{r=1}^{n} \left[2 - 2z_i \cos\left(\frac{2\pi r}{n}\right)\right] = \prod_i 2^n \frac{(1-\xi_i^n)^2}{(1+\xi_i^2)^n}, \tag{A.8}$$

where ξ_i is defined as

$$z_i = 2\xi_i/(\xi_i^2 + 1). \tag{A.9}$$

From (A.4) and (A.8) we obtain

$$\mathrm{Tr}\rho_{0\Omega}^n = \prod_i \frac{(1-\xi_i)^n}{(1-\xi_i^n)}. \tag{A.10}$$

Thus we obtain the Rényi entropies $S_{0\Omega}^{(n)} = (1-n)^{-1}\ln \mathrm{Tr}\rho_{0\Omega}^n$ and the entanglement entropy $S_{0\Omega} = -\mathrm{Tr}\rho_{0\Omega}\ln\rho_{0\Omega} = -\frac{\partial}{\partial n}\ln \mathrm{Tr}\rho_{0\Omega}^n|_{n=1}$ as follows:

$$S_{0\Omega}^{(n)} = \sum_i (1-n)^{-1}[n\ln(1-\xi_i) - \ln(1-\xi_i^n)], \tag{A.11}$$

$$S_{0\Omega} = \sum_i \left[-\ln(1-\xi_i) - \frac{\xi_i}{1-\xi_i}\ln\xi_i\right]. \tag{A.12}$$

In order to show that (A.11) and (A.12) are the same results as those of the real time approach, we rewrite (A.11) and (A.12). We define the matrix

$$\tilde{C} = \frac{1}{2}(DA)^{1/2}, \tag{A.13}$$

and rewrite the matrix Z in (A.7) as

$$Z = (1 + 4(\tilde{C}^T)^2)^{-1}(4(\tilde{C}^T)^2 - 1). \tag{A.14}$$

From (A.9) and (A.14) we obtain

$$\tilde{C}_i^T = \frac{1}{2}(1+\xi_i)(1-\xi_i)^{-1}. \tag{A.15}$$

From (A.15), we rewrite (A.11) and (A.12) as

$$\begin{aligned} S_{0\Omega}^{(n)} &= \sum_i (n-1)^{-1} \ln[(\tilde{C}_i^T + 1/2)^n - (\tilde{C}_i^T - 1/2)^n] \\ &= (n-1)^{-1} \mathrm{tr} \ln[(\tilde{C}^T + 1/2)^n - (\tilde{C}^T - 1/2)^n] \\ &= (n-1)^{-1} \mathrm{tr} \ln[(\tilde{C} + 1/2)^n - (\tilde{C} - 1/2)^n], \end{aligned} \tag{A.16}$$

$$S_{0\Omega} = \mathrm{tr}[(\tilde{C} + 1/2) \ln(\tilde{C} + 1/2) - (\tilde{C} - 1/2) \ln(\tilde{C} - 1/2)]. \tag{A.17}$$

(A.16) and (A.17) are the same results as those of the real time approach (see e.g. [13]).

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