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A new quasidilaton theory of massive gravity

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Abstract. We present a new quasidilaton theory of Poincare invariant massive gravity, based on the recently proposed framework of matter coupling that makes it possible for the kinetic energy of the quasidilaton scalar to couple to both physical and fiducial metrics simultaneously. We find a scaling-type exact solution that expresses a self-accelerating de Sitter universe, and then analyze linear perturbations around it. It is shown that in a range of parameters all physical degrees of freedom have non-vanishing quadratic kinetic terms and are stable in the subhorizon limit, while the effective Newton's constant for the background is kept positive.

Keywords: modified gravity, gravity

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Contents

1 Introduction

It has been a long standing fundamental question in theoretical physics whether the graviton, a spin-2 field that mediates the gravitational force, can have a finite mass or not. While Fierz and Pauli's pioneering work in 1939 [1] found a consistent linear theory of massive gravity, Boulware and Deser in 1972 [2] showed that generic nonlinear extensions of the theory exhibit ghost-type instability, often called Boulware-Deser (BD) ghost. It took almost 40 years since then until de Rham, Gabadadze and Tolley (dRGT) in 2010 [3, 4] finally found a nonlinear completion of the Fierz and Pauli's theory without the BD ghost. While the theory was initially found by demanding the absence of the BD ghost in the so called decoupling limit, it was later proved by Hassan and Rosen [5, 6] that the theory is free from the BD ghost at the fully non-linear level even away from the decoupling limit.

Despite the recent theoretical progress in massive gravity, it is still fair to say that cosmology in massive gravity has not been established yet. In this respect, two no-go results are currently known against simple realization of viable cosmology in massive gravity. The first one forbids the flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology in the original dRGT theory [7]. This no-go can be avoided either by considering open FLRW cosmology in the original theory [8] or by slightly extending the theory with a de Sitter or FLRW fiducial metric [9] (see [10] for self-accelerating FLRW solutions and [11–13] for non self-accelerating FLRW solutions with a generalized fiducial metric). However, the second no-go tells that all homogeneous and isotropic FLRW solutions in the dRGT theory, either in its original form or with a more general fiducial metric, are unstable [14]. There then seem (at least) three possible options to go around the second no-go: (i) to relax either homogeneity [7] or isotropy [15, 16] of the background solution; (ii) to extend the theory either by introducing extra degree(s) of freedom [17, 18] or by abandoning the direct connection with the Fierz and Pauli's theory [19–21]; or (iii) to change the way how matter fields couple to gravity [22, 23].

The quasidilaton theory [17] introduces an extra scalar degree of freedom to the dRGT theory and thus falls into the category (ii). There exists a scaling-type solution that expresses a self-accelerating de Sitter universe in the flat FLRW chart. However, the scaling solution in the original theory turned out to be unstable [24, 25]. (See [26] for another type of self-accelerating solution in the decoupling limit.) Fortunately, the scaling solution can be stabilized in a range of parameters by introducing a new coupling constant corresponding to the amount of disformal transformation to the fiducial metric [27]. For the minimal model of this type, stability of cosmological evolution in the presence of matter fields was recently studied in [28]. The theory can be further generalized as in [29], allowing for a larger set of parameters.

In the present paper we shall propose yet another extension of the quasidilaton theory of massive gravity, motivated by the new matter coupling [22]. The role of the new matter coupling is to make it possible for the kinetic energy of the quasidilaton scalar to couple to both physical and fiducial metrics simultaneously.

The rest of the present paper is organized as follows. In section 2 we describe the dRGT theory, the original quasidilaton theory and the new quasidilaton theory step by step. In section 3 we analyze the background equations of motion with the FLRW ansatz and find an exact scaling-type solution that expresses a self-accelerating de Sitter universe. This is a continuous deformation of the same type of solution that was already found in the original quasidilaton theory. What is interesting is that, unlike the extension considered in [27], properties of the scaling-type solution depends crucially on a new parameter introduced by the extension in the present paper. For example, in the limit of a small Hubble expansion rate, the effective Newton's constant for the FLRW background evolution is positive as far as the new parameter is non-zero, irrespective of other parameters of the theory. In section 4 we analyze tensor, vector and scalar perturbations around the de Sitter solution. Based on the result of the perturbative analysis, in section 5 we study the stability of subhorizon perturbations around the de Sitter solution. It is shown that all physical degrees of freedom have finite quadratic kinetic terms and are stable in a range of parameters while the effective Newton's constant for the background is positive, even when the genuine cosmological constant is set to zero. Section 6 is devoted to a summary and discussions.

2 From dRGT to new quasidilaton theory

In this section we describe the dRGT theory, the original quasidilaton theory and the new quasidilaton theory step by step.

2.1 dRGT

We begin with describing the dRGT massive gravity theory [4]. In the covariant formulation the theory is described by a physical metric $g_{\mu\nu}$ and four scalar fields called Stückelberg fields, ϕ^a $(a = 0, 1, 2, 3)$. The theory enjoys the Poincare symmetry in the Stückelberg field space, i.e. the action is invariant under the following transformation

$$\phi^a \to \phi^a + c^a, \quad \phi^a \to \Lambda^a_b \phi^b, \tag{2.1}$$

where c^a are constants and Λ^a_b represents a Lorentz transformation. Hence the Stückelberg fields enter the action only through the pull-back of the Minkowski metric in the field space to the spacetime defined as

$$f_{\mu\nu} = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b, \quad \eta_{ab} = \mathrm{diag}(-1,1,1,1). \tag{2.2}$$

Using the tensor $f_{\mu\nu}$, often called a fiducial metric, it is convenient to define

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu{}_\nu. \tag{2.3}$$

The graviton mass terms that describe interactions between the physical metric and the Stückelberg fields are then constructed as

$$I_{\mathrm{dRGT}}[g_{\mu\nu}, f_{\mu\nu}] = M_{\mathrm{Pl}}^2 m_g^2 \int d^4x\sqrt{-g}\,\left[\mathcal{L}_2(\mathcal{K}) + \alpha_3\mathcal{L}_3(\mathcal{K}) + \alpha_4\mathcal{L}_4(\mathcal{K})\right], \tag{2.4}$$

where

$$\begin{aligned}\mathcal{L}_2(\mathcal{K}) &= \frac{1}{2}\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \quad \mathcal{L}_3(\mathcal{K}) = \frac{1}{6}\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right),\\ \mathcal{L}_4(\mathcal{K}) &= \frac{1}{24}\left([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]\right),\end{aligned} \tag{2.5}$$

and a square bracket in (2.5) denotes trace operation. The theory is free from BD ghost at the fully non-linear level [5, 6]. However, it has been a rather non-trivial task to find stable cosmological solutions.

2.2 Original quasidilaton

The quasidilaton theory is an extension of dRGT theory that involves an extra scalar field, called a quasidilaton. In its covariant formulation the theory is thus described by the physical metric $g_{\mu\nu}$, the four Stückelberg fields ϕ^a and the quasidilaton scalar σ. In addition to the Poincare symmetry as described in the previous subsection, the theory is invariant under the global transformation

$$\sigma \to \sigma + \sigma_0, \quad \phi^a \to e^{-\sigma_0/M_{\mathrm{Pl}}}\phi^a, \tag{2.6}$$

where σ_0 is an arbitrary constant. One can construct graviton mass terms that are invariant under the global transformation by simply replacing $\mathcal{K}^\mu_\nu$ in the dRGT mass terms with

$$\bar{\mathcal{K}}^\mu_\nu = \delta^\mu_\nu - e^{\sigma/M_{\mathrm{Pl}}}\left(\sqrt{g^{-1}f}\right)^\mu{}_\nu. \tag{2.7}$$

Adding a kinetic term of the quasidilaton scalar, one then obtains

$$I_{\mathrm{QD}}[g_{\mu\nu}, f_{\mu\nu}, \sigma] = M_{\mathrm{Pl}}^2 m_g^2 \int d^4x\sqrt{-g}\,\left[\mathcal{L}_2(\bar{\mathcal{K}}) + \alpha_3\mathcal{L}_3(\bar{\mathcal{K}}) + \alpha_4\mathcal{L}_4(\bar{\mathcal{K}})\right] - \frac{\omega}{2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma, \tag{2.8}$$

where ω is a dimensionless constant. Thanks to the global symmetry (2.6), the quasidilaton theory allows for a scaling-type solution that describes a self-accelerating de Sitter universe in the flat FLRW chart. While the self-accelerating de Sitter solution in the original quasidilaton theory is unstable [24, 25], an extension of the theory makes the same solution stable in a range of parameters [27].

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2.3 New quasidilaton

In the present paper we propose yet another extension of the quasidilaton theory of Poincare invariant massive gravity. In the original theory, the kinetic term of the quasidilaton scalar σ is given in terms of the physical metric $g_{\mu\nu}$. In the new theory we consider the following effective metric to construct the kinetic term of the quasidilaton scalar.

$$g^{\rm eff}_{\mu\nu} = g_{\mu\nu} + 2\beta e^{\sigma/M_{\rm Pl}} g_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} + \beta^2 e^{2\sigma/M_{\rm Pl}} f_{\mu\nu}, \tag{2.9}$$

where β is a dimensionless constant. This is a simple extension of the effective metric proposed by [22]. (See also [30].) Hereafter, it is assumed that β is non-negative in order to avoid signature change of the effective metric. It is evident that this effective metric respects the global quasidilaton symmetry (2.6). We thus propose the action of the new quasidilaton theory as

$$I_{\rm NQD}[g_{\mu\nu}, f_{\mu\nu}, \sigma] = M_{\rm Pl}^2 m_g^2 \int d^4x\sqrt{-g}\left[\mathcal{L}_2(\bar{\mathcal{K}}) + \alpha_3\mathcal{L}_3(\bar{\mathcal{K}}) + \alpha_4\mathcal{L}_4(\bar{\mathcal{K}})\right] - \frac{\omega}{2}\int d^4x\sqrt{-g_{\rm eff}}\, g^{\mu\nu}_{\rm eff}\partial_\mu\sigma\partial_\nu\sigma, \tag{2.10}$$

where $g_{\rm eff}$ and $g^{\mu\nu}_{\rm eff}$ are the determinant and the inverse of $g^{\rm eff}_{\mu\nu}$. The new quasidilaton theory is thus parameterized by (m_g, α_3, α_4, ω, β). Adding the Einstein-Hilbert action, the total action is then

$$I_{\rm tot} = I_{\rm EH} + I_{\rm NQD}, \quad I_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\,(R - 2\Lambda). \tag{2.11}$$

3 de Sitter background

We consider a flat FLRW ansatz

$$g_{\mu\nu}dx^{\mu\nu} = -N(t)^2dt^2 + a(t)^2\delta_{ij}dx^idx^j, \quad \phi^0 = f(t), \quad \phi^i = a_0x^i, \quad \sigma = \bar{\sigma}(t), \tag{3.1}$$

where a_0 is a constant. The fiducial metric and the effective metric are then

$$\begin{aligned} f_{\mu\nu}dx^\mu dx^\nu &= -(\dot{f})^2dt^2 + a_0^2\delta_{ij}dx^idx^j, \\ g^{\rm eff}_{\mu\nu}dx^\mu dx^\nu &= -(1+\beta rX)^2N^2dt^2 + (1+\beta X)^2a^2\delta_{ij}dx^idx^j, \end{aligned} \tag{3.2}$$

where an over-dot represents derivative with respect to t and we have introduced the following quantities

$$X = \frac{e^{\bar{\sigma}/M_{\rm Pl}}a_0}{a}, \quad r = \frac{\dot{f}a}{Na_0}. \tag{3.3}$$

The independent background equations of motion are

$$\begin{aligned} 0 &= \frac{\dot{J}}{N} + 4HJ, \\ 3H^2 &= \Lambda + m_g^2\rho_X + \frac{\omega}{2}\frac{(1+\beta X)^3}{(1+\beta rX)^2}\left(\frac{\dot{X}}{NX} + H\right)^2, \\ -\frac{2\dot{H}}{N} &= (1-r)Xm_g^2J_X + \frac{\omega}{2}\frac{(1+\beta X)^2[(1+\beta X) + (1+\beta rX)]}{(1+\beta rX)^2}\left(\frac{\dot{X}}{NX} + H\right)^2, \end{aligned} \tag{3.4}$$

where $H = \dot{a}/(Na)$ is the Hubble expansion rate and

$$
\begin{aligned}
J &= m_g^2 X(1-X)\left[3+3(1-X)\alpha_3+(1-X)^2\alpha_4\right] + \frac{\omega}{2}\frac{\beta X(1+\beta X)^3}{(1+\beta r X)^2}\left(\frac{\dot{X}}{NX}+H\right)^2, \\
\rho_X &= (X-1)\left[(X-1)(X-4)\alpha_3+(X-1)^2\alpha_4-3(X-2)\right], \\
J_X &= (X-1)(X-3)\alpha_3+(X-1)^2\alpha_4+3-2X,
\end{aligned}
\tag{3.5}
$$

The first equation in (3.4) implies that J decays as $\propto 1/a^4$ as the universe expands. We thus have an attractor de Sitter solution at $J=0$ as

$$
H = m_g h, \quad X = X_0, \quad r = r_0, \tag{3.6}
$$

where h, X_0 and r_0 are constants satisfying

$$
\begin{aligned}
X_0(1-X_0)\left[3+3(1-X_0)\alpha_3+(1-X_0)^2\alpha_4\right] + \frac{\omega}{2}\frac{\beta X_0(1+\beta X_0)^3}{(1+\beta r_0 X_0)^2}h^2 &= 0, \\
-3h^2+\lambda+\rho_{X0}+\frac{\omega}{2}\frac{(1+\beta X_0)^3}{(1+\beta r X_0)^2}h^2 &= 0, \\
(1-r_0)X_0 J_{X0}+\frac{\omega}{2}\frac{(1+\beta X_0)^2[(1+\beta X_0)+(1+\beta r_0 X_0)]}{(1+\beta r X_0)^2}h^2 &= 0.
\end{aligned}
\tag{3.7}
$$

Here, ρ_{X0} and J_{X0} are ρ_X and J_X, respectively, evaluated at $X = X_0$ and we have defined $\lambda = \Lambda/m_g^2$.

By using the set of equations, one can express $(\lambda,\, \alpha_3,\, \alpha_4)$ in terms of $(h^2,\, X_0,\, r_0)$ as

$$
\begin{aligned}
\lambda &= 3h^2+(1-X_0)^2+\frac{\omega h^2(1+\beta X_0)^2 A_\lambda}{2(1+\beta r_0 X_0)^2(r_0-1)X_0^2}, \\
\alpha_3 &= \frac{2}{X_0-1}+\frac{\omega h^2(1+\beta X_0)^2 A_3}{2(1+\beta r_0 X_0)^2(r_0-1)X_0^2}, \\
\alpha_4 &= \frac{3}{(X_0-1)^2}+\frac{\omega h^2(1+\beta X_0)^2 A_4}{2(1+\beta r_0 X_0)^2(r_0-1)X_0^2},
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
A_\lambda &= (1-r_0)X_0^2\beta^2+2[-r_0X_0^2+(1+r_0)X_0-r_0]X_0\beta-(1+r_0)X_0^2+4X_0-2, \\
A_3 &= \frac{(1-r_0)X_0^2\beta^2+[(1+r_0)X_0-2r_0]X_0\beta+2(X_0-1)}{(X_0-1)^2}, \\
A_4 &= \frac{(r_0-1)(X_0-3)X_0^2\beta^2+2[(2r_0+1)X_0-3r_0]X_0\beta+6(X_0-1)}{(X_0-1)^3}.
\end{aligned}
\tag{3.9}
$$

One can then calculate partial derivatives of $(\lambda,\, \alpha_3,\, \alpha_4)$ w.r.t. $(h^2,\, X_0,\, r_0)$. By inverting the Jacobian matrix, one obtains

$$
\left(\frac{\partial h^2}{\partial \lambda}\right)_{\alpha_3,\alpha_4} = \frac{1}{3}\left[1+\frac{c_3\omega^2h^2}{c_1\omega h^2+c_2}\right]^{-1}, \tag{3.10}
$$

where

$$\begin{aligned}
c_1 &= (1+\beta X_0)\left\{X_0^5(1-r_0)^3\beta^5+X_0^4(1-r_0)\left[(r_0X_0^2+2(-r_0^2-r_0+1)X_0+r_0(4r_0-3)\right]\beta^4\right.\\
&\quad -X_0^3\left[(2r_0+1)(r_0^2+2r_0-2)X_0^2+2(-3r_0^3-6r_0^2+7r_0-1)X_0+(5r_0^3+4r_0^2-9r_0+3)\right]\beta^3\\
&\quad +X_0^2(1-X_0)[(10r_0^2+5r_0-6)X_0-(7r_0+2)(2r_0-1)]\beta^2\\
&\quad \left.+3X_0(1-X_0)^2(1-4r_0)\beta-3(1-X_0)^2\right\},\\
c_2 &= 2X_0^5(1-r_0)^3(1-X_0)(1+\beta r_0X_0)^2\beta^2,\\
c_3 &= \frac{1}{2}(1-X_0)^2(1+\beta X_0)^6. \qquad (3.11)
\end{aligned}$$

This quantity must be positive in order for the Hubble expansion rate to be an increasing function of the energy density coupling to the physical metric. In other words, the positivity of this quantity is nothing but the positivity of the effective Newton's constant for the background FLRW cosmology. For $\beta=0$, the expression (3.10) reduces to the result known in the original quasidilaton as

$$\left(\frac{\partial h^2}{\partial\lambda}\right)_{\alpha_3,\alpha_4}=\frac{2}{6-\omega},\qquad \text{for } \beta=0. \qquad (3.12)$$

The positivity of this quantity is incompatible with the stability of the de Sitter attractor solution in the original quasidilaton theory, i.e. with $\beta=0$. On the other hand, with $\beta>0$ (see subsection 2.3 for the reason why we do not consider a negative β), we shall see that the positivity of $(\partial h^2/\partial\lambda)_{\alpha_3,\alpha_4}$ can be compatible with the stability of the de Sitter attractor solution. For example, if we take the Minkowski limit ($h\to 0$) while keeping β non-zero then we reach the following universal value, which is positive:

$$\left(\frac{\partial h^2}{\partial\lambda}\right)_{\alpha_3,\alpha_4}\to\frac{1}{3},\qquad (h\to 0 \text{ with } \beta \text{ kept finite and positive}). \qquad (3.13)$$

4 Perturbations

In this section we analyze tensor, vector and scalar perturbations around the de Sitter solution that we described in the previous section.

4.1 Tensor perturbations

For tensor perturbations

$$\delta g_{ij}=a^2h_{ij}^{\mathrm{TT}} \qquad (4.1)$$

with $\delta^{ij}h_{ij}^{\mathrm{TT}}=0$ and $\delta^{ki}\partial_k h_{ij}^{\mathrm{TT}}=0$, we expand the total action (2.11) up to quadratic order in perturbations. After decomposing the perturbations into Fourier modes, we obtain the quadratic Lagrangian as

$$L_T=\frac{M_{\mathrm{Pl}}^2}{8}a^3N\left[\frac{|\dot h_{ij}^{\mathrm{TT}}|^2}{N^2}-\left(\frac{k^2}{a^2}+M_{\mathrm{GW}}^2\right)|h_{ij}^{\mathrm{TT}}|^2\right], \qquad (4.2)$$

where

$$M_{\mathrm{GW}}^2=\left[\frac{(1+\beta X_0)(\mu_3\beta^3+\mu_2\beta^2+\mu_1\beta+\mu_0)}{(X_0-1)^2(r_0-1)(1+\beta r_0X_0)^2}\omega h^2+\frac{X_0^3(r_0-1)}{X_0-1}\right]m_g^2. \qquad (4.3)$$

and

$$
\begin{aligned}
\mu_3 &= -X_0^3(1-r_0)^2, \\
\mu_2 &= 2X_0^2[r_0X_0^2 + (r_0^2 - 2r_0 - 1)X_0 + r_0(3-2r_0)], \\
\mu_1 &= X_0[(r_0+1)^2X_0^2 - 8X_0 + 6 - 2r_0^2], \\
\mu_0 &= 2(X_0-1)(X_0r_0 + r_0 - 2).
\end{aligned} \tag{4.4}
$$

4.2 Vector perturbations

For vector perturbations

$$
\delta g_{0i} = aNB_i^{\mathrm{T}}, \quad \delta g_{ij} = \frac{a^2}{2}(\partial_i E_j^{\mathrm{T}} + \partial_j E_i^{\mathrm{T}}), \tag{4.5}
$$

with $\delta^{ij}\partial_i B_j^{\mathrm{T}} = 0$ and $\delta^{ij}\partial_i E_j^{\mathrm{T}} = 0$, we expand the total action (2.11) up to quadratic order in perturbations. We find that the quadratic action does not depend on time derivatives of B_i^{T}. After decomposing the perturbations into Fourier modes, one can then eliminate B_i^{T} by solving its equation of motion as

$$
B_i^{\mathrm{T}} = \frac{c_V^2\frac{k^2}{a^2}}{c_V^2\frac{k^2}{a^2} + M_{GW}^2}\frac{a\dot{E}_i^{\mathrm{T}}}{2N}, \tag{4.6}
$$

where

$$
c_V^2 = \frac{(r_0+1)^2(r_0-1)(1+\beta r_0X_0)}{2(1+\beta X_0)(1+r_0+2\beta r_0X_0)}\frac{M_{\mathrm{GW}}^2}{\omega h^2 m_g^2}. \tag{4.7}
$$

The reduced quadratic Lagrangian is then

$$
L_V = \frac{M_{\mathrm{Pl}}^2}{16}\int d^4x a^3 N \frac{k^2 M_{\mathrm{GW}}^2}{c_{\mathrm{V}}^2\frac{k^2}{a^2} + M_{\mathrm{GW}}^2}\left[\frac{|\dot{E}_i^{\mathrm{T}}|^2}{N^2} - \left(c_V^2\frac{k^2}{a^2} + M_{\mathrm{GW}}^2\right)|E_i^{\mathrm{T}}|^2\right], \tag{4.8}
$$

4.3 Scalar perturbations

For scalar perturbations

$$
\delta g_{00} = -2N^2\Phi, \quad \delta g_{0i} = aN\partial_i B, \quad \delta g_{ij} = a^2\left[2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_i j\delta^{kl}\partial_k\partial_l\right)E\right], \tag{4.9}
$$

and

$$
\delta\sigma = M_{\mathrm{Pl}}\sigma_1, \tag{4.10}
$$

we expand the total action (2.11) up to quadratic order in perturbations. We find that the quadratic action does not depend on time derivatives of Φ and B. After decomposing the perturbations into Fourier modes, one can then eliminate Φ and B by solving their equations of motion. We then change the variables from (σ_1, E, Ψ) to $(\tilde{\sigma}_1, E, \Psi)$ by

$$
\sigma_1 = \frac{\Psi}{1+\beta r_0 X_0} + \tilde{\sigma}_1, \tag{4.11}
$$

to find that Ψ is also non-dynamical, i.e. the quadratic Lagrangian does not contain time derivatives of Ψ. One can thus eliminate Ψ as well by using its equation of motion. Finally, we obtain the reduced quadratic Lagrangian for the two dynamical variables $(\tilde{\sigma}_1, E)$ of the form

$$
L_S = \frac{M_{\mathrm{Pl}}^2}{2}Na^3\left(\frac{1}{N^2}\dot{y}^T K\dot{y} + \frac{2m_g}{N}\dot{y}^T My - m_g^2 y^T Vy\right), \tag{4.12}
$$

where $K = K^T$, $M = -M^T$ and $V = V^T$ are 2×2 matrices and

$$y = \begin{pmatrix} \tilde{\sigma}_1 \\ \frac{k^2 E}{6(1+\beta r_0 X_0)} \end{pmatrix}. \tag{4.13}$$

Hereafter, we consider subhorizon modes, i.e. modes with $k/a \gg H$. We suppose that $H \sim |m_g|$ up to a factor of order unity, meaning that subhorizon modes satisfy $k/a \gg |m_g|$ as well, and that $\beta > 0$. (See subsection 2.3 for the reason why we do not consider a negative β.) It is convenient to introduce

$$\kappa \equiv \frac{k}{m_g a}, \quad |\kappa| \gg 1 \tag{4.14}$$

as a bookkeeping parameter. With $H \sim |m_g|$ and $\beta > 0$, the matrices K, M and V are expanded as

$$\begin{aligned}
K &= \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = K^{(0)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \kappa^{-2} \begin{pmatrix} K_{11}^{(-2)} & K_{12}^{(-2)} \\ K_{12}^{(-2)} & K_{22}^{(-2)} \end{pmatrix} + O(\kappa^{-4}), \\
M &= \begin{pmatrix} 0 & M_{12} \\ -M_{12} & 0 \end{pmatrix} = M^{(0)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O(\kappa^{-2}), \\
V &= \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} = \kappa^2 V^{(2)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} V_{11}^{(0)} & V_{12}^{(0)} \\ V_{12}^{(0)} & V_{22}^{(0)} \end{pmatrix} + O(\kappa^{-2}).
\end{aligned} \tag{4.15}$$

The leading-order components are

$$\begin{aligned}
K^{(0)} &= \frac{(1+\beta X_0)^3 \omega}{1+\beta r_0 X_0}, \\
M^{(0)} &= -\frac{3}{2} \frac{\beta^2 X_0^2 (1+\beta X_0)^2 (r_0-1)^2 \omega h}{2X_0^2 \beta^2 + (2+3r_0-r_0^2) X_0 \beta + r_0 + 1}, \\
V^{(2)} &= \frac{(1+\beta X_0)^3 (1+r_0+2\beta r_0 X_0) \omega}{2X_0^2 \beta^2 + (2+3r_0-r_0^2) X_0 \beta + r_0 + 1}.
\end{aligned} \tag{4.16}$$

The following combination of the sub-leading components will also be needed for the stability analysis in the next section.

$$\begin{aligned}
K^{(-2)} &\equiv K_{11}^{(-2)} + K_{22}^{(-2)} + 2K_{12}^{(-2)} = \frac{9\beta^2 X_0^2 (1-r_0)(1+\beta X_0)^3 \omega h^2}{2X_0^2 \beta^2 + (2+3r_0-r_0^2) X_0 \beta + r_0 + 1}, \\
V^{(0)} &\equiv V_{11}^{(0)} + V_{22}^{(0)} + 2V_{12}^{(0)} = 0.
\end{aligned} \tag{4.17}$$

5 Subhorizon stability

In order to avoid instabilities whose time scales are parametrically shorter than the cosmological time scale H^{-1}, we require that modes with $k/a \gg H$, i.e. subhorizon modes, be stable. Other types of instabilities, if exist, would be as slow as the standard Jeans instability and thus could be harmless. Throughout this section we assume that $H \sim |m_g|$ and that $\beta > 0$. (See subsection 2.3 for the reason why we do not consider a negative β.)

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5.1 No-ghost condition

For scalar perturbations, we impose that both of the two eigenvalues of the matrix K be positive for subhorizon modes. Since

$$K_{22} = K^{(0)} + O(\kappa^{-2}), \quad \frac{\det K}{K_{22}} = K^{(-2)}\kappa^{-2} + O(\kappa^{-4}), \tag{5.1}$$

where κ is defined in (4.14), the necessary and sufficient condition for the positivity of the two eigenvalues in the subhorizon limit is that

$$K^{(0)} > 0, \quad K^{(-2)}\kappa^{-2} > 0. \tag{5.2}$$

For vector perturbations, we shall see in the next subsection that the absence of gradient instability for subhorizon modes requires that $c_V^2 > 0$. Under this condition, the coefficient of the kinetic term is positive for subhorizon modes if and only if

$$M_{\rm GW}^2 > 0. \tag{5.3}$$

For tensor modes, the coefficient of kinetic term is constant and always positive.

5.2 Positivity of sound speed squared

For scalar modes, the kinetic matrix K is diagonalized by the change of variables from y to $\tilde{y}$ through

$$y = \begin{pmatrix} 1 & 0 \\ -\frac{K_{12}}{K_{22}} & 1 \end{pmatrix} \tilde{y}. \tag{5.4}$$

By employing the ansatz[1]

$$\tilde{y} \propto \exp\left(i|m_g| \int \Omega N dt \right), \tag{5.5}$$

and neglecting the time dependence of Ω, a and N (we are interested in modes with $k/a \gg H$), the equations of motion is reduced to the dispersion relation

$$(\det K)\Omega^4 - [K_{11}V_{22} + K_{22}V_{11} - 2K_{12}V_{12} + 4(M_{12})^2]\Omega^2 + \det V = 0. \tag{5.6}$$

It is easy to estimate the order of each coefficient as

$$\begin{aligned} \det K &= \kappa^{-2}K^{(0)}K^{(-2)} + O(\kappa^{-4}) = O(\kappa^{-2}), \\ K_{11}V_{22}+K_{22}V_{11}-2K_{12}V_{12}+4(M_{12})^2 &= [K^{(0)}V^{(0)}+K^{(-2)}V^{(2)}+4(M^{(0)})^2]+O(\kappa^{-2}) = O(\kappa^0), \\ \det V &= \kappa^2 V^{(2)}V^{(0)} + O(\kappa^0) = O(\kappa^0), \end{aligned} \tag{5.7}$$

where we have used $V^{(0)} = 0$ to show the last equality. Thus there is a pair of positive and negative frequency modes with $\Omega^2 = O(\kappa^0)$, corresponding to a vanishing sound speed. The other pair of modes corresponds to

$$\Omega^2 = \frac{K^{(0)}V^{(0)} + K^{(-2)}V^{(2)} + 4(M^{(0)})^2}{K^{(0)}K^{(-2)}}\kappa^2 + O(\kappa^0). \tag{5.8}$$

[1] This ansatz is appropriate for $m_g^2 > 0$. For $m_g^2 < 0$, one can simply replace κ^2 and Ω^2 by $-\kappa^2$ and $-\Omega^2$, respectively, and then all results below hold. In particular, (5.9), (5.10) and (5.12) are unchanged by this replacement.

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Hence, we obtain the sound speed squared for this pair as

$$c_s^2 = \lim_{\kappa\to\infty} \kappa^{-2}\Omega^2 = \frac{K^{(0)}V^{(0)} + K^{(-2)}V^{(2)} + 4(M^{(0)})^2}{K^{(0)}K^{(-2)}} = \left(\frac{1+\beta r_0 X_0}{1+\beta X_0}\right)^2. \tag{5.9}$$

This is always positive and thus there is no classical instability for subhorizon modes. This value of c_s^2 corresponds to the speed limit set by the light cone of the background effective metric $g^{\rm eff}_{\mu\nu}$ (see (3.2)).

For vector perturbations, from the action (4.8) one can easily read off the dispersion relation for modes with $k/a \gg H$ as

$$\Omega^2 = c_V^2\kappa^2 + O(\kappa^0). \tag{5.10}$$

Hence the absence of classical instability for subhorizon modes requires that

$$c_V^2 > 0. \tag{5.11}$$

As is clear from the quadratic action (4.2), the subhorizon dispersion relation for tensor perturbations is

$$\Omega^2 = \kappa^2 + O(\kappa^0). \tag{5.12}$$

Thus tensor subhorizon modes are always classically stable.

5.3 Subhorizon stability and self-acceleration

In summary, supposing that $H \sim |m_g|$ and that $\beta > 0$, all subhorizon modes are stable if and only if

$$K^{(0)} > 0, \quad K^{(-2)}\kappa^{-2} > 0, \quad c_V^2 > 0, \quad M_{\rm GW}^2 > 0. \tag{5.13}$$

In addition to these conditions, we require that the effective Newton's constant for the FLRW background be positive, i.e.

$$\left(\frac{\partial h^2}{\partial\lambda}\right)_{\alpha_3,\alpha_4} > 0, \tag{5.14}$$

where the left hand side was calculated in section 3 and the result is shown in (3.10).

Hereafter, we assume that $m_g^2 > 0$. Among the five conditions shown in (5.13) and (5.14), the first three can be restated as

$$\omega > 0, \quad r_0 > 2 + 2\sqrt{2}, \quad x_- < X_0\beta < x_+, \tag{5.15}$$

where

$$x_\pm = \frac{1}{4}\left[r_0^2 - 3r_0 - 2 \pm (r_0 - 1)\sqrt{r_0^2 - 4r_0 - 4}\right]. \tag{5.16}$$

The remaining two conditions are complicated but can be satisfied simultaneously in a range of parameters. (See explicit self-accelerating examples below.)

Under the condition $r_0 > 2 + 2\sqrt{2}$, it is easy to show that $x_- > 0$, meaning that $\beta = 0$ is excluded. This is consistent with the result of [24, 25]: in the original quasidilaton theory ($\beta = 0$) subhorizon modes always suffer from ghost instability if the effective Newton's constant for the FLRW background evolution is positive. On the other hand, if β is non-zero and is between x_-/X_0 and x_+/X_0 then subhorizon modes are stable in a range of parameters.

The subhorizon behavior of the new quasidilaton theory considered in this paper is quite different from that of the original quasidilaton theory. This is because in the subhorizon limit, various quantities such as $\det K$ are dominated by terms that are absent for $\beta = 0$, where β is the new parameter that measures the strength of the coupling of the kinetic energy of the quasidilaton scalar to the fiducial metric. Hence, the $\beta \to 0$ limit and the subhorizon limit do not commute. In other words, the subhorizon limit of the new quasidilaton theory with $\beta > 0$ is quite different from that of the original theory. (See subsection 2.3 for the reason why we do not consider a negative β.)

So far, we kept the cosmological constant Λ (or its dimensionless version $\lambda = \Lambda/m_g^2$) as a placeholder for ordinary matter in order to calculate the response of the Hubble expansion rate to the energy density coupling to the physical metric $g_{\mu\nu}$. On the other hand, since one of the modern motivations for massive gravity is to explain the origin of the current acceleration of the universe, it is favorable if the graviton mass term (as well as the quasidilaton kinetic action) can hold the de Sitter expansion without the genuine cosmological constant. For this reason we set $\lambda = 0$ from now on.

By setting $\lambda = 0$ in (3.8), one obtains

$$\omega = \frac{2(1+\beta r_0 X_0)^2 (r_0 - 1) X_0^2}{-(1+\beta X_0)^2 A_\lambda} \left[3 + \frac{(1-X_0)^2}{h^2} \right]. \tag{5.17}$$

We thus consider the subspace of the parameter space defined by this relation. This subspace is 4-dimensional and can be spanned by $(\beta,\ h,\ r_0,\ X_0)$. In this subspace there are many examples that satisfy the all five conditions shown in (5.13) and (5.14). For example,

$$\beta = 1, \quad h = 1, \quad r_0 = 5, \quad X_0 = 2, \tag{5.18}$$

and

$$\beta = \frac{1}{200}, \quad h = 1, \quad r_0 = 200, \quad X_0 = 2, \tag{5.19}$$

satisfy all five conditions shown in (5.13) and (5.14). The corresponding parameters in the action are, respectively,

$$\Lambda = 0, \quad \beta = 1, \quad \omega = \frac{7744}{387}, \quad \alpha_3 = \frac{66}{43}, \quad \alpha_4 = \frac{165}{43}, \tag{5.20}$$

and

$$\Lambda = 0, \quad \beta = \frac{1}{200}, \quad \omega = \frac{1910400000000}{27542016533}, \quad \alpha_3 = \frac{5426534}{2699933}, \quad \alpha_4 = \frac{24700201}{8099799}. \tag{5.21}$$

All five conditions are satisfied in neighborhoods of these points, at least.

6 Summary and discussions

We have presented a new quasidilaton theory of Poincare invariant massive gravity, based on the recently proposed framework of matter coupling that makes it possible for the kinetic energy of the quasidilaton scalar to couple both physical and fiducial metrics. We have found a scaling-type exact solution that expresses a self-accelerating de Sitter universe, and then analyzed linear perturbations around it. We have shown that in a range of parameters all physical degrees of freedom have non-vanishing quadratic kinetic terms and are stable in the subhorizon limit, while the effective Newton's constant for the background is kept positive.

The proposal of the present paper relies on a simple extension of the new matter coupling in massive gravity that was recently introduced in [22]. Based on the analysis in the

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decoupling limit, it was argued in [22, 31] that the BD ghost is absent up to $\Lambda_3 = (M_{\rm Pl}m_g^2)^{1/3}$ but it may show up at some higher scale. The mass of the BD ghost is expected to be around $m_{\rm ghost} \sim m_g^3 M_{\rm Pl}^2/(\sqrt{\beta}\dot{\chi}\partial_i\chi)$, where χ is a canonical scalar field that couples to the effective metric [22, 32]. The mass of the BD ghost is higher for smaller β. (This is consistent with the fact that there is no BD ghost up to arbitrarily high scale at classical level if $\beta = 0$.) Simply replacing χ with the quasidilaton σ and noticing that $\dot{\sigma} \sim M_{\rm Pl}m_g$ on the self-accelerating background (we still assume that $H \sim m_g$), we obtain $m_{\rm ghost} \sim (\Lambda_3/\sqrt{\beta}) \times (\Lambda_3^2/\partial_i\sigma)$. This means that for $\partial_i\sigma$ below Λ_3^2, the lowest possible mass of the BD ghost would be $\sim \Lambda_3/\sqrt{\beta}$. This can be above Λ_3 if β is small enough, and the BD ghost can be integrated out. (On the other hand, $\dot{\sigma}$ is above Λ_3^2 and thus the self-accelerating solution cannot be described by the standard Λ_3-decoupling limit.)

For β of order unity or higher, it is expected that the BD ghost reappears in some ways. In the present paper we have explictly shown that the would-be BD degree of freedom (Ψ in subsection 4.3) has a vanishing time kinetic term and thus is non-dynamical at the level of the quadratic action for any values of β and k/a. This may be due to high symmetry of the FLRW background or for other subtle reasons. It is worth while investigating this issue in more details. For example, as in [14, 33], one may consider linear perturbations around a Bianchi-I background with axisymmetry as a consistent truncation of nonlinear perturbations around the self-accelerating de Sitter solution in the flat FLRW chart. The sixth degree of freedom may or may not show up in the linear perturbations around the Bianchi-I background. If it does then an important question is how heavy the mass gap is. If and only if it is heavy enough then one can safely integrate it out. While we admit that this is a rather important issue, we consider it as outside the scope of the present paper and leave it for a future work.

It is also worthwhile investigating more general quasidilaton theories by combining the proposal in the present paper with extra terms considered in [27, 29].

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References

[1] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc. Roy. Soc. Lond.* **A 173** (1939) 211 [INSPIRE].

[2] D.G. Boulware and S. Deser, *Can gravitation have a finite range?*, *Phys. Rev.* **D 6** (1972) 3368 [INSPIRE].

[3] C. de Rham and G. Gabadadze, *Generalization of the Fierz-Pauli Action*, *Phys. Rev.* **D 82** (2010) 044020 [arXiv:1007.0443] [INSPIRE].

[4] C. de Rham, G. Gabadadze and A.J. Tolley, *Resummation of Massive Gravity*, *Phys. Rev. Lett.* **106** (2011) 231101 [arXiv:1011.1232] [INSPIRE].

[5] S.F. Hassan and R.A. Rosen, *Resolving the Ghost Problem in non-Linear Massive Gravity*, *Phys. Rev. Lett.* **108** (2012) 041101 [arXiv:1106.3344] [INSPIRE].

[6] S.F. Hassan and R.A. Rosen, *Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity*, *JHEP* **04** (2012) 123 [arXiv:1111.2070] [INSPIRE].

[7] G. D'Amico et al., *Massive Cosmologies*, *Phys. Rev.* **D 84** (2011) 124046 [arXiv:1108.5231] [INSPIRE].

[8] A.E. Gumrukcuoglu, C. Lin and S. Mukohyama, *Open FRW universes and self-acceleration from nonlinear massive gravity*, *JCAP* **11** (2011) 030 [arXiv:1109.3845] [INSPIRE].

[9] S.F. Hassan, R.A. Rosen and A. Schmidt-May, *Ghost-free Massive Gravity with a General Reference Metric*, *JHEP* **02** (2012) 026 [arXiv:1109.3230] [INSPIRE].

[10] A.E. Gumrukcuoglu, C. Lin and S. Mukohyama, *Cosmological perturbations of self-accelerating universe in nonlinear massive gravity*, *JCAP* **03** (2012) 006 [arXiv:1111.4107] [INSPIRE].

[11] S.F. Hassan and R.A. Rosen, *On Non-Linear Actions for Massive Gravity*, *JHEP* **07** (2011) 009 [arXiv:1103.6055] [INSPIRE].

[12] M. Fasiello and A.J. Tolley, *Cosmological perturbations in Massive Gravity and the Higuchi bound*, *JCAP* **11** (2012) 035 [arXiv:1206.3852] [INSPIRE].

[13] D. Langlois and A. Naruko, *Cosmological solutions of massive gravity on de Sitter*, *Class. Quant. Grav.* **29** (2012) 202001 [arXiv:1206.6810] [INSPIRE].

[14] A. De Felice, A.E. Gumrukcuoglu and S. Mukohyama, *Massive gravity: nonlinear instability of the homogeneous and isotropic universe*, *Phys. Rev. Lett.* **109** (2012) 171101 [arXiv:1206.2080] [INSPIRE].

[15] A.E. Gumrukcuoglu, C. Lin and S. Mukohyama, *Anisotropic Friedmann-Robertson-Walker universe from nonlinear massive gravity*, *Phys. Lett.* **B 717** (2012) 295 [arXiv:1206.2723] [INSPIRE].

[16] A. De Felice, A.E. Gumrukcuoglu, C. Lin and S. Mukohyama, *Nonlinear stability of cosmological solutions in massive gravity*, *JCAP* **05** (2013) 035 [arXiv:1303.4154] [INSPIRE].

[17] G. D'Amico, G. Gabadadze, L. Hui and D. Pirtskhalava, *Quasidilaton: Theory and cosmology*, *Phys. Rev.* **D 87** (2013) 064037 [arXiv:1206.4253] [INSPIRE].

[18] Q.-G. Huang, Y.-S. Piao and S.-Y. Zhou, *Mass-Varying Massive Gravity*, *Phys. Rev.* **D 86** (2012) 124014 [arXiv:1206.5678] [INSPIRE].

[19] D. Comelli, F. Nesti and L. Pilo, *Massive gravity: a General Analysis*, *JHEP* **07** (2013) 161 [arXiv:1305.0236] [INSPIRE].

[20] D. Langlois, S. Mukohyama, R. Namba and A. Naruko, *Cosmology in rotation-invariant massive gravity with non-trivial fiducial metric*, *Class. Quant. Grav.* **31** (2014) 175003 [arXiv:1405.0358] [INSPIRE].

[21] C. de Rham, M. Fasiello and A.J. Tolley, *Stable FLRW solutions in Generalized Massive Gravity*, arXiv:1410.0960 [INSPIRE].

[22] C. de Rham, L. Heisenberg and R.H. Ribeiro, *On couplings to matter in massive (bi-)gravity*, arXiv:1408.1678 [INSPIRE].

[23] A.E. Gumrukcuoglu, L. Heisenberg and S. Mukohyama, *Cosmological perturbations in massive gravity with doubly coupled matter*, arXiv:1409.7260 [INSPIRE].

[24] A.E. Gumrukcuoglu, K. Hinterbichler, C. Lin, S. Mukohyama and M. Trodden, *Cosmological Perturbations in Extended Massive Gravity*, *Phys. Rev.* **D 88** (2013) 024023 [arXiv:1304.0449] [INSPIRE].

[25] G. D'Amico, G. Gabadadze, L. Hui and D. Pirtskhalava, *On Cosmological Perturbations of Quasidilaton*, *Class. Quant. Grav.* **30** (2013) 184005 [arXiv:1304.0723] [INSPIRE].

[26] G. Gabadadze, R. Kimura and D. Pirtskhalava, *Self-acceleration with Quasidilaton*, *Phys. Rev.* **D 90** (2014) 024029 [arXiv:1401.5403] [INSPIRE].

[27] A. De Felice and S. Mukohyama, *Towards consistent extension of quasidilaton massive gravity*, *Phys. Lett.* **B 728** (2014) 622 [arXiv:1306.5502] [INSPIRE].

[28] H. Motohashi and W. Hu, *Stability of cosmological solutions in extended quasidilaton massive gravity*, *Phys. Rev.* **D 90** (2014) 104008 [arXiv:1408.4813] [INSPIRE].

[29] A. De Felice, A. Emir Gumrukcuoglu and S. Mukohyama, *Generalized quasidilaton theory*, *Phys. Rev.* **D 88** (2013) 124006 [arXiv:1309.3162] [INSPIRE].

[30] J. Noller and S. Melville, *The coupling to matter in Massive, Bi- and Multi-Gravity*, arXiv:1408.5131 [INSPIRE].

[31] C. de Rham, L. Heisenberg and R.H. Ribeiro, *Ghosts & Matter Couplings in Massive (bi-& multi-)Gravity*, arXiv:1409.3834 [INSPIRE].

[32] C. de Rham and L. Heisenberg, private communication.

[33] Y. Yamashita, A. De Felice and T. Tanaka, *Appearance of Boulware-Deser ghost in bigravity with doubly coupled matter*, arXiv:1408.0487 [INSPIRE].

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