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TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE
REPRESENTATIONS

TOMOYUKI ARAKAWA

Abstract. We prove the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence of two-sided BGG resolutions of $G$-integrable admissible representations of affine Kac-Moody algebras at fractional levels. As an application we establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras which enables an inductive study of admissible representations.

1. Introduction

Wakimoto modules are representations of non-twisted affine Kac-Moody algebras introduced by Wakimoto [Wak] in the case of $\mathfrak{sl}_2$ and by Feigin and Frenkel [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In these applications it is important to have a resolution of an irreducible highest weight representation $L(\lambda)$ of an affine Kac-Moody algebra $\mathfrak{g}$ in terms of Wakimoto modules, that is, a complex

$$C^\bullet(\lambda) : \rightarrow C^{i-1}(\lambda) \xrightarrow{d_{i-1}} C^i(\lambda) \xrightarrow{d_i} C^{i+1}(\lambda) \rightarrow \ldots$$

with a differential $d_i$ which is a $\mathfrak{g}$-module homomorphism such that $C^i(\lambda)$ is a direct sum of Wakimoto modules and

$$H^i(C^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a resolution has been proved by Feigin and Frenkel [FF2] for any integrable representations over arbitrary $\mathfrak{g}$ and by Bernard and Felder [BF] and Feigin and Frenkel [FF3] for any admissible representation [FKM] over $\mathfrak{sl}_2$. In their study of $\mathcal{W}$-algebras Frenkel, Kac and Wakimoto [FKW, Conjecture 3.5.1] conjectured the existence of such a resolution for any principle admissible representations over arbitrary $\mathfrak{g}$. In this paper we prove the existence of a two-sided resolution in terms of Wakimoto modules for any $\mathfrak{g}$-integrable admissible representations over arbitrary $\mathfrak{g}$ (Theorem 6.11), where $\hat{\mathfrak{g}}$ is the classical part of $\mathfrak{g}$. For a general principal admissible representation of $\mathfrak{g}$ we obtain the two-sided resolution in terms of twisted Wakimoto modules (Theorem 6.15).

Let us sketch the proof of our result briefly. By Fiebig’s equivalence [Fie], the block of the category $\mathcal{O}$ of $\mathfrak{g}$ containing an admissible representation $L(\lambda)$ is equivalent to the block containing an integrable representation. Therefore an admissible representation $L(\lambda)$ is a block of another Kac-Moody algebra.

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1In the case $L(\lambda)$ is a non-principal $G$-integrable admissible representation this is a block of another Kac-Moody algebra.
representation admits a usual BGG type resolution in terms of Verma modules by the result of \cite{Ark1}. Hence the idea of Arkhipov \cite{Ark1} is applicable in our situation: One can obtain a twisted BGG resolution of $L(\lambda)$ in terms of twisted Verma modules by applying the twisting functor $T_w$ \cite{Ark1} to the BGG resolution of $L(\lambda)$ as we have the “Borel-Weil-Bott” vanishing property \cite{AS}:

$$L_iT_wL(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = \ell(w), \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the integral Weyl group of $\lambda$ and $\ell : \mathcal{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ is the length function, see Theorem \cite{AS}. It remains to show that one can construct an inductive system of twisted BGG resolutions $\{B_w^\bullet(\lambda)\}$ of $L(\lambda)$ such that the complex $\lim \overset{\leftarrow}{B_w^\bullet(\lambda)}$ gives the required two-sided resolution of $L(\lambda)$, see \cite{A6} for the details.

We note that by applying the (generalized) quantum Drinfeld-Sokolov reduction functor \cite{FKW,KRW} to the (duals of the) two-sided BGG resolutions of admissible representations we obtain resolutions of some of simple modules over $W$-algebras in terms of free field realizations due to the vanishing of the associated BRST cohomology \cite{A1,A2,A3,A4,A5}. In particular we obtain two-sided resolutions of all the minimal series representations \cite{FKW,A7} of the $W$-algebras associated with principal nilpotent elements in terms of free bosonic realizations.

As an application of the existence of two-sided BGG resolution for admissible representations we prove a semi-infinite analogue of the generalized Borel-Weil theorem \cite{Kos} for minimal parabolic subalgebras (Theorem \cite{A8}). This result is important since it enable an inductive study of admissible representations, see our subsequent paper \cite{A6}.

This paper is organized as follows. In \S2 we collect and prove some basic results about semi-infinite cohomology \cite{Fe} and semi-regular bimodules \cite{Vor1} which are needed for later use. In particular we establish an important property of semi-regular bimodules in Proposition \cite{Vor1}. In \S3 we collect basic results on the semi-infinite Bruhat ordering (or the generic Bruhat ordering) of an affine Weyl group defined by Lusztig \cite{Lus} and study the semi-infinite analogue of parabolic subgroups. Semi-infinite Bruhat ordering is important for us since it (conjecturally) describes the space of homomorphisms between Wakimoto modules, see Proposition \cite{Vor2} and Conjecture \cite{Vor2}. The semi-infinite analogue of the minimal (or maximal) length representatives (Theorem \cite{Vor2}) is important for describing the semi-infinite restriction functors studied in \S6. In \S4 we define Wakimoto modules and twisted Verma modules following \cite{Vor2} and study some of their basic properties. In particular we prove the uniqueness of Wakimoto modules which was stated in \cite{FF2} without a proof (Theorem \cite{Vor2}). In \S5 we generalize the Borel-Weil-Bott vanishing property of the twisting functor established in \cite{AS} to the affine Kac-Moody algebra cases. In \S6 we state and prove the main results of this paper. In \S7 we study the semi-infinite restriction functor and establish the semi-infinite analogue of the generalized Borel-Weil theorem \cite{Kos} for minimal parabolic subalgebras. This is a non-trivial fact since admissible representations are not unitarizable unless they are integrable.

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2. Semi-regular bimodules and semi-infinite cohomology

2.1. Some notation. For $\mathbb{Z}$-graded vector spaces $M = \bigoplus_{n \in \mathbb{Z}} M_n, N = \bigoplus_{n \in \mathbb{Z}} N_n$ with finite-dimensional homogeneous components let

$$\text{Hom}_C(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_C(M, N)_n,$$

$$\text{Hom}_C(M, N)_n = \{ f \in \text{Hom}_C(M, N) ; f(M_i) \subset N_{i+n} \},$$

$$\text{End}_C(M) = \text{Hom}_C(M, M).$$

We denote by $M = \bigoplus_{n \in \mathbb{Z}} (M)_n$ the space $\text{Hom}_C(M, C)$, where $C$ is considered as a graded vector space concentrated in the degree 0 component. If $M, N$ are module over an algebra $A$ we denote by $\text{Hom}_A(M, N)$ the space of all $A$-homomorphisms in $\text{Hom}_C(M, N)$.

2.2. Semi-infinite structure. Let $\mathfrak{g}$ be a complex Lie algebra. A semi-infinite structure of $\mathfrak{g}$ is the following data:

(i) a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ of $\mathfrak{g}$ with finite-dimensional homogeneous components, $\dim_C \mathfrak{g}_n < 1$ for all $n$,

(ii) a semi-infinite 1-cochain $\gamma : \mathfrak{g} \to \mathbb{C}$.

Here by a semi-infinite 1-cochain we mean the following: Decompose $\mathfrak{g}$ into the direct sum of two subalgebras

1. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$,

2. $\mathfrak{g}_+ = \bigoplus_{i \geq 0} \mathfrak{g}_i$, $\mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i$.

A linear map $\gamma : \mathfrak{g} \to \mathbb{C}$ is called a semi-infinite 1-cochain if $\gamma$ satisfies

$$\gamma([x, y]) = \text{tr}((\text{ad} x)_+ - (\text{ad} y)_- - (\text{ad} y)_+ (\text{ad} x)_-)$$

for $x, y \in \mathfrak{g}$, where $(\text{ad} x)_\pm \mathfrak{g}$ denotes the composition $\mathfrak{g} \xrightarrow{\text{ad} x} \mathfrak{g} \xrightarrow{\text{projection}} \mathfrak{g}_\pm$.

In the rest of this section we assume that $\mathfrak{g}$ is equipped with a semi-infinite structure such that $\gamma(\sum_{i \neq 0} \mathfrak{g}_i) = 0$.

We denote by $U$, $U_-$, $U_+$, the enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}_+$, $\mathfrak{g}_-$ by respectively. These algebras inherit a $\mathbb{Z}$-grading from the corresponding Lie algebras.

Let $\mathcal{O}_\mathfrak{g}$ be the category of $\mathbb{Z}$-graded $\mathfrak{g}$-modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ with $\dim M_n < \infty$ for all $m$ on which $\bigoplus_{j > 0} \mathfrak{g}_+$ acts locally nilpotently and $\mathfrak{g}_0$ acts locally finitely.

2.3. Semi-infinite cohomology. Choose a basis $\{x_i ; i \in \mathbb{Z}\}$ of $\mathfrak{g}$ such that $\{x_i ; i \geq 0\}$ and $\{x_i ; i < 0\}$ are bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively, and let $\{c^k_{ij}\}$ be the structure constant: $[x_i, x_j] = \sum_k c^k_{ij} x_k$. 

Denote by $\mathfrak{cl}(\mathfrak{g})$ the Clifford algebra associated with $\mathfrak{g} \oplus \mathfrak{g}^*$, which has the following generators and relations:

- generators: $\psi_i, \psi_i^*$ for $i \in \mathbb{Z}$,
- relations: $[\psi_i, \psi_j^*] = \delta_{i,j}$, $[\psi_i, \psi_j] = [\psi_i^*, \psi_j^*] = 0$.

Here $\{X,Y\} = XY + YX$. The space of the semi-infinite forms $\wedge^{\infty + *}(\mathfrak{g})$ of $\mathfrak{g}$ is by definition the irreducible representation of $\mathfrak{cl}(\mathfrak{g})$ generated by the vector $1$ satisfying

$$\psi_i 1 = 0 \quad \text{for } i \geq 0, \quad \psi_i^* 1 = 0 \quad \text{for } i > 0.$$  

It is graded by $\deg 1 = 0$, $\deg \psi_i^* = 1$, and $\deg \psi_i = -1$: $\wedge^{\infty + *}(\mathfrak{g}) = \bigoplus_{p \in \mathbb{Z}} \wedge^{\infty + p}(\mathfrak{g})$.

For $A \in End_{\mathbb{C}}(\wedge^{\infty + *}(\mathfrak{g}))$ of degree $n$ set

$$(3) \quad : \psi_k A := \begin{cases} \psi_k A & \text{if } k < 0, \\ (-1)^n A \psi_k & \text{if } k \geq 0, \end{cases} \quad : \psi_k^* A := \begin{cases} \psi_k^* A & \text{if } k \leq 0, \\ (-1)^n A \psi_k^* & \text{if } k > 0. \end{cases}$$

The following defines a $\mathfrak{g}$-module structure on $\wedge^{\infty + *}(\mathfrak{g})$:

$$(4) \quad x_i \mapsto \text{ad}(x_i) := + \gamma(x_i),$$

where

$$\text{ad}.x_i := \sum_{j,k} c_{ij}^k : \psi_k \psi_i^* : .$$

For $M \in \mathcal{O}^\mathfrak{g}$, define $d \in End(M \otimes \wedge^{\infty + *}(\mathfrak{g}))$ by

$$d = \sum_{i \in \mathbb{Z}} x_i \otimes \psi_i^* - \frac{1}{2} \sum_{i,j,k \in \mathbb{Z}} c_{ij}^k : \psi_i^* (\psi_j^* \psi_k) : + 1 \otimes \sum_{i \in \mathbb{Z}} \gamma(x_i) \psi_i^* .$$

Then

$$d^2 = 0, \quad d(M \otimes \wedge^{\infty + p}(\mathfrak{g})) \subset M \otimes \wedge^{\infty + p+1}(\mathfrak{g}).$$

The cohomology of the complex $(M \otimes \wedge^{\infty + *}(\mathfrak{g}), d)$ is called the semi-infinite $\mathfrak{g}$-cohomology with coefficients in $M$ and denoted by $H^{\infty + *}(\mathfrak{g}, M)$ (\cite{2}, \cite{3}).

2.4. Semi-regular bimodules. We consider the full dual space $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ of $U$ as a $U$-bimodule by $(Xf)(u) = f(Xu), \ (fX)(u) = f(Xu)$ for $X \in \mathfrak{g}$, $f \in \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, $u \in U$. The graded duals $U^*_+$ of $U_+$ are $\mathfrak{g}_\pm$-submodule of $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$.

By abuse of notation we denote by $U^*$ the image of the embedding $U^*_+ \otimes_{\mathbb{C}} U^*_- \hookrightarrow \text{Hom}_{\mathbb{C}}(U, \mathbb{C}), \ f_+ \otimes f_- \mapsto (u_- u_+ \mapsto f_+ (u_+) f_-(u_-)).$  

Then $U^*$ is a $U$-bimodule of $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ and coincides with the image of the embedding $U^*_+ \otimes_{\mathbb{C}} U^*_+ \hookrightarrow \text{Hom}_{\mathbb{C}}(U, \mathbb{C}), \ f_- \otimes f_+ \mapsto (u_+ u_- \mapsto f_-(u_+) f_+(u_-)).$  

Following \cite{5},\cite{6}, define

$$US(\mathfrak{g}) = H^{\infty + 0}(\mathfrak{g}, U^* \otimes_{\mathbb{C}} U),$$

where $\mathfrak{g}$ is given the opposite semi-infinite structure and the semi-infinite $\mathfrak{g}$-cohomology is taken with respect to the diagonal left action on $U^* \otimes_{\mathbb{C}} U$. Here by the opposite semi-infinite structure we mean the one obtained by replacing $\mathfrak{g}_+$ with $\mathfrak{g}_-$ and $\gamma$.
with $-\gamma$. The space $US(g)$ inherits the $U$-bimodule structure from $U^* \otimes U$ defined by

$$X(f \otimes u) = -(fX) \otimes u, \quad (f \otimes u)X = f \otimes (uX)$$

for $X \in g, u \in U$. The $U$-bimodule $US(g)$ is called the semi-regular bimodule of $g$. One has

$$US(g) \cong U^*_+ \otimes_{U_+} U \quad (5)$$

as left $g_+$-modules and right $g$-modules, and the left $g$-module structure of $US(g)$ is defined through the isomorphism

$$U_+ \otimes_U U \cong \text{Hom}_C(U_+, U) \cong \text{Hom}_{U_-}(U, U_- \otimes C_{-\gamma}) \quad (6)$$

Let $M$ be a $g$-module and consider the following four left $g$-module structures on $US(g) \otimes_C M$:

$$\pi_1(X)(s \otimes m) = -(sX) \otimes m + s \otimes Xm, \quad \pi_2(X)(s \otimes m) = (Xs) \otimes m, \quad \pi'_1(X)(s \otimes m) = -(sX) \otimes m, \quad \pi'_2(X)(s \otimes m) = (Xs) \otimes m + s \otimes Xm,$$

for $X \in g, s \in US(g), m \in M$. Clearly, the two actions $\pi_1$ and $\pi_2$ (resp. $\pi'_1$ and $\pi'_2$) commute.

**Proposition 2.1** (cf. [72, 6.4]). For $M \in \mathcal{O}_g$ the two $U$-bimodule structures $(\pi_1, \pi_2)$ and $(\pi'_1, \pi'_2)$ on $US(g) \otimes_C M$ are equivalent. Namely there exists a linear isomorphism $\Phi : US(g) \otimes_C M \cong US(g) \otimes_C M$ such that $\Phi \circ \pi'_i(X) = \pi_i(X) \circ \Phi$ for $i = 1, 2, X \in g$.

**Proof.** Define the linear isomorphism

$$\Phi_1 : U^* \otimes_C U \otimes_C M \cong U^* \otimes_C U \otimes_C M$$

by $\Phi_1(f \otimes u \otimes m) = f \otimes (\Delta(u)(1 \otimes m))$ for $f \in U^*, u \in U, m \in M$, where $\Delta : U \to U \otimes U$ is the coproduct. We have

$$\Phi_1 \circ \pi_{2,L}(X) = (\pi_{2,L}(X) + \pi_{3,L}(X)) \circ \Phi_1$$

$$\Phi_1 \circ (\pi_{2,R}(X) + \pi_{3,R}(X)) = \pi_{2,R}(X) \circ \Phi_1,$$

where $\pi_{1,L}$ (resp. $\pi_{1,R}$) denotes the left action (resp. the right action) of $g$ on the $i$-th factor of $U^* \otimes U \otimes M$, and $M$ is considered as a right $g$-module by the action $mx = -xm$ for $m \in M, x \in g$.

Next consider the graded dual $M^* = \bigoplus_n (M^*)_n$ as a right module by the action $(fX)(m) = f(Xm)$. Let

$$\Psi : U^* \otimes_C M \cong U^* \otimes_C M$$

be the linear isomorphism defined by $\Psi(f \otimes m)(u \otimes g) = (f \otimes m)((1 \otimes g)\Delta(u))$ for $f \in U^*, m \in M, u \in U, g \in M^*$, where $M$ is identified with $(M^*)^*$. Extend this to the linear isomorphism

$$\Phi_2 : U^* \otimes_C U \otimes_C M \cong U^* \otimes_C U \otimes_C M$$
by setting $\Phi_2(f \otimes u \otimes m) = \sum_i f_i \otimes u \otimes m_i$ if $\Psi(f \otimes m) = \sum_i f_i \otimes m_i$ with $f_i \in U^*$, $m_i \in M$. Then

$$\Phi_2 \circ \pi_1(R) = (\pi_1(R) + \pi_3(R)) \circ \Phi_2,$$

$$\Phi_2 \circ (\pi_1(L) + \pi_3(L)) = \pi_1(L) \circ \Phi_2.$$

Set

$$\Phi = \Phi_2 \circ \Phi_1 : U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M \cong U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M.$$

Then

$$\Phi \circ (\pi_1(L) + \pi_2(L)) = \Phi_2 \circ (\pi_1(L) + \pi_2(L) + \pi_3(L)) \circ \Phi_1$$

$$= (\pi_1(L) + \pi_2(L)) \circ \Phi,$$

$$\Phi \circ (\pi_2(R) + \pi_3(R)) = \Phi_2 \circ \pi_2(R) \circ \Phi_1 = \pi_2(R) \circ \Phi,$$

$$\Phi \circ \pi_1(R) = \Phi_2 \circ \pi_1(R) \circ \Phi_1 = (\pi_1(R) + \pi_3(R)) \circ \Phi.$$

By (9) and the definition of $US(\mathfrak{g})$, $\Phi$ gives rise to a linear isomorphism

$$\Phi : US(\mathfrak{g}) \otimes_{\mathbb{C}} M \cong US(\mathfrak{g}) \otimes_{\mathbb{C}} M.$$

Moreover $\Phi$ satisfies the required properties by (10) and (11). \hfill \Box

2.5. Semi-infinite induction. Let $\mathfrak{h} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}_n$ be a graded Lie subalgebra of $\mathfrak{g}$ such that $\gamma|_{\mathfrak{h}}$ is a semi-infinite 1-cochain of $\mathfrak{h}$. Following [72, 77, 78] we define the semi-induced $\mathfrak{g}$-module $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M$ as

$$S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M := H^{\mathfrak{g} + \mathfrak{h}} \otimes_{\mathfrak{h}} US(\mathfrak{g}) \otimes_{\mathbb{C}} M,$$

where $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ is considered as an $\mathfrak{h}$-module by the action $\pi_1$ defined in (10). The space $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M$ inherits the structure of a $\mathfrak{g}$-module from the action $\pi_2$ defined in (11).

**Lemma 2.2.** The assignment $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M \mapsto S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M$ defines an exact functor from $\mathcal{O}^\mathfrak{h}$ to $\mathcal{O}^\mathfrak{g}$.

**Proof.** Clearly $\text{S-ind} M$ is an object of $\mathcal{O}^\mathfrak{h}$ since $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ is. By Proposition [72, 77, 78] we may replace the actions of $\pi_1$ and $\pi_2$ on $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ with $\pi'_1$ and $\pi'_2$, simultaneously. It follows that

$$H^{\mathfrak{g} + \mathfrak{h}} \otimes_{\mathfrak{h}} US(\mathfrak{g}) \otimes_{\mathbb{C}} M \cong H^{\mathfrak{g} + \mathfrak{h}} \otimes_{\mathfrak{h}} US(\mathfrak{g}) \otimes_{\mathbb{C}} M.$$

Since $US(\mathfrak{g})$ is free over $\mathfrak{h}_-$ and cofree over $\mathfrak{h}_+$, $H^{\mathfrak{g} + \mathfrak{h}} \otimes_{\mathfrak{h}} US(\mathfrak{g}) = 0$ for $i \neq 0$ by [72, 77, 78, Theorem 2.1]. (Note that the spectral sequence on converges since the complex $US(\mathfrak{g}) \otimes \mathfrak{f}^{\mathfrak{g} + \mathfrak{h}}$ is a direct sum of finite-dimensional subcomplexes consisting of homogeneous vectors.) We have shown that the functor $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M$ is exact.

In the case that $\mathfrak{h} = \mathfrak{g}$ and $\gamma_0 = \gamma$, we have the following assertion.

**Proposition 2.3 ([72, (1.9)])**. The functor $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M : \mathcal{O}^\mathfrak{h} \to \mathcal{O}^\mathfrak{g}$ is isomorphic to the identity functor.

**Proof.** As $H^{\mathfrak{h} + \mathfrak{g}} \otimes_{\mathfrak{h}} US(\mathfrak{g})$ is isomorphic to the trivial representation $\mathbb{C}$ of $\mathfrak{g}$ ([72, 77, 78, Theorem 2.1]), (12) gives the $\mathfrak{g}$-module isomorphism $S\text{-ind}_{\mathfrak{h}} \mathfrak{g} M \cong M$ as required. \hfill \Box
2.6. Suppose that $\mathfrak{g}$ admits a decomposition
\[ \mathfrak{g} = \mathfrak{a} \oplus \tilde{\mathfrak{a}} \]
with graded subalgebras $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ such that the restrictions $\gamma|_\mathfrak{a}$ and $\gamma|_{\tilde{\mathfrak{a}}}$ of $\gamma$ are semi-infinite 1-cochains of $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$, respectively.

**Lemma 2.4.** $US(\mathfrak{g}) \cong US(\mathfrak{a}) \otimes_US(\tilde{\mathfrak{a}})$ as left $\mathfrak{a}$-modules and right $\tilde{\mathfrak{a}}$-modules.

**Proof.** We have $U^+_n \cong U(\mathfrak{a}_+)^* \otimes_US(\tilde{\mathfrak{a}})_*$ as left $\mathfrak{a}_+$-modules and right $\tilde{\mathfrak{a}}_+$-modules. Consider the composition
\[ US(\mathfrak{a}) \otimes_US(\tilde{\mathfrak{a}}) = (U(\mathfrak{a}_-) \otimes_US(\tilde{\mathfrak{a}})_+)^* \otimes_US(\tilde{\mathfrak{a}})_+ \overset{\cong}{\rightarrow} U(\mathfrak{a}_+) \otimes_US(\tilde{\mathfrak{a}})_+ \rightarrow US(\mathfrak{g}), \]
where the last map is the multiplication map. From the description (1), (2) of the $\mathfrak{g}$-bimodule structure of semi-regular bimodules one sees that the image of the above map is stable under the left and the right action of $\mathfrak{g}$ on $US(\mathfrak{g})$. Hence the image must coincides with $US(\mathfrak{g})$ since it contains $U^+_n$. By the equality of the graded dimensions it follows that above map is an isomorphism. \[ \square \]

**Lemma 2.5.** For $M \in \hat{\mathfrak{h}}^\mathfrak{a}$, $S\text{-ind}_\mathfrak{a}^\mathfrak{g} M \cong US(\mathfrak{a}) \otimes_US M$ as $\mathfrak{a}$-modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ of $US(\mathfrak{a}) \otimes_US M$.

**Proof.** We have
\[ S\text{-ind}_\mathfrak{a}^\mathfrak{g} M \cong H^{\mathfrak{a} \rightarrow 0}(\mathfrak{g}, US(\mathfrak{a}) \otimes_US(\tilde{\mathfrak{a}}) \otimes_US M) \cong US(\mathfrak{a}) \otimes_US S\text{-ind}_\mathfrak{a}^\mathfrak{g} M \]
by Lemmas 2.4 and 2.6. \[ \square \]

### 3. Semi-infinite Bruhat ordering

#### 3.1. Affine Kac-Moody algebras and affine Weyl groups. We first fix some notation which are used for the rest of the paper.

Let $\hat{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, and fix a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$. Let $\hat{\Delta} \subset \hat{\mathfrak{h}}^*$ be the set of roots of $\hat{\mathfrak{g}}$. Choose a subset $\Delta_+ \subset \hat{\mathfrak{h}}^*$ of positive roots and the set $\hat{\Pi} = \{\alpha_i; i \in \hat{I}\} \subset \hat{\Delta}_+$, $\hat{I} = \{1, 2, \ldots, l\}$, of simple roots. Let $\theta$ be the highest root, $\theta_+$ the highest short root, $\Delta_- = -\Delta_+$, $\check{\rho} = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$.

Let $\check{Q} = \sum_{\alpha \in \Delta_+} \mathbb{Z} \alpha \subset \hat{\mathfrak{h}}^*$, the root lattice of $\hat{\mathfrak{g}}$. Set $\check{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta_+} \check{\mathfrak{g}}_{\alpha}$, $\check{\mathfrak{n}}_- = \bigoplus_{\alpha \in \Delta_-} \check{\mathfrak{g}}_{\alpha}$, where $\check{\mathfrak{g}}_{\alpha}$ is the root space of $\hat{\mathfrak{g}}$ with root $\alpha$. We have the triangular decomposition
\[ \hat{\mathfrak{g}} \cong \check{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \check{\mathfrak{n}}. \]

Let $(\mid \ )$ be the normalized invariant bilinear form of $\hat{\mathfrak{g}}$. We identify $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$ using $(\mid \ )$. Let $\hat{\Delta}^\vee = \{\alpha^\vee; \alpha \in \hat{\Delta}\}$, the set of coroots , $\hat{Q}^\vee = \sum_{\alpha \in \hat{\Delta}} \mathbb{Z} \alpha^\vee \subset \hat{\mathfrak{h}} = \hat{\mathfrak{h}}^*$, the coroot lattice of $\hat{\mathfrak{g}}$, $\check{\rho}^\vee = \frac{1}{2} \sum_{\alpha \in \hat{\Delta}_+} \alpha^\vee$, where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$.
Let \( \hat{W} \subset GL(\hat{\mathfrak{h}}^+) \) be the Weyl group of \( \hat{\mathfrak{g}}, \) \( s_\alpha \in \hat{W} \) be the reflection corresponding to \( \alpha \in \Delta: \) \( s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha. \)

Let \( \mathfrak{g} \) be the affine Kac-Moody algebra associated with \( \hat{\mathfrak{g}}: \)
\[
\mathfrak{g} = \hat{\mathfrak{g}} [t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D.
\]
The commutation relations of \( \hat{\mathfrak{g}} \) are given by
\[
[x t^n, y t^m] = [x, y] t^{m+n} + m \delta_{m+n,0} (x|y) K, \quad [K, \mathfrak{g}] = 0, \quad [D, x t^n] = n x t^n.
\]
We consider \( \hat{\mathfrak{g}} \) as a subalgebra of \( \hat{\mathfrak{g}} \) by the natural embedding \( \hat{\mathfrak{g}} \hookrightarrow \hat{\mathfrak{g}}, \) \( x \mapsto x t^0. \) Let
\[
\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D,
\]
the Cartan subalgebra of \( \mathfrak{g}, \) The bilinear form \( (\ , \) \) from \( \mathfrak{h} \) to \( \mathfrak{h} \) by letting \( (K|\mathfrak{h}) = (D|\mathfrak{h}) = (K|K) = (D|D) = 0 \) and \( (D|K) = 1. \) We identify \( \mathfrak{h}^* \) with the subspace of \( \mathfrak{h}^* \) consisting of elements which vanishes on \( \mathbb{C}K \oplus \mathbb{C}D. \) Thus,
\[
\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,
\]
where \( \Lambda_0 \) and \( \delta \) are defined by \( \Lambda_0(K) = \delta(D) = 1, \) \( \Lambda_0(\mathfrak{h} \oplus \mathbb{C}\delta) = \delta(\mathfrak{h} \oplus \mathbb{C}K) = 0. \)

The number \( (\lambda, K) \) is called the level of \( \lambda. \)

Let \( \Delta^c = \hat{\Delta} \cup \{ \alpha + n \delta; \alpha \in \hat{\Delta}, \ n \in \mathbb{N}\}, \) the set of positive real roots of \( \mathfrak{g}, \)
\( \Delta^c = -\Delta^c, \ 
\Delta^c = \Delta^c \cup \Delta^c \) the set of real roots, \( \Pi = \{ \alpha_0 = \theta + \delta, \alpha_1, \ldots, \alpha_\ell \} \) the set of simple roots.

Let \( \mathcal{W} \) be the Weyl group of \( \mathfrak{g}, \) or the affine Weyl group of \( \hat{\mathfrak{g}}. \) We have
\[
\mathcal{W} = \hat{\mathcal{W}} \rtimes \hat{Q}^\vee.
\]
The extended affine Weyl group \( \mathcal{W}^e \) of \( \mathfrak{g} \) is the semidirect product
\[
\mathcal{W}^e = \hat{\mathcal{W}} \rtimes \hat{P}^\vee
\]
where \( \hat{P}^\vee = \{ \lambda \in \hat{\mathfrak{h}}; (\alpha, \lambda) \in \mathbb{Z} \) for all \( \alpha \in \hat{\Delta} \}, \) the coweight lattice of \( \hat{\mathfrak{g}}. \) We have
\[
\mathcal{W}^e = \mathcal{W}^e_+ \rtimes \mathcal{W},
\]
where \( \mathcal{W}^e_+ \) subgroup of \( \mathcal{W}^e \) consisting of elements which fix the set \( \Pi. \)

We denote by \( t_\alpha \) or simply by \( \alpha \) for the element of \( \mathcal{W}^e \) corresponding to \( \alpha \in \hat{P}^\vee. \)
The reflection \( s_\alpha \) corresponding \( \alpha = \lambda + n \delta \in \Delta^c \) is given by \( s_\alpha = t_{-n\delta} s_\alpha. \) We set \( s_i = s_{\alpha_i} \) for \( i \in I := \{ 0, 1, \ldots, l \}, \) so that \( \mathcal{W} = \langle s_i; i \in I \rangle. \) The action of \( \mathcal{W} \) in \( \mathfrak{h}^* \) is extended to the action of \( \mathcal{W}^e \) on \( \mathfrak{h}^* \) by
\[
w(\Lambda_0) = \Lambda_0, \quad w(\delta) = \delta \quad w \in \hat{\mathcal{W}},
\]
\[
t_\alpha(\lambda) = \lambda + \langle \Lambda, K \rangle \alpha - \langle (\Lambda, \alpha) + \frac{(\alpha|\alpha)}{2} \langle \Lambda, K \rangle \delta, \lambda \in \mathfrak{h}^*.
\]
For \( \lambda \in \mathfrak{h}^* \) let \( \bar{\lambda} \in \mathfrak{h}^* \) be its restriction to \( \hat{\mathfrak{h}}. \)
3.2. Twisted Bruhat ordering. Let $\ell : \mathcal{W}^\circ \to \mathbb{Z}_{\geq 0}$ the length function: $\ell(w) = \sharp(\Delta^+ \cap w(\Delta^+))$. We have

$$\ell(t_{\mu}y) = \sum_{\alpha \in \Delta_+ \cap y(\Delta_+)} |(\alpha|\mu)| + \sum_{\alpha \in \Delta_+ \cap y(\Delta_-)} |1 - (\alpha|\mu)|$$

for $\mu \in \mathfrak{g}^\circ, y \in \mathcal{W}$.

The twisted length function $\ell^y : \mathcal{W}^\circ \to \mathbb{Z}$ with the twist $y \in \mathcal{W}^\circ$ is defined by

$$\ell^y(w) = \sharp(\Delta^+ \cap w(\Delta^+ \cap y(\Delta^+)) - \sharp(\Delta^+_+ \cap w(\Delta^+ \cap y(\Delta^+_+))).$$

**Lemma 3.1.** Let $w, y \in \mathcal{W}^\circ$.

(i) Suppose that $\ell(y_{s_i}) = \ell(y) + 1$ for $i \in I$. Then

$$\ell^y_{s_i}(w) = \begin{cases} \ell^y(w) & \text{if } w^{-1}y(\alpha_i) \in \Delta^+_+ \\ \ell^y(w) - 2 & \text{if } w^{-1}y(\alpha_i) \in \Delta^-_. \end{cases}$$

(ii) $\ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}).$

**Proof.** (i) The assertion follows from the definition and the fact that

$$\Delta^+_+ \cap y_{s_i}(\Delta^-_.) = \Delta^+_+ \cap y(\Delta^-_.) \cup \{y(\alpha_i)\} \quad \text{if } \ell(y_{s_i}) = \ell(y) + 1.$$  

(ii) We prove by induction on $\ell(y)$. If $\ell(y) = 0$ then $\ell^y(w) = \ell(w) = \ell(y^{-1}w)$. Suppose that $\ell(y_{s_i}) = \ell(y) + 1$. If $w^{-1}y(\alpha_i) \in \Delta^+_+$ then $\ell(s_iy^{-1}w) = \ell(y^{-1}w) + 1$. Hence by (i) and induction hypothesis,

$$\ell^y_{s_i}(w) = \ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}) = \ell(s_iy^{-1}w) + \ell(s_iy^{-1}).$$

If $w^{-1}y(\alpha_i) \in \Delta^-_.$, then $\ell(s_iy^{-1}w) = \ell(y^{-1}w) - 1$. Again by (i) and induction hypothesis,

$$\ell^y_{s_i}(w) = \ell^y(w) - 2 = \ell(y^{-1}w) - 2 - \ell(y^{-1}) = \ell(s_iy^{-1}w) - \ell(s_iy^{-1}).$$

This completes the proof. $\square$

For $w_1, w_2, y \in \mathcal{W}$ and $\gamma \in \Delta^\circ$, write $w_1 \xrightarrow{\gamma} y w_2$ if $w_1 = s_\gamma w_2$ and $\ell^y(w_1) > \ell^y(w_2)$. Below, we shall often omit the symbol $\gamma$ above the arrow. Also, we shall omit the symbol $y$ under the arrow if $y = 1$. By Lemma (ii) we have

$$w_1 \xrightarrow{\gamma} y w_2 \iff y^{-1}w_1 \xrightarrow{\gamma} y^{-1}w_2.$$  

In particular for $\beta = y(\alpha_i) \in \Delta^+_+, \alpha_i \in \Pi$, and $w_1, w_2 \in \mathcal{W}$ such that $\ell^y(w_2) - \ell^y(w_1) = 1$ we have the equivalence

$$w_1 \xrightarrow{\beta} y w_2 \iff w_2 = s_\beta w_1 \iff \ell^y(w_2) = \ell^y(w_1) + 1.$$

by [K-L, Lemma 11.3].
Define \( w \succeq_y w' \) if there exists a sequence \( w_1, w_2, \ldots, w_k \) of elements of \( W \) such that
\[
\begin{array}{cccc}
w & \to & y & \to w_1 & \to y & \to w_2 & \to y & \cdots & \to y & \to w_k & \to y & \to w'.
\end{array}
\]

Note that
\[
(w \succeq_y w' \iff y^{-1}w \succeq_y y^{-1}w').
\]

by (16), where \( \succeq \equiv \succeq_1 \), the usual Bruhat ordering of \( W \). It follows that \( \succeq_y \) defines a partial ordering of \( W \).

We will use the symbol \( w \triangleright_y w' \) to denote a covering in the twisted Bruhat order \( \succeq_y \). Thus \( w \triangleright_y w' \) means that \( w \succeq_y w' \) and \( \ell^y(w) = \ell^y(w') + 1 \).

3.3. Semi-infinite Bruhat ordering. Define the semi-infinite length \( \ell \overset{\circ}{\circ}(w) \) of \( w \in W^c \) by
\[
\ell \overset{\circ}{\circ}(w) = \sharp \{ \alpha \in \Delta^c \cap w(\Delta^c); \delta \in \overset{\circ}{\Delta}_+ \} - \sharp \{ \alpha \in \Delta^c \cap w(\Delta^c); \delta \in \overset{\circ}{\Delta}_- \}.
\]
We have
\[
\ell \overset{\circ}{\circ}(t_\lambda y) = \ell(y) - 2 \langle \hat{\nu} | \lambda \rangle
\]
for \( \lambda \in \overset{\circ}{\nu}_+ \), \( w \in \overset{\circ}{\nu}_+ \).

Set
\[
\overset{\circ}{\nu}_+ = \{ \lambda \in \overset{\circ}{\nu}_+; \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \overset{\circ}{\Delta}_+ \},
\]
We say that \( \lambda \in \overset{\circ}{\nu}_+ \) is sufficiently large if \( \alpha(\lambda) \) is sufficiently large for all \( \alpha \in \overset{\circ}{\Delta}_+ \).

By (15) and (17) we have the following assertion.

Lemma 3.2. \( \ell \overset{\circ}{\circ}(w) = \ell^\lambda(w) = -\ell^\lambda(w) \) for a sufficiently large \( \lambda \in \overset{\circ}{\nu}_+ \).

We write
\[
w_1 \rightarrow \overset{\circ}{\gamma} \rightarrow w_2
\]
for \( w_1, w_2 \in W \) and \( \gamma \in \Delta^c \) if \( w_1 = w_2s_\gamma \) and \( \ell \overset{\circ}{\circ}(w_1) < \ell \overset{\circ}{\circ}(w_2) \). (We shall often omit the symbol \( \gamma \) above the arrow.) Define \( w \succeq \overset{\circ}{\gamma} w' \) if there exists a sequence \( w_1, w_2, \ldots, w_k \) of elements of \( W \) such that
\[
w \rightarrow \overset{\circ}{\gamma} \rightarrow w_1 \rightarrow \overset{\circ}{\gamma} \rightarrow w_2 \rightarrow \overset{\circ}{\gamma} \rightarrow \cdots \rightarrow \overset{\circ}{\gamma} \rightarrow w_k \rightarrow \overset{\circ}{\gamma} \rightarrow w'.
\]

By Lemma 15
\[
w \succeq \overset{\circ}{\gamma} w' \iff w' \succeq \overset{\circ}{\epsilon}_{\lambda} w \text{ for a sufficiently large } \lambda \in \overset{\circ}{\nu}_+,
\]
\[
\iff w \succeq \overset{\circ}{\epsilon}_{-\lambda} w' \text{ for a sufficiently large } \lambda \in \overset{\circ}{\nu}_+.
\]
It follows that \( \succeq \overset{\circ}{\gamma} \) defines a partial ordering of \( W \). Following Arkhipov [Ark1], we call it the semi-infinite Bruhat ordering on \( W \). By [Ark1], Claim 4.14] the semi-infinite Bruhat ordering coincides with the generic Bruhat ordering defined by Lusztig [Las].

We will use the symbol \( w \triangleright \overset{\circ}{\gamma} w' \) to denote a covering in the twisted Bruhat order \( \succeq \overset{\circ}{\gamma} \). Thus \( w \triangleright \overset{\circ}{\gamma} w' \) means that \( w \succeq \overset{\circ}{\gamma} w' \) and \( \ell \overset{\circ}{\circ}(w) = \ell \overset{\circ}{\circ}(w') - 1 \).
3.4. Semi-infinite analogue of parabolic subgroups and minimal (maximal) length representatives. Let $S$ be a subset of $\Pi$, $\hat{\Delta}_S$ the subroot system of $\hat{\Delta}$ generated by $\alpha_i \in S$, $\hat{\Delta}_S = \bigsqcup_{i=1}^r \hat{\Delta}_{S,i}$, the decomposition into the simple subroot systems $\hat{\Delta}_{1,S}, \ldots, \hat{\Delta}_{r,S}$. Let $\theta_i$ be the longest root of $\hat{\Delta}_{S,i}$.

Set
\[ \Delta_S = \{ \alpha + n\delta \in \Delta^{re}; \alpha \in \hat{\Delta}_S, n \in \mathbb{Z} \}, \quad \mathcal{W}_S = \langle s_\alpha; \alpha \in \Delta_S \rangle \subset \mathcal{W}. \]

Then $\Delta_S$ is a subroot system of $\Delta^{re}$ isomorphic to the affine root system associated with $\hat{\Delta}_S$. Put $\Delta_{S,+} = \Delta_S \cap \Delta^{re}_+$, the set of positive root of $\Delta_S$. Then $\Pi_S = S \cup \{-\theta_1 + \delta, \ldots, -\theta_r + \delta\}$ is a set of simple roots of $\Delta_S$. We have $\mathcal{W}_S = \hat{\mathcal{W}}_S \ltimes t_{\mathcal{Q}_S'}$, where $\mathcal{Q}_S' = \sum_{\alpha \in \Delta_S} \mathbb{Z}\alpha$. By (\ref{co}), the restriction of the semi-infinite length function to $\mathcal{W}_S$ coincides with the semi-infinite length function of the affine Weyl group $\mathcal{W}_S$.

Define
\[ \mathcal{W}^S = \{ w \in \mathcal{W}; w^{-1}(\Delta_{S,+}) \subset \Delta^{re}_+ \}. \]

**Theorem 3.3** (\ref{co}). *The multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$, $(u,v) \mapsto uv$, is a bijection. Moreover, we have*
\[ \ell(\hat{\Xi}(uv)) = \ell(\hat{\Xi}(u)) + \ell(\hat{\Xi}(v)) \text{ for } u \in \mathcal{W}_S, v \in \mathcal{W}^S. \]

**Proof.** First, we show the injectivity of the multiplication map. Suppose that $u_1v_1 = u_2v_2$ with $u_i \in \mathcal{W}_S$, $v_i \in \mathcal{W}^S$. Then $v_1 = w_2$ with $w = u_1^{-1}u_2 \in \mathcal{W}_S$. If $u \neq 1$ then there exists $\alpha \in \Delta_{S,+}$ such that $w^{-1}(\alpha) \in -\Delta_{S,+}$. But then $v_2 \in \mathcal{W}^S$ implies that $v_1^{-1}(\alpha) = v_2^{-1}u^{-1}(\alpha) \in \Delta^{re}$, and this contradicts that $v_1 \in \mathcal{W}^S$. Hence $u = u_2$, and so $v_1 = v_2$.

Second, we show that the multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$ is surjective. We will prove by induction on $\mathcal{P}(w^{-1}(\Delta_{S,+}) \cap \Delta^{re})$ that there exists $u \in \mathcal{W}_S$ such that $w^{-1}w \in \mathcal{W}^S$. If $\mathcal{P}(w^{-1}(\Delta_{S,+}) \cap \Delta^{re}) = 0$, $w \in \mathcal{W}^S$ there is nothing to show. Suppose that $\mathcal{P}(w^{-1}(\Delta_{S,+}) \cap \Delta^{re}) > 0$. Then there exists $\beta \in \Pi_S$ such that $w^{-1}(\beta) \in \Delta^{re}$. Indeed, any element $\alpha \in \Delta_{S,+}$ is expressed as $\alpha = \sum_{\beta \in \Pi_S} n_\beta \beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Thus $w^{-1}(\alpha) = \sum_{\beta \in \Pi_S} n_\beta w^{-1}(\beta) \in \Delta^{re}$ implies that one of $w^{-1}(\beta)$ must belong to $\Delta^{re}$. Now because $(s_\beta w)^{-1}(\Delta_{S,+}) = w^{-1}s_\beta(\Delta_{S,+}) = w^{-1}(\Delta_{S,+} \setminus \{\beta\} \cup \{-\beta\}) = w^{-1}(\Delta_{S,+}) \setminus \{w^{-1}(\beta)\} \cup \{-w^{-1}(\beta)\}$.

Hence by applying the induction hypothesis to $s_\beta w$ we find an element $u \in \mathcal{W}_S$ such that $u^{-1}s_\beta w \in \mathcal{W}^S$.

Finally, we prove the equality of the semi-infinite length. By (\ref{co}), we have $\ell(\hat{\Xi}(t_{\mu}w)) = \ell(\hat{\Xi}(t_{\mu})) + \ell(\hat{\Xi}(w))$ for any $\mu \in \mathcal{Q}_S'$. Hence we may assume that $u \in \mathcal{W}_S$.

We will prove by induction on the length $\ell(u)$ of $u \in \mathcal{W}_S$ that $\ell(\hat{\Xi}(uv)) = \ell(\hat{\Xi}(u)) + \ell(\hat{\Xi}(v))$ for any $v \in \mathcal{W}^S$. Suppose that $\ell(u) = 1$, so that $u = s_\alpha$ for some $\alpha \in S$. Let
Let \( w \in \mathcal{W} \), \( y \in \mathcal{W} \). Note that \( v \in \mathcal{W} \) is equivalent to that

\[
\text{if } \alpha \in \Delta_{S,+} \text{ then } (18) \quad \alpha(\mu) = \begin{cases} 0 & \text{if } y^{-1}(\alpha) \in \Delta_+, \\ 1 & \text{if } y^{-1}(\alpha) \in \Delta_. \end{cases}
\]

Since

\[
(18) \quad \ell_{\mathcal{W}}(s_t u_\mu y) = \ell(s_t(\mu) s_t y) = \ell(s_t y) - 2(\rho|\mu - \alpha_i(\mu)\alpha_i^\vee) = \ell(s_i y) - 2(\rho|\mu) + 2\alpha_i(\mu),
\]

implies that \( \ell_{\mathcal{W}}(s_i y) = \ell_{\mathcal{W}}(v) + 1 \). Next let \( u = s_t u_1 \in \mathcal{W} \) with \( u_1 \in \mathcal{W}_S \), \( \alpha_i \in S, \ell(u) = \ell(u_1) + 1 \), so that \( u_1^{-1}(\alpha_i) \in \Delta_+ \). Let \( v = t_{\mu} y \in \mathcal{W} \) as above. We have

\[
\ell_{\mathcal{W}}(uv) = \ell_{\mathcal{W}}(t_{s_1 u_1(\mu)} s_1 u_1 y) = \ell(s_1 u_1 y) - 2(\rho|s_1 u_1(\mu)).
\]

If \( \ell(s_1 u_1 y) = \ell(u_1 y) + 1 \), then \( \Delta_+ \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i)) \). Hence \( (\mu|u_1^{-1}(\alpha_i)) = 0 \) by (18), which means \( s_1 u_1(\mu) = u_1(\mu) \). By the induction hypothesis, this proves that \( \ell_{\mathcal{W}}(uv) = \ell_{\mathcal{W}}(u) + \ell_{\mathcal{W}}(v) \). If \( \ell(s_1 u_1 y) = \ell(u_1 y) - 1 \), then \( \Delta_+ \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i)) \). So (18) gives \( (\mu|u_1^{-1}(\alpha_i)) = 1 \), which means \( s_1 u_1(\mu) = u_1(\mu) - \alpha_i^\vee \). By the induction hypothesis, this proves that \( \ell_{\mathcal{W}}(uv) = \ell_{\mathcal{W}}(u) + \ell_{\mathcal{W}}(v) \) as required.

4. Wakimoto modules and twisted Verma modules

4.1. The category \( \mathcal{O} \) of \( \mathfrak{g} \). For any \( \mathfrak{h} \)-module \( M \) we set \( M_\mu = \{ m \in M; hm = \mu(h)m \text{ for all } h \in \mathfrak{h} \} \).

Let \( \mathcal{O}^\mathfrak{g} \) be the full subcategory of \( \mathcal{O}^\mathfrak{h} \) consisting of modules on which \( \mathfrak{h} \) acts semisimply. The formal character of \( M \in \mathcal{O}^\mathfrak{g} \) is defined by

\[
\text{ch} M = \sum_{\mu \in \mathfrak{h}^*} (\dim \mathcal{M}_\mu) e^\mu.
\]

Let \( \mathcal{O}_k^\mathfrak{g} \) be the full subcategory of \( \mathcal{O}^\mathfrak{g} \) consisting of objects of level \( k \), where a \( \mathfrak{g} \)-module \( M \) is said to be of level \( k \) if \( K \) acts as the multiplication by \( k \).

4.2. Twisting functors and twisted Verma modules. By abuse of notation we denote also by \( w \) a Tits lifting of \( w \in \mathcal{W} \) to \( \text{Aut}(\mathfrak{g}) \).

For each \( w \in \mathcal{W} \) the twisting functor \( T_w : \mathcal{O}^\mathfrak{g} \to \mathcal{O}^\mathfrak{g} \) is defined as follows (19): Let \( n_w = n_\infty \cap w^{-1}(n_+ +) \) and set \( N_w = U(n_w) \). Put

\[
S_w = U \otimes_{N_w} N_w^*.
\]

The space \( S_w \) has a \( U \)-bimodule structure, which is described as follows: Let \( f \in n_\infty \setminus \{0\} \), and set \( U(f) = U \otimes_{\mathcal{C}[f]} \mathcal{C}[f,f^{-1}] \). Then \( U(f) \) is an associative algebra which contains \( U \) as a subalgebra. We set \( S_f = U(f)/U \). Choose a filtration \( n_w = F^0 \supset F^1 \supset \cdots \supset F^p > 0 \), \( r = \ell(w) \), consisting of ideals \( F^p \subset n_w \) of codimension \( p \). If \( f_r \in F^{p-1} \setminus F^p \) we have an isomorphism of \( U \)-bimodules

\[
S_w = S_{f_r} \otimes_U S_{f_r} \otimes_U \cdots \otimes_U S_{f_r}.
\]

We have

\[
S_w \cong N_w^* \otimes_{N_w} U
\]
as right $U$-modules and left $N_w$-modules. Put
\[ 1_w^* = f_1^{-1} \otimes f_2^{-1} \otimes \ldots \otimes f_r^{-1} \in S_w. \]

For $M \in \mathcal{O}^\mathfrak{g}$ define
\[ T_w(M) = \phi_w(S_w \otimes_U \mathfrak{g}) M, \]
where $\phi_w$ means that the action of $\mathfrak{g}$ is twisted by the automorphism $w$ of $\mathfrak{g}$. This define a right exact functor $T_w : \mathcal{O}^\mathfrak{g} \to \mathcal{O}^\mathfrak{g}$ such that
\[ T_w \circ \text{id} = T_w T_i \quad \text{if} \quad i \in \Pi \quad \text{and} \quad \ell(ws_i) = \ell(w) + 1, \]
where $T_i = T_{\sigma_i}$.

The functor $T_w$ admits a right adjoint functor $G_w$ in the category $\mathcal{O}^\mathfrak{g}$ such that
\[ T_w G_w = \text{id}. \]

The following assertion follows in the same manner as \cite[Theorem 2.1]{20}.

**Lemma 4.1.** Let $M \in \mathcal{O}^\mathfrak{g}$, $w \in \mathcal{W}^\mathfrak{g}$.

(i) Suppose that $M$ is free over $\mathfrak{n}_w$. Then $M \cong G_w T_w(M)$.

(ii) Suppose that $M$ is cofree over $w(\mathfrak{n}_w)$. Then $M \cong T_w G_w(M)$.

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of $\mathfrak{g}$ with highest weight $\lambda$. Set
\[ M_w(\lambda) = T_w M(w^{-1} \circ \lambda). \]

The $\mathfrak{g}$-module $M_w(\lambda) \in \mathcal{O}^\mathfrak{g}$ is called the **twisted Verma module** $M^w(\lambda)$ with highest weight $\lambda$ and twist $w \in \mathcal{W}^\mathfrak{g}$. Note that by (22) we have
\[ M_w(\lambda)_{\mu} \cong \phi_w(N_w^+ \otimes N_w^- U(\mathfrak{n}_-))_{\mu-\lambda} \cong (U(w(\mathfrak{n}_-) \cap \mathfrak{n}_+)^* \otimes U(w(\mathfrak{n}_-) \cap \mathfrak{n}_-))_{\mu-\lambda} \]
as $\mathfrak{h}$-modules. Hence
\[ \text{ch} M^w(\lambda) = \text{ch} M(\lambda). \]

In particular $M^w(\lambda)$ is an object of $\mathcal{O}^\mathfrak{g}$.

By Lemma 4.1 (1) we have
\[ M(\mu) \cong G_w M^w(w \circ \mu). \]

Hence the functor $T_w$ gives the isomorphism
\[ \text{Hom}_\mathfrak{g}(M(\lambda), M(\mu)) \cong \text{Hom}_\mathfrak{g}(M^w(w \circ \lambda), M^w(w \circ \mu)) \]
for $\lambda, \mu \in \mathfrak{h}^*$.

We have \cite[Proposition 6.3]{20}.

\[ M^w(\lambda) \cong M(\lambda) \quad \text{if} \quad \langle \lambda + \rho, \alpha^\vee \rangle \not\in \mathbb{N} \quad \text{for all} \quad \alpha \in \Delta^\mathfrak{g}_+ \cap w(\Delta^\mathfrak{g}_+). \]
4.3. Hom spaces between twisted Verma modules. For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ and $W(\lambda)$ be its integral root system and integral Weyl group, respectively:

$$\Delta(\lambda) = \{ \alpha \in \Delta^\text{re}; (\lambda + \rho, \alpha^\vee) \in \mathbb{Z} \},$$

$$W(\lambda) = \langle s_\alpha; \alpha \in \Delta(\lambda) \rangle \subset W.$$

Let $\Delta(\lambda)_+ = \Delta(\lambda) \cap \Delta^\text{re}_+$ the set of positive roots of $\Delta(\lambda)$, $\Pi(\lambda) \subset \Delta(\lambda)_+$ the set of simple roots of $\Delta(\lambda)$, $\ell : W(\lambda) \to \mathbb{Z}_{\geq 0}$ the length function.

For $y \in W(\lambda)$ the twisted length function $\ell^y$ and the twisted Bruhat ordering $\succeq_{\lambda, y}$ are defined for $W(\lambda)$. We will use the symbol $w \succeq_{\lambda, y} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda, y}$.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called regular dominant if $(\lambda + \rho, \alpha^\vee) \not\in \{0, -1, -2, \ldots \}$ for all $\alpha \in \Delta^\text{re}$. It is called regular anti-dominant if $(\lambda + \rho, \alpha^\vee) \not\in \{0, 1, 2, \ldots \}$ for all $\alpha \in \Delta^\text{re}$.

**Theorem 4.2.** Let $w, w', y \in W(\lambda)$.

(i) If $\lambda$ is regular dominant then

$$\dim \mathbb{C} \text{Hom}_g(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\lambda, y} w', \\
0 & \text{otherwise}. \end{cases}$$

(ii) If $\lambda$ is regular anti-dominant then

$$\dim \mathbb{C} \text{Hom}_g(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\lambda, y} w', \\
0 & \text{otherwise}. \end{cases}$$

**Proof.** (i) By (i) the assertion follows from (ii) and [25], Proposition 2.5.5 (ii)]. Proof of (ii) is similar. \(\Box\)

4.4. Wakimoto modules. Let $\mathfrak{g}, \mathfrak{h}$ be as in §3.4 and let us consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ of root $\alpha$. Let $\rho = \frac{1}{2} h^\vee \Delta_0 \in \mathfrak{h}^*$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Then $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Pi$ and $2\rho$ define a semi-infinite 1-cochain of $\mathfrak{g}$.

Let $L^n_0$, $L^n_-$, $a$ and $\tilde{a}$ be graded subalgebras of $\mathfrak{g}$ defined by

$$L^n_0 = \mathfrak{h}[t, t^{-1}], \quad L^n_- = \mathfrak{h}_- [t, t^{-1}],$$

$$a = L^n_0 \oplus \mathfrak{h}[t^{-1}] t^{-1}, \quad \tilde{a} = L^n_0 \oplus \mathfrak{h}[t] \oplus \mathbb{C} K \oplus CD.$$

Then $0 = 2\rho|_{L^n_0} = 2\rho|_{L^n_-} = 2\rho|_a$ gives semi-infinite 1-cochains of $L^n_0$, $L^n_-$, $a$, and $2\rho|_{\tilde{a}}$ gives a semi-infinite 1-cochain of $\tilde{a}$.

Following [26] we define the **Wakimoto module** $W(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ by

$$W(\lambda) = S\text{-ind}^\mathfrak{g}_\mathfrak{h} C_\lambda,$$

where $C_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to $\lambda$ regarded as a $\mathfrak{a}$-module by the natural projection $\tilde{a} \to \mathfrak{h}$. By Lemma 3.3.2 we have

$$W(\lambda) \cong US(\mathfrak{a})$$

as $\mathfrak{a}$-modules,

(25)
and hence
\begin{equation}
H^i(\mathfrak{n}, W(\lambda)) \cong \begin{cases} 
\text{C}_\lambda & \text{if } i = 0, \\
0 & \text{otherwise}
\end{cases}
\end{equation}

as \( \mathfrak{g} \)-modules,

\begin{equation}
\text{ch} W(\lambda) = \text{ch} M(\lambda).
\end{equation}

In particular \( W(\lambda) \) is an object of \( \mathcal{O}_\mathfrak{g} \).

Theorem \[\text{1}\] below shows that the above definition of Wakimoto module coincides with that of Feigin and Frenkel \[\text{26, 27}\].

4.5. **Wakimoto modules as inductive limits of twisted Verma modules.**

Let \( y, w, u \in \mathcal{W} \) such that \( w = yu \) and \( \ell(w) = \ell(y) + \ell(u) \). Then \( T_w = T_y T_u \) and \( S_w \cong S_y \otimes U \phi_y(S_u) \). Let

\[ j_{w,y} : S_y \longrightarrow S_w \]

be the homomorphism of left \( U \)-modules which maps \( s \in S_y \) to \( s \otimes 1_y \in S_y \otimes U \phi_y(S_u) = S_w \). Define \( \nu^\lambda_{w,y} \in \text{Hom}_\mathfrak{g}(M^\mu(\lambda), M^w(\lambda)) \) by

\[ \nu^\lambda_{w,y}(s \otimes v_{w^{-1}o}) = j_{w,y}(s) \otimes v_{w^{-1}o} \]

for \( s \in S_y \), where \( v_\mu \) denotes the highest weight vector of \( M(\mu) \) for \( \mu \in \mathfrak{h}^* \). Then

\[ \text{Hom}_\mathfrak{g}(M^\mu(\lambda), M^w(\lambda)) = \mathbb{C} \nu^\lambda_{w,y} \]

by \((\text{2})\). We have

\begin{equation}
\nu^\lambda_{w_3,w_2} \circ \nu^\lambda_{w_2,w_1} = \nu^\lambda_{w_3,w_1}
\end{equation}

if \( w_3 = w_2 w_1, w_2 = w_1 w_1 \) with \( \ell(w_1) = \ell(w_2) + \ell(u_2), \ell(w_2) = \ell(w_1) + \ell(u_1) \).

Let \( \{ \gamma_1, \gamma_2, \ldots \} \) be a sequence in \( P_+^\mathfrak{g} \) such that \( \gamma_i - \gamma_{i-1} \in \hat{P}_+^\mathfrak{g} \) and \( \lim_{n \to \infty} \alpha(\gamma_n) = \infty \) for all \( \alpha \in \hat{\Delta}_+ \). Then \( t_{-\gamma_i+1} = t_{-\gamma_i} t_{-\gamma_{i+1}+\gamma_i} \) with \( \ell(t_{-\gamma_i+1}) = \ell(t_{-\gamma_i}) + \ell(t_{-\gamma_{i+1}+\gamma_i}) \) for all \( i \). It follows that \( \{ M^{-\gamma_n}(\lambda) : \nu^\lambda_{\gamma_m,-\gamma_n} \} \) forms an inductive system of \( \mathfrak{g} \)-modules.

**Proposition 4.3** \([\text{28, 29}, \text{Lemma } 6.1.7]\). **There is an isomorphism of \( \mathfrak{g} \)-modules**

\[ W(\lambda) \cong \lim_{n \to \infty} M^{-\gamma_n}(\lambda). \]

**Proof.** For the reader’s convenience we shall give a proof of Proposition \[\text{28}\] here. Set \( W(\lambda)^{\prime} = \lim_{n} M^{-\gamma_n}(\lambda) \). First note that

\[ t_{-\gamma_i}(n^-) = t_{-\gamma_i}(n^-) \cap n^+ = \text{span}_\mathbb{C}\{ x_\alpha t^n ; \alpha \in \Delta_+, 0 \leq n < \alpha(\gamma_i) \}, \]

\[ t_{-\gamma_i}(n^-) \cap n^- = (\mathfrak{g} \oplus \mathfrak{n}) t^{-1} \text{span}_\mathbb{C}\{ x_\alpha t^{-n} ; \alpha \in \Delta_+, n > \alpha(\gamma_i) \}, \]

where \( x_\alpha \) is a root vector of \( \mathfrak{g} \) of root \( \alpha \). Thus we have \( t_{-\gamma_i}(n^-) \subset t_{-\gamma_2}(n^-) \subset \cdots \subset a_+ \). and \( \mathfrak{a}_+ = \bigcup_{j \geq 1} t_{-\gamma_j}(n^-) \). The map \( j_{-\gamma_i,-\gamma_j} : S_{-\gamma_i} \to S_{-\gamma_j} \) restricts to the embedding \( j_{-\gamma_i,-\gamma_j} : N^-_{-\gamma_i} \hookrightarrow N^-_{-\gamma_j} \) for \( i < j \), and we have

\[ U(a_+)^* \cong \lim_{i} \phi_{-\gamma_i}(N^-_{-\gamma_i}) \]

as left \( \mathfrak{a}_+ \)-modules. Let \( j_{-\gamma_i} : \phi_{-\gamma_i}(N^-_{-\gamma_i}) \hookrightarrow U(a_+)^* \) be the embedding of left \( \phi_{-\gamma_i}(N^-_{-\gamma_i}) \)-modules under the above identification.
Since \( t_{-\gamma_i}(n_{-\gamma_i}) = \text{span}_C \{ x_\alpha t^{-\alpha}; \alpha \in \Delta_+; 0 < \alpha \leq \alpha(\gamma_i) \} \subset a, \)
\[ W(\lambda) \cong T_{-\gamma_i}G_{-\gamma_i}(W(\lambda)) \]
by Lemma (ii). Hence \( \text{Hom}_g(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \text{Hom}_g(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(W(\lambda))). \)

As \( \text{ch} G_{-\gamma_i}(W(\lambda)) = \text{ch} M(t_{\gamma_i} \circ \lambda), \) there exists a unique \( g \)-module homomorphism \( \psi_1 : M(t_{\gamma_i} \circ \lambda) \rightarrow G_{-\gamma_i}(M) \) which sends \( v_{t_{\gamma_i} \circ \alpha} \) to \( w_i \), a vector of \( G_{-\gamma_i}(W(\lambda)) \) of weight \( t_{\gamma_i} \circ \alpha \). Up to a non-zero constant multiplication, \( w_i \) equals to the element of \( G_{-\gamma_i}(W(\lambda)) = \mathcal{H}om_{N_{-\gamma_i}}(N_{-\gamma_i}^*, \phi^{-1}_{-\gamma_i}(W(\lambda))) \) which sends \( f \in N_{-\gamma_i}^* \) to \( j_{-\gamma_i}(f) \otimes 1 \lambda \in US(a) \otimes C_\lambda = W(\lambda) \). The corresponding homomorphism \( T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \rightarrow W(\lambda) \) is given by

\[ T_{-\gamma_i}(\psi_i)(f \otimes v_{t_{\gamma_i} \circ \alpha}) = j_{-\gamma_i}(f) \otimes 1 \lambda \quad \text{for} \quad f \in N_{-\gamma_i}^*. \]

It follows that \( T_{-\gamma_i}(\psi_i) \circ v_{\gamma_i} = T_{-\gamma_i}(\psi_i) \) for \( i < j \), and the sequence \( \{ T_{-\gamma_i}(\psi_j) \} \) yields a \( g \)-module homomorphism

\[ \Phi : W(\lambda)' = \lim_i M^{-\gamma_i}(\lambda) \rightarrow W(\lambda). \]

Fix \( \mu \in \mathfrak{h}^* \). Since \( W(\lambda) \cong US(a) \) as an \( a \)-module, it follows from (29) that \( T_{-\gamma_i} \) restricts to the isomorphism \( M^{-\gamma_i}(\lambda)_\mu \cong W(\lambda)_\mu \) for a sufficiently large \( i \). This completes the proof. \( \square \)

### 4.6. Endmorphisms of Wakimoto modules.

**Proposition 4.4.** Let \( \alpha \in 0^\vee, \lambda \in \mathfrak{h}^* \).

(i) \( T_{-\alpha} W(\lambda) \cong W(t_{-\alpha} \circ \lambda). \)

(ii) \( G_{-\alpha} W(\lambda) \cong W(t_{\alpha} \circ \lambda). \)

**Proof.** (i) Let \( \{ \gamma_1, \gamma_2, \ldots \} \) be a sequence in \( 0^\vee \) such that \( \gamma_i - \gamma_{i-1} \in 0^\vee \) and \( \lim_{n \to \infty} \beta(\gamma_n) = \infty \) for all \( \beta \in \Delta_+ \). Set \( \gamma'_i = \gamma_i + \alpha \). Then the sequence \( \{ \gamma'_1, \gamma'_2, \ldots \} \) satisfies the same property. Hence by Proposition 4(ii) and the fact that a homology functor commutes with inductive limits we have \( T_{-\alpha}W(\lambda) = \lim_i T_{-\alpha}M^{-\gamma_i}(\lambda) = \lim_i T_{-\alpha}T_{-\gamma_i}M(t_{\gamma_i} \circ \lambda) = \lim_i T_{-\gamma_i}M(t_{\gamma_i} \circ \lambda) = \lim_i M^{-\gamma'_i}(t_{\alpha} \circ \lambda) = W(t_{\alpha} \circ \lambda). \) (ii) Since \( n_{-\gamma_i} \subset a_-, W(\lambda) \) is free over \( n_{-\gamma_i} \). Hence \( W(t_{\alpha} \circ \lambda) = G_{-\alpha}T_{-\alpha}W(t_{\alpha} \circ \lambda) \cong G_{-\alpha}W(\lambda) \) by Lemma 4(i) and (ii).

**Corollary 4.5.** Let \( \alpha \in 0^\vee. \) The functor \( G_{-\alpha} \) gives the isomorphism

\[ \text{Hom}_g(W(\lambda), W(\mu)) \cong \text{Hom}_g(W(t_{\alpha} \circ \lambda), W(t_{\alpha} \circ \mu)). \]

for \( \lambda, \mu \in \mathfrak{h}^*. \)

**Proposition 4.6.** For \( \lambda \in \mathfrak{h}^* \) we have \( \text{End}_g(W(\lambda)) = C. \)

**Proof.** Let \( \{ \gamma_1, \gamma_2, \ldots \} \) be in Subsection 4(i). Then

\[ \text{End}_g(W(\lambda)) = \text{Hom}_g(\lim_i M^{-\gamma_i}(\lambda), W(\lambda)) \] (by Proposition 4(ii))

\[ = \lim_i \text{Hom}_g(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \lim_i \text{Hom}_g(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}W(\lambda)) \]

\[ \cong \lim_i \text{Hom}_g(M(t_{\gamma_i} \circ \lambda), W(t_{\gamma_i} \circ \lambda)) \] (by Proposition 4(i)).
As we have seen in the proof of Proposition \ref{11}, the space $\text{Hom}_\mathfrak{g}(M(t_\gamma \circ \lambda), W(t_\gamma \circ \lambda))$ is one-dimensional and $\nu_{-\gamma_m, \gamma_n}^\lambda$ induces the isomorphism

$$\text{Hom}_\mathfrak{g}(M^{-\gamma_n}(\lambda), W(\lambda)) \cong \text{Hom}_\mathfrak{g}(M^{-\gamma_n}(\lambda), W(\lambda)).$$

This completes the proof. \hfill \Box

4.7. Uniqueness of Wakimoto modules. A finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset M_r = M$ of a $\mathfrak{g}$-module $M$ is called a Wakimoto flag if each successive quotient $M_i/M_{i-1}$ is isomorphic to $W(\lambda_i)$ for some $\lambda_i$.

**Theorem 4.7.** Suppose that $k$ is non-critical, that is, $k \neq -h^\vee$. For an object $M$ of $\mathcal{O}_\mathfrak{g}$ the following conditions are equivalent.

(i) $M$ admits a Wakimoto flag.

(ii) $H^{\bar{\pi}+i}(a, M) = 0$ for $i \neq 0$ and $H^{\bar{\pi}+0}(a, M)$ is finite-dimensional.

If this is the case the multiplicity $(M : W(\lambda))$ of $W(\lambda)$ in a Wakimoto flag of $M$ equals to $\dim H^{\bar{\pi}+0}(a, M)_\lambda$. In particular if

$$H^{\bar{\pi}+i}(a, M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

as $\mathfrak{h}$-modules, $M$ is isomorphic to $W(\lambda)$.

The proof of Theorem \ref{11} will be given in Subsection \ref{11}. We put on record some of consequences of Theorem \ref{11}.

**Proposition 4.8.** A tilting module in $\mathcal{O}_\mathfrak{g}$ at a non-critical level admits a Wakimoto flag.

**Proof.** By definition a tilting module $M$ admits both a Verma flag and a dual Verma flag. It follows that $M$ is free over $\mathfrak{n}_-$ and cofree over $\mathfrak{n}_+$. In particular $M$ is free over $\mathfrak{n}[t^{-1}]$ and cofree over $\mathfrak{n}[t]$. Hence by \ref{21}, Theorem 2.1, we have $H^{\bar{\pi}+i}(a, M) = 0$ for $i \neq 0$. The fact that $H^{\bar{\pi}+0}(a, M)$ is finite-dimensional follows from the Euler-Poincaré principle. \hfill \Box

**Proposition 4.9.** Suppose that $(\lambda + \rho, K) \not\in \mathbb{Q}_{\geq 0}$. Then $W(t_\alpha \circ \lambda) \cong M(t_\alpha \circ \lambda)$ for a sufficiently large $\alpha \in \hat{P}_+^\vee$.

**Proof.** Let $\alpha$ be sufficiently large. By the hypothesis $(t_\alpha(\lambda + \rho), \beta^\vee) \not\in \mathbb{N}$ for all $\beta \in \Delta_\mathfrak{g}^+$ such that $\beta \in \hat{\Delta}_+$. It follows from \ref{31}, Theorem 3.1 that $M(t_\alpha \circ \lambda)$ is cofree over $\mathfrak{n}[t] = \mathfrak{a}_+$. Because $M(t_\alpha \circ \lambda)$ is obviously free over $\mathfrak{a}_-$ we have $H^{\bar{\pi}+i}(a, M(t_\alpha \circ \lambda)) \cong \begin{cases} \mathbb{C}_{t_\alpha \circ \lambda} & \text{for } i = 0, \\ 0 & \text{otherwise}. \end{cases}$ \hfill \Box

The following assertion follows from Proposition \ref{11} and Corollary \ref{11}.

**Proposition 4.10.** Let $\lambda, \mu \in \mathfrak{h}^*$ be of level $k$, and suppose that $k + h^\vee \not\in \mathbb{Q}_{\geq 0}$. Then

$$\text{Hom}_\mathfrak{g}(W(\lambda), W(\mu)) \cong \text{Hom}_\mathfrak{g}(M(t_\alpha \circ \lambda), M(t_\alpha \circ \mu)).$$
for a sufficiently large $\alpha \in \text{P}^n_k$. In particular if $\lambda \in \mathfrak{h}^*$ is integral, regular antidominant, then

$$\dim \mathbb{C} \text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq \frac{\mathfrak{h}}{\mathfrak{h}} y \\ 0 & \text{else} \end{cases}$$

for $w, y \in \mathcal{W}$.

**Conjecture 4.11.** Let $\lambda \in \mathfrak{h}^*$ be integral, regular dominant. Then

$$\dim \mathbb{C} \text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq \frac{\mathfrak{h}}{\mathfrak{h}} y \\ 0 & \text{else} \end{cases}$$

for $w, y \in \mathcal{W}$.

In Theorem 4.11.4 below we prove Conjecture 6.11 in the case that $w \triangleright \frac{\mathfrak{h}}{\mathfrak{h}} y$ (in a slightly more general setting).

4.8. **Proof of Theorem 4.11.** Let

$$\mathcal{H} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g},$$

the Heisenberg subalgebra. Denote by $\pi_\lambda$ the irreducible representation of $\mathcal{H}$ with highest weight $\lambda$. We have $\pi_\lambda \cong U(\mathfrak{h}[t^{-1}t^{-1}])$ as a module over $\mathfrak{h}[t^{-1}t^{-1}] \subset \mathcal{H}$ provided that $\lambda(K) \neq 0$.

For $M \in \mathcal{O}_k^\mathfrak{g}$ one knows that $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)$ is naturally an $\mathcal{H}$-module of level $k + \mathfrak{h}^\vee$ (1.1.2).

**Lemma 4.12.** Let $M$ be an object of $\mathcal{O}_k^\mathfrak{g}$ with $k \neq -\mathfrak{h}^\vee$. Then the following conditions are equivalent:

1. $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(\mathfrak{a}, M) = 0$ for $i \neq 0$;
2. $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M) = 0$ for $i \neq 0$.

**Proof.** The assumption that $k \neq -\mathfrak{h}^\vee$ implies that $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)$ is semi-simple as an $\mathcal{H}$-module and is a direct sum of $\pi_\lambda$s. Consider the Hochschild-Serre spectral sequence for the ideal $L\mathfrak{h} \subset \mathfrak{a}$ to compute $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(\mathfrak{a}, M)$. By definition, we have

$$E_2^{p, q} = \begin{cases} H_{-p}(\mathfrak{h}[t^{-1}t^{-1}], H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)) & \text{for } p \leq 0, \\ 0 & \text{for } p > 0. \end{cases}$$

By the above mentioned fact $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)$ is free over $U(\mathfrak{h}[t^{-1}t^{-1}])$. Hence

$$E_2^{p, q} = \begin{cases} H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)/\mathfrak{h}[t^{-1}t^{-1}](H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M)) & \text{for } p = 0, \\ 0 & \text{for } p \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_2 = E_\infty$, and $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(\mathfrak{a}, M) = 0$ for $i \neq 0$ if and only if $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(L\mathfrak{h}, M) = 0$ for $i \neq 0$. This completes the proof.

**Proposition 4.13.** Let $M$ be an object of $\mathcal{O}_k$ at a non-critical level $k$ such that $H_{\mathfrak{g}}^{\mathfrak{g} \bullet}(\mathfrak{a}, M) = 0$ for $i \neq 0$. Then

$$M \cong US(\mathfrak{a}) \otimes \mathbb{C} H_{\mathfrak{g}}^{\mathfrak{g} 0}(\mathfrak{a}, M)$$

as $\mathfrak{a}$-modules and $\mathfrak{h}$-modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ and $\mathfrak{h}$ acts as $\mathfrak{h}(s \otimes m) = \text{ad}(h)(s) \otimes m + s \otimes \mathfrak{h}m$. 
Proof. By Proposition 4.12 it suffices to show that $\text{S-ind}_a^b M \cong US(a) \otimes_C H_{\mathfrak{z}^+}^0(a, M)$. As in the proof of Lemma 4.13 we shall consider the Hochschild-Serre spectral sequence for the ideal $\mathcal{L} a \subset a$ to compute $H_{\mathfrak{z}^+}^\bullet(a, US(a) \otimes M)$. By definition we have

\begin{align}
E^{0, q}_1 &= H_{\mathfrak{z}^+}^{q+0}(\mathcal{L} \mathfrak{h}, US(a) \otimes_C M) \otimes_C \bigwedge \phi \mathfrak{h} [t^{-1}] [t^{-1}], \\
E^{p, q}_2 &= H_{-p} \mathfrak{h} [t^{-1}] [t^{-1}], H_{\mathfrak{z}^+}^{q+0}(\mathcal{L} \mathfrak{h}, US(a) \otimes_C M)).
\end{align}

To compute the $E_1$-term set

$$F^p US(a) = \bigoplus_{(\mu, \rho) \geq p} US(a)_\mu,$$

where $US(a)$ is considered as an $\mathfrak{h}$-module by the adjoint action. Then

$$US(a) = F^0 US(a) \supset F^1 US(a) \supset \ldots, \bigcap F^p US(a) = 0,$$

$$F^p US(a) \cdot \mathcal{L} \mathfrak{h} \subset F^{p+1} US(a).$$

Define the filtration $F^\bullet(US(a) \otimes_C M \otimes_C \bigwedge \mathfrak{z}^+ \mathfrak{m}(\mathcal{L} \mathfrak{h}))$ by setting

$$F^p(US(a) \otimes_C M \otimes_C \bigwedge \mathfrak{z}^+ \mathfrak{m}(\mathcal{L} \mathfrak{h})) = F^p US(a) \otimes_C M \otimes_C \bigwedge \mathfrak{z}^+ \mathfrak{m}(\mathcal{L} \mathfrak{h}).$$

This defines a decreasing, weight-wise regular filtration of the complex. Consider the associated spectral sequence $E'_i \Rightarrow H_{\mathfrak{z}^+}^\bullet(\mathcal{L} \mathfrak{h}, US(a) \otimes_C M)$. Because the associated graded space $\text{gr} US(a)$ with respect to this filtration is a trivial $\mathcal{L} \mathfrak{h}$-module the $E_1$-term of the spectral sequence $E'_i$ is isomorphic to $US(a) \otimes_C H_{\mathfrak{z}^+}^\bullet(\mathcal{L} \mathfrak{h}, M)$. Hence by the hypothesis and Lemma 4.13 the spectral sequence $E'_i$ collapses at $E'_1 = E'_\infty$ and we obtain the isomorphism of $\mathfrak{h}$-modules

$$H_{\mathfrak{z}^+}^{i+0}(\mathcal{L} \mathfrak{h}, US(a) \otimes_C M) \cong \begin{cases} US(a) \otimes_C H_{\mathfrak{z}^+}^{i+0}(\mathcal{L} \mathfrak{h}, M) & \text{for } i = 0, \\
0 & \text{for } i \neq 0. \end{cases}$$

This is also an isomorphism of $\mathfrak{a}$-modules since $US(a) \cong \text{gr} US(a)$ as left $\mathfrak{a}$-modules, where $\tau_x \tau^a \in a$ is considered as an operator on $\text{gr} US(a) = \bigoplus_p F^p US(a)/F^{p+1} US(a)$ which maps $F^p US(a)/F^{p+1} US(a)$ to $F^{p+\rho}(\mathfrak{h}) US(a)/F^{p+\rho}(\mathfrak{h})+1 US(a)$. We have computed the $E_1$-term (13):

$$E^{p, q}_1 \cong \begin{cases} US(a) \otimes_C H_{\mathfrak{z}^+}^{p+0}(\mathcal{L} \mathfrak{h}, M) \otimes_C \bigwedge \mathfrak{z}^+ \mathfrak{m}(\mathfrak{h}) \cdot [t^{-1}] [t^{-1}] & \text{for } q = 0, \\
0 & \text{for } q \neq 0. \end{cases}$$

It follows that

$$E^{p, q}_2 \cong \begin{cases} US(a) \otimes_C H_{\mathfrak{z}^+}^{p+0}(a, M) & \text{for } p = q = 0, \\
0 & \text{otherwise} \end{cases}$$

as $\mathfrak{h}$-modules and $\mathfrak{a}$-modules, see the proof of Lemma 4.13. The spectral sequence collapses at $E_2 = E_\infty$ and we obtain the required isomorphism.

Set

$$Q_{\mathfrak{z}^+, +} = \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0} \alpha + \mathbb{Z}_{\geq 0} \delta \subset \mathfrak{h}^*.$$
and define the partial ordering $\preceq_\varpi$ on $\mathfrak{h}^*$ by $\mu \preceq_\varpi \lambda$ if and only if $t_\alpha \circ \mu \leq t_\alpha \circ \lambda$ for a sufficiently large $\alpha \in \check{Q}^\vee$. Note that $\mu \preceq_\varpi \lambda$ if and only if $t_\alpha \circ \mu \leq t_\alpha \circ \lambda$.

Theorem. Since the direction (i) $\Rightarrow$ (ii) in Theorem is obvious by (22), we shall prove that (ii) implies (i). Let $\{\lambda_1, \ldots, \lambda_r\}$ be the set of weights of $H^{\check{\varpi} + 0}(\mathfrak{a}, M)$ with multiplicities counted, so that 

$$M \cong \bigoplus_{i=1}^r US(a) \otimes C_{\lambda_i}$$

as $a$-modules and $\mathfrak{h}$-modules by Proposition 24. We may assume that if $\lambda_i \preceq_\varpi \lambda_j$ then $j < i$.

Set $\lambda = \lambda_1$. We shall show that there is a $g$-module embedding $W(\lambda) \hookrightarrow M$. Let $\{\gamma_1, \gamma_2, \ldots\}$ be a sequence in $\check{P}^\vee_+$ such that $\gamma_i - \gamma_i - 1 \in \check{P}^\vee_+$ and $\lim_{n \to \infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \Delta_+$, so that $W(\lambda) = \text{lim} M^{-\gamma_i}(\lambda)$ by Proposition 24. By Lemma 24 (ii) we have $M \cong T_{-\gamma_i} G_{-\gamma_i}(M)$, and hence,

$$\text{Hom}_g(M^{-\gamma_i}(\lambda), M) \cong \text{Hom}_g(M(\gamma_i, \gamma_i, \lambda), G_{-\gamma_i}(M)).$$

By (24), $\text{ch} G_{-\gamma_i}(M) = \sum_{i=1}^n \text{ch} M(t_{\gamma_i} \circ \lambda)$. Let $i$ be sufficiently large so that $t_{\gamma_i} \circ \lambda$ is maximal in $G_{-\gamma_i}(M)$. Denote by $\Phi_i$ the $g$-module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \to G_{-\gamma_i}(M)$ which sends $\psi(t_{\gamma_i} \circ \lambda)$ to a vector of $G_{-\gamma_i}(M)$ of weight $t_{\gamma_i} \circ \lambda$. As in the proof of Proposition 24 $\{T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \to M\}$ yield an injective $g$-module homomorphism 

$$\Phi : W(\lambda) = \text{lim}_{i} M^{-\gamma_i}(\lambda) \hookrightarrow M.$$ 

The map $\Phi$ induces the homomorphism $H^{\check{\varpi} + 0}(\mathfrak{a}, W(\lambda)) = C_{\lambda} \to H^{\check{\varpi} + 0}(\mathfrak{a}, M)$ which is certainly injective. It follows from the long exact sequence associated with the exact sequence $0 \to W(\lambda) \xrightarrow{\Phi} M \to M/W(\lambda) \to 0$ we obtain that $H^{\check{\varpi} + i}(\mathfrak{a}, M/W(\lambda)) = 0$ for $i \neq 0$ and $\dim H^{\check{\varpi} + 0}(\mathfrak{a}, M/W(\lambda)) = \dim H^{\check{\varpi} + 0}(\mathfrak{a}, M) - 1$. Theorem 24 follows by the induction on $\dim H^{\check{\varpi} + 0}(\mathfrak{a}, M)$.  

\section{Twisted Wakimoto modules.} For $w \in \check{W}$ we have the decomposition $g = w(\mathfrak{a}) \oplus w(\check{\mathfrak{a}})$, and $2\rho$ defines a semi-infinite 1-cochain of the graded subalgebra $w(\check{\mathfrak{a}})$. Hence we can define the twisted Wakimoto module $W^w(\lambda)$ with highest weight $\lambda$ and twist $w \in \check{W}$ by 

$$W^w(\lambda) = \text{S-ind}_{w(\check{\mathfrak{a}})}^{w(\mathfrak{a})} C_{\lambda},$$

where $C_{\lambda}$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to $\lambda$ regarded as a $a$-module by the projection $\check{\mathfrak{a}} \to \mathfrak{h}$. We have 

$$W^w(\lambda) \cong US(w(\mathfrak{a})) \text{ as } w(\mathfrak{a})\text{-modules and } \text{ch} W^w(\lambda) = \text{ch} M(\lambda),$$

$$H^{\check{\varpi} + i}(w(\mathfrak{a}), W^w(\lambda)) \cong \begin{cases} C_{\lambda} & \text{for } i = 0, \\ 0 & \text{otherwise}, \end{cases} \text{ as } \mathfrak{h}\text{-modules.}$$
Let \( \{ \gamma_1, \gamma_2, \ldots \} \) be a sequence in \( \hat{P}_+ \) such that \( \gamma_i - \gamma_{i-1} \in \hat{P}_+ \) and \( \lim_{n \to \infty} \alpha(\gamma_n) = \infty \) for all \( \alpha \in \hat{\Delta}_+ \). The following assertion can be proved in the same manner as Proposition 4.2.

**Proposition 4.14.** Let \( \lambda \in \mathfrak{h}^* \), \( w \in \hat{W} \). There is an isomorphism of \( \mathfrak{g} \)-modules
\[
W^w(\lambda) \cong \lim_{n \to \infty} M^{-w(\gamma_n)}(\lambda).
\]

The following assertion can be proved in the same manner as Theorem 5.1.

**Theorem 5.15.** Let \( \lambda \in \mathfrak{h}^* \) be non-critical, \( w \in \hat{W} \). Let \( M \) be an object of \( \mathcal{O}^\emptyset \) such that
\[
H^{\hat{\mathfrak{h}}^+}(w(\mathfrak{a}), M) \cong \begin{cases} 
C_\lambda & \text{if } i = 0, \\
0 & \text{otherwise},
\end{cases}
\]
as \( \mathfrak{h} \)-modules. Then \( M \) is isomorphic to \( W^w(\lambda) \).

5. **Borel-Weil-Bott vanishing property of Twisting functors**

5.1. **Left derived functors of twisting functors.** The functor \( T_w, w \in \hat{W} \), admits the left derived functor \( \mathcal{L}_* T_w \) in the category \( \mathcal{O}^\emptyset \) since it is a Lie algebra homology functor:
\[
\mathcal{L}_i T_w(M) = \phi_w(H_i(\mathfrak{g}, \mathfrak{g}_w \otimes \mathbb{C} M)),
\]
where \( \mathfrak{g} \) acts on \( N^w_+ \otimes \mathbb{C} M \) by \( X(f \otimes m) = -f X \otimes m + f \otimes X m \). Because \( \mathcal{L}_i T_w(M) \cong \phi_w(H_i(n_w, N^w_+ \otimes \mathbb{C} M)) \) as \( w(n_w) \)-modules, we have the following assertion.

**Lemma 5.1.** Suppose \( M \in \mathcal{O}^\emptyset \) is free over \( n_w \). Then \( \mathcal{L}_i T_w(M) = 0 \) for \( i \geq 1 \).

Let \( \{ e_i, h_i, f_i; i \in I \} \), \( e_i \in \mathfrak{g}_{\alpha_i} \), \( f_i \in \mathfrak{g}_{-\alpha_i} \), be the Chevalley generators of \( \mathfrak{g} \). For \( i \in I \), let \( \mathfrak{s}_{\mathfrak{L}_2}^{(i)} \) denote the copy of \( \mathfrak{s}_{\mathfrak{L}_2} \) in \( \mathfrak{g} \) spanned by \( \{ e_i, h_i, f_i \} \).

**Proposition 5.2.** Let \( M \in \mathcal{O}^\emptyset, i \in I \). Denote by \( N \) the largest \( \mathfrak{s}_{\mathfrak{L}_2}^{(i)} \)-integrable submodule of \( M \). Then \( T_i(M) \cong T_i(M/N), \) \( \text{ch} \mathcal{L}_1 T_i(M) \cong \text{ch} N \) and \( \mathcal{L}_p T_i(M) = 0 \) for \( p \geq 2 \).

**Proof.** Let \( T_i^{(j)} \) denote the twisting functor for \( \mathfrak{s}_{\mathfrak{L}_2}^{(j)} \) corresponding to the reflection \( s_{\alpha_j} \). Because \( T_i(M) \cong T_i^{(j)}(M) \) as \( \mathfrak{s}_{\mathfrak{L}_2}^{(j)} \)-modules and \( \mathfrak{h} \)-modules, we have
\[
\mathcal{L}_p T_i(M) \cong \mathcal{L}_p T_i^{(j)}(M) \text{ as } \mathfrak{s}_{\mathfrak{L}_2}^{(j)} \text{-modules and } \mathfrak{h} \text{-modules.}
\]
In particular, \( \mathcal{L}_p T_i(M) = 0 \) for \( p \geq 2 \). It follows that the exact sequence
\[
0 \to N \to M \to M/N \to 0
\]
yields the long exact sequence
\[
0 \to \mathcal{L}_1 T_i(N) \to \mathcal{L}_1 T_i(M) \to \mathcal{L}_1 T_i(M/N)
\]
\[
\to T_i(N) \to T_i(M) \to T_i(M/N) \to 0.
\]
Since \( M/N \) is free as \( \mathbb{C}[f_i] \)-module \( \mathcal{L}_1 T_i(M/N) = 0 \) by Lemma 5.1. Also, \( T_i(N) = 0 \) and \( \mathcal{L}_1 T_i(N) \cong N \) as \( \mathfrak{h} \)-modules by Theorem 6.1 and (55). This completes the proof. \( \square \)
Let $L(\lambda) \in \mathcal{O}^\theta$ be the irreducible highest weight representation of $g$ with highest weight $\lambda \in \mathfrak{h}^*$. 

**Theorem 5.3** ([2], Theorem 6.1). Let $\lambda \in \mathfrak{h}^*$ and suppose that $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ with $i \in I$. Then

$$\mathcal{L}_\mu T_i(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 1. \end{cases}$$

**Proof.** The hypothesis implies that $L(\lambda)$ is $s_{\alpha_i}^{(i)}$-integrable. Therefore $\mathcal{L}_\mu T_i(L(\lambda)) = 0$ for $p \neq 1$ and $\text{ch} \mathcal{L}_1 T_i(L(\lambda)) = \text{ch} L(\lambda)$ by Proposition [2].

5.2. Twisting functors associated with integral Weyl group.

**Lemma 5.4.** Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$. There exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_\alpha = xs_i x^{-1}$, $\ell(s_\alpha) = 2\ell(x) + 1$ and $\Delta_\mu^c \cap x(\Delta_\mu^c) \cap \Delta(\lambda) = \emptyset$.

**Proof.** Let $s_\alpha = s_{j_1} s_{j_2} \cdots s_{j_r}$ be a reduced expression of $s_\alpha$ in $\mathcal{W}$. Then

$$\Delta_\mu^c \cap s_\alpha(\Delta_\mu^c) = \{ \alpha_1, s_{j_1}(\alpha_{j_2}), \ldots, s_{j_1} \cdots s_{j_2}(\alpha_{j_3}), \ldots, s_{j_1} \cdots s_{j_2} \cdots s_{j_r}(\alpha_{j_{r+1}}) \}$$

Since $\ell_\lambda(\alpha) = 1$, $\Delta_\mu^c \cap s_\alpha(\Delta_\mu^c) \cap \Delta(\lambda) = \{ \alpha \}$. Thus there exists $r$ such that $\alpha = s_{j_1} \cdots s_{j_r-1}(\alpha_{j_r})$. Set $x = s_{j_1} \cdots s_{j_r-1}$, $i = j_r$. Then $s_\alpha = s_\alpha(\alpha) = xs_ix^{-1}$. It follows that $s_{j_1} \cdots s_{j_r-1} = x$ and $\ell(s_\alpha) = 2\ell(x) + 1$. Also $\Delta_\mu^c \cap s_\alpha(\Delta_\mu^c) \cap \Delta(\lambda) = \{ \alpha \}$ implies that $\Delta_\mu^c \cap x(\Delta_\mu^c) \cap \Delta(\lambda) = \emptyset$.

Note that if $\lambda$, $\alpha$, $\alpha_i$, $x$ are as in Lemma [2] then

$$T_\alpha = T_x \circ T_i \circ T_{x^{-1}}.$$

Let $\mathcal{O}^\theta_{[\alpha]}$ be the block of $\mathcal{O}^\theta$ corresponding to $\lambda$, that is, the full subcategory of $\mathcal{O}^\theta$ consisting of objects $M$ such that $[M : L(\mu)] \neq 0 \Rightarrow \mu \in \mathcal{W}(\lambda) \circ \mu$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of $M$.

**Lemma 5.5.** Let $\lambda \in \mathfrak{h}^*$, $y \in \mathcal{W}$, and suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \not\in \mathbb{Z}$ for all $\alpha \in \Delta_\mu^c \cap y^{-1}(\Delta_\mu^c)$. Then $T_y M(w \circ \lambda) \cong M(yw \circ \lambda)$, $T_y L(w \circ \lambda) \cong L(yw \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$. Moreover $T_w$ gives an equivalence of categories $\mathcal{O}^\theta_{[\lambda]} \cong \mathcal{O}^\theta_{[\rho \circ \lambda]}$. The same is true for $G_w$.

**Proof.** First note that the assumption implies that $\mathcal{W}(y \circ \lambda) = y \mathcal{W}(\lambda)y^{-1}$.

We prove by induction on $\ell(y)$. Let $\ell(y) = 1$, so that $y = s_i$ for $i \in I$. Then the fact that $T_i M(w \circ \lambda) \cong M(s_i w \circ \lambda)$ with $w \in \mathcal{W}(\lambda)$ follow from [2]. By Theorems 3.1, 3.2 any object of $\mathcal{O}^\theta_{[\lambda]}$ and $\mathcal{O}^\theta_{[\rho \circ \lambda]}$ is free over $\mathbb{C}[f_i]$ and cofree over $\mathbb{C}[e_i]$. Hence by Lemma [2] $T_i$ gives an equivalence of categories $\mathcal{O}^\theta_{[\lambda]} \cong \mathcal{O}^\theta_{[\rho \circ \lambda]}$ with a quasi-inverse $G_i$. It follows that $T_i L(\lambda)$ is a simple $g$-module which is a quotient of $T_i M(\lambda) = M(s_i \circ \lambda)$, and hence is isomorphic to $L(s_i \circ \lambda)$. Next let $y = s_iz$ with $z \in \mathcal{W}$, $\ell(y) = \ell(z) + 1$. Then $\Delta_\mu^c \cap y^{-1}(\Delta_\mu^c) = \{ z^{-1}(\alpha_i) \} \cup (\Delta_\mu^c \cap z^{-1}(\Delta_\mu^c))$. The assertion follows from the induction hypothesis.

**Corollary 5.6.** Let $\lambda$, $\alpha$, $\alpha_i$, $x$ be as in Lemma [2]. Then $T_x$ gives an equivalence of categories $\mathcal{O}^\theta_{[\alpha]} \cong \mathcal{O}^\theta_{[\rho \circ \lambda]}$ such that $T_x M(\mu) \cong M(x \circ \mu)$, $T_x L(\mu) \cong M(x \circ \mu)$ for $\mu \in \mathcal{W}(x^{-1} \circ \lambda \circ x^{-1}) = \mathcal{W}(\lambda) \circ \lambda$.

**Lemma 5.7.** Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \not\in \mathbb{Z}$. Then $T_i M^w(\lambda) \cong M^{s_i w(s_i \circ \lambda)}$ for $w \in \mathcal{W}(\lambda)$. 

Proof. By Lemma 5.8, \( T_i M^w(\lambda) \cong T_i T_w M(w^{-1} \circ \lambda) \cong T_i T_w T_i M(s_i w^{-1} \circ \lambda) \cong T_i w M(s_i w^{-1} s_i w \circ \lambda) \).

Lemma 5.8. Let \( \lambda \in \mathfrak{h}^* \), \( \alpha_i \in \Pi \) such that \( \langle \lambda + \rho, \alpha_i^{\vee} \rangle \not\in \mathbb{Z} \). Then \( T_i^2 : \mathcal{O}_{[\lambda]}^g \to \mathcal{O}_{[\lambda]}^g \) is isomorphic to the identity functor, and so is \( G_i^2 : \mathcal{O}_{[\lambda]}^g \to \mathcal{O}_{[\lambda]}^g \).

Proof. By Lemma 5.8, \( T_i^2 \) induces an auto-equivalence of the category \( \mathcal{O}_{[\lambda]}^g \) such that \( T_i^2 M(w \circ \lambda) \cong M(w \circ \lambda) \) and \( T_i^2(L(w \circ \lambda)) \cong L(w \circ \lambda) \) for all \( w \in \mathcal{W}(\lambda) \). The standard argument shows that such a functor must be isomorphic to the identity functor.

Corollary 5.9. Let \( \lambda \in \mathfrak{h}^* \), \( w = s_a y \in \mathcal{W}(\lambda) \), \( \alpha_i \in \Pi(\lambda) \), \( y \in \mathcal{W}(\lambda) \), \( \ell_\lambda(w) = \ell_\lambda(y) + 1 \). Then \( T_w : \mathcal{O}_{[\lambda]}^g \to \mathcal{O}_{[w \circ \lambda]}^g \) is isomorphic to the functor \( T_{w} \circ T_{y} : \mathcal{O}_{[\lambda]}^g \to \mathcal{O}_{[w \circ \lambda]}^g \).

Proposition 5.10. Let \( \lambda \in \mathfrak{h}^* \), \( w \in \mathcal{W}(\lambda) \), \( \alpha_i \in \Pi(\lambda) \) and suppose that \( \langle w(\lambda + \rho), \alpha_i^{\vee} \rangle \not\in \mathbb{N} \). Then the following sequence is exact:

\[
0 \to M(s_a w \circ \lambda) \xrightarrow{\varphi_1} M(w \circ \lambda) \xrightarrow{\varphi_2} M^{s_a}(w \circ \lambda) \xrightarrow{\varphi_3} M^{s_a}(s_a w \circ \lambda) \to 0,
\]

where \( \varphi_1, \varphi_2, \varphi_3 \) are any non-trivial \( g \)-homomorphisms.

Proof. First observe that \( \text{Hom}_g(M(s_a w \circ \lambda), M(w \circ \lambda)) \), \( \text{Hom}_g(M(w \circ \lambda), M(s_a w \circ \lambda)) \) and \( \text{Hom}_g(M^{s_a}(w \circ \lambda), M^{s_a}(s_a w \circ \lambda)) \) are all one-dimensional. (The first and the third are one-dimensional by Theorem 5.6.) By Lemma 5.8 there exists \( x \in \mathcal{W} \) and \( \alpha_i \in \Pi \) such that \( s_a = xs_i x^{-1} \), \( \ell(s_a) = 2\ell(x) + 1 \), and \( \Delta^{x}_x \cap x(\Delta^{x}_x) \cap \Delta(\lambda) = \emptyset \).

We have

\[
M(y \circ \lambda) \cong T_x M(x^{-1} y \circ \lambda),
\]

\[
M^{s_a}(y \circ \lambda) = T_x T_y T_{x^{-1}} M(xs_i x^{-1} y \circ \lambda) \cong T_x T_i M(s_i x^{-1} y \circ \lambda) \cong T_x M^{x_i}(x^{-1} y \circ \lambda)
\]

for \( y \in \mathcal{W}(\lambda) \) by Lemma 5.8. Since \( \langle x^{-1} w(\lambda + \rho), \alpha^{\vee}_i \rangle = \langle w(\lambda + \rho), \alpha^{\vee}_i \rangle \not\in \mathbb{N} \) there is an exact sequence

\[
0 \to M(s_a x^{-1} w \circ \lambda) \to M(x^{-1} w \circ \lambda) \to M^{x_i}(x^{-1} w \circ \lambda) \to M^{x_i}(s_a x^{-1} w \circ \lambda) \to 0
\]

by 5.2 Proposition 6.2. The required exact sequence is obtained by applying the exact functor \( T_x : \mathcal{O}_{[w^{-1} \circ \lambda]}^g \to \mathcal{O}_{[\lambda]}^g \) to the above.

Proposition 5.11. Let \( \lambda \in \mathfrak{h}^* \), \( \alpha \in \Pi(\lambda) \), \( M \in \mathcal{O}_{[\lambda]}^g \). Take \( \alpha_i \in \Pi \), \( x \in \mathcal{W} \) such that \( \alpha = x(\alpha_i) \) and \( x^{-1} \Delta(\lambda)_{+} \subset \Delta^{x}_x \) as in Lemma 5.3. Let \( N' \) be the largest \( G_1^{(x)} \)-integrable submodule of \( T_{x^{-1}}(M) \) and set \( N = T_x(N') \subset M \). Then \( T_\alpha(M) \cong T_{s_a}(M/N) \), \( \text{ch} L_1 T_{s_a}(M) = \text{ch} N \) and \( L_p T_{s_a}(M) = 0 \) for \( p \geq 2 \).

Proof. We have \( T_\alpha = T_x T_i T_{x^{-1}} \) and \( T_{x^{-1}} : \mathcal{O}_{[\lambda]}^g \to \mathcal{O}_{[x^{-1} \circ \lambda]}^g \), \( T_x : \mathcal{O}_{[x^{-1} \circ \lambda]}^g \to \mathcal{O}_{[\lambda]}^g \) are exact functors by Corollary 5.8. Therefore

\[
L_p T_{s_a}(M) = T_x (L_p T_i(T_{x^{-1}} M)).
\]

Hence Proposition 5.8 gives that

\[
T_{s_a}(M) = T_x T_i T_{x^{-1}}(M) \cong T_x T_i (T_{x^{-1}}(M)/N') \cong T_x T_i T_{x^{-1}}(M/N) = T_{s_a}(M/N),
\]

\[
\text{ch} L_1 T_{s_a}(M) = \text{ch} T_x T_{x^{-1}}(N) = \text{ch} N,
\]

\[
L_p T_{s_a}(M) = 0 \quad \text{for} \ p \geq 0.
\]

This completes the proof.
Theorem 5.12. Let $\lambda \in \mathfrak{h}^*$ be regular dominant weight, $w \in \mathcal{W}(\lambda)$. Then
\[
\mathcal{L}_p T_w(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = \ell(w), \\ 0 & \text{otherwise.} \end{cases}
\]

Proof. Let $\alpha \in \Pi(\lambda)$. Since $T_{x^{-1}} L(\lambda) = L(x^{-1} \circ \lambda)$ and $\langle x^{-1} \circ \lambda + \rho, \alpha \rangle = \langle \lambda + \rho, \alpha \rangle \in \mathbb{N}$, $T_{x^{-1}} L(\lambda)$ is $\mathfrak{sl}_2(\mathbb{C})$-integrable. Thus,
\[
\mathcal{L}_p T_x L(\lambda) \cong \begin{cases} T_{x^{-1}} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 0 \end{cases}
\]
by Theorem 5.11. It follows from (38) that
\[
\mathcal{L}_p T_{s_n} L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}
\]
Finally the assertion follows in the same manner as in [18, Corollary 6.2] by Corollary 6.2.

\[ \square \]

6. Two-sided BGG resolutions of admissible representations

6.1. Admissible representations. A weight $\lambda \in \mathfrak{h}^*$ is called admissible if it is regular dominant and
\[ \mathbb{Q} \Delta(\lambda) = \mathbb{Q} \Delta^{\text{re}}. \]
The irreducible representation $L(\lambda)$ is called admissible if $\lambda$ is admissible. A complex number $k$ is called an admissible number for $\mathfrak{g}$ if the weight $k \lambda_0$ is admissible.

Let $r^\vee$ be the lacing number of $\check{\mathfrak{g}}$, that is, the maximal number of the edges of the Dynkin diagram of $\check{\mathfrak{g}}$. Also, let $h$ be the Coxeter number of $\check{\mathfrak{g}}$.

Proposition 6.1 (18, 20). A complex number $k$ is admissible if and only if
\[
k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ \hat{h} & \text{if } (r^\vee, q) = r^\vee. \end{cases}
\]

A complex number $k$ of the form (39) is called an admissible number with denominator $q$. For an admissible number $k$ with denominator $q$, we have
\[ \Delta(k \lambda_0) = \{ \alpha + n q \delta; \alpha \in \Delta, \ n \in \mathbb{Z} \} \cong \Delta^\vee \text{ and } \mathcal{W}(k \lambda_0) \cong \mathcal{W} \text{ if } (r^\vee, q) = 1, \]
\[ \Delta(k \lambda_0)^\vee = \{ \alpha^\vee + n q \delta; \alpha \in \Delta, \ n \in \mathbb{Z} \} \cong L \Delta^{\text{re}} \text{ and } \mathcal{W}(k \lambda_0) \cong L \mathcal{W} \text{ if } (r^\vee, q) = r^\vee, \]
where $\Delta(\lambda)^\vee = \{ \alpha^\vee; \alpha \in \Delta(\lambda) \}$ and $L \Delta^{\text{re}}$ and $L \mathcal{W}$ are the real root system and the Weyl group of the non-twisted affine Kac-Moody algebra $L \check{\mathfrak{g}}$ associated with the Langlands dual $L \check{\mathfrak{g}}$ of $\check{\mathfrak{g}}$, respectively. Set
\[ \hat{\alpha}_0 = \begin{cases} -\theta + q \delta & \text{if } (r^\vee, q) = 1, \\ -\theta^\vee + \frac{q}{r^\vee} \delta & \text{if } (r^\vee, q) = r^\vee. \end{cases} \]
Then $\Pi(k \lambda_0) = \{ \alpha_1, \ldots, \alpha_t, \hat{\alpha}_0 \}$. Put $s_{\hat{\alpha}_0} \in \mathcal{W}(k \lambda_0)$, so that $\mathcal{W}(k \lambda_0) = \langle s_1, \ldots, s_t, \hat{\alpha}_0 \rangle$.

For an admissible number $k$ let $Pr_k^+$ be the set of admissible weights $\lambda$ of level $k$ such that $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \hat{\Delta}_+$. Then $\{ L(\lambda); \lambda \in Pr_k^+ \}$ is the set of
irreducible admissible representations of level $k$ which are integrable over $\hat{g} \subset g$. We have $\Delta(\lambda) = \Delta(k\lambda_0)$ for $\lambda \in Pr_k$.

For an admissible number $k$ denote by $Pr_k$ the set of admissible weights $\lambda$ of level $k$ such that $\Delta(\lambda) \cong \Delta(k\lambda_0)$ as root systems. Then

$$Pr_k = \bigcup_{y \in V(\lambda)} Pr_{k,y}, \quad Pr_{k,y} = y \circ Pr_k^+.$$  

Note that

$$V(\lambda) = yW(k\lambda_0)y^{-1} \text{ for } \lambda \in Pr_{k,y}.$$  

For $\lambda \in Pr_k$, let $\Xi(\lambda)$ be the semi-infinite length function of the affine Weyl group $W(\lambda)$. The semi-infinite Bruhat ordering $\preceq_{\lambda, \Xi}$ are also defined for $W(\lambda)$. We will use the symbol $w \triangleright_{\lambda, \Xi} w'$ to denote a covering in the twisted Bruhat order $\preceq_{\lambda, \Xi}$.

**Remark 6.2.** The admissible weight $\lambda \in Pr_k$ is called the principal admissible weight $\Xi(\lambda)$ if $\Delta(\lambda) \cong \Delta^{\infty}$, that is, if the denominator $q$ of $k$ is prime to $r^\vee$.

### 6.2. Fiebig’s equivalence and BGG resolution of admissible representations

The following theorem is the special case of a result of Fiebig [11, Theorem 11].

**Theorem 6.3.** Let $\lambda$ be regular dominant. Suppose that there exists a symmetricizable Kac-Moody algebra $g'$ whose Weyl group $W'$ is isomorphic to $W(\lambda)$. Let $\lambda'$ be an integral dominant weight of $g'$, $O_{[\lambda]}^{\lambda'}$ the block of $O^{\lambda'}$ containing the irreducible highest weight representation $L^{\lambda'}(\lambda')$ of $g'$ with highest weight $\lambda'$. Then there is an equivalence of categories

$$O_{[\lambda]}^{\lambda} \cong O_{[\lambda']}^{\lambda'}$$

which maps $M(w \circ \lambda)$ and $L(w \circ \lambda)$, $w \in W(\lambda)$, to $M'\left(\phi(w) \circ \lambda'\right)$ and $L\left(\phi(w) \circ \lambda'\right)$, respectively. Here $M^{\phi}(\lambda')$ is the Verma module of $g'$ with highest weight $\lambda'$ and $\phi : W(\lambda) \to W'$ is the isomorphism.

Let $k$ be an admissible number with denominator $q$, $\lambda \in Pr_k$. By Theorem the block $O_{[\lambda]}^{\lambda}$ is equivalent to a block of the category $O$ of $g$ or $^Lg$ containing an integrable representation. In particular the existence of a BGG resolution of an integrable representation of an affine Kac-Moody algebra [17] implies the existence of a BGG resolution for $L(\lambda)$:

**Theorem 6.4.** Let $k$ be an admissible number, $\lambda \in Pr_k$. Then there exists a complex

$$B_\bullet(\lambda) : \cdots d_3 B_2(\lambda) d_2 B_1(\lambda) d_1 B_0(\lambda) d_0 0$$

of the form $B_i(\lambda) = \bigoplus_{w \in W(\lambda)} M(w \circ \lambda)$, $d_i = \sum_{w, w', w'' \in W(\lambda)} d_{w', w} d_{w''}$, $w, w', w'' \in \text{Hom}_g(M(w \circ \lambda), M(w' \circ \lambda))$, such that

$$H_i(B_\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$
The resolution of $L(\lambda)$ in Theorem 6.3 can be combinatorially constructed as follows [28]: Fix a $g$-homomorphisms
\[ i_{w',w}^\lambda : M(w \circ \lambda) \rightarrow M(w' \circ \lambda) \]
for $w, w' \in \mathcal{W}(\lambda)$ with $w \geq w'$ in such a way that $i_{w',w}^\lambda \circ i_{w',w}^\lambda = i_{w',w}^\lambda$ if $w \geq w' \geq w$.

A quadruple $(w_1, w_2, w_3, w_4)$ in $\mathcal{W}(\lambda)$ is called a square if $w_1 \triangleright \lambda, w_2 \triangleright \lambda, w_4$, $w_1 \triangleright \lambda, w_3 \triangleright \lambda, w_4$ and $w_2 \neq w_3$.

**Theorem 6.5.** Let $k$ be an admissible number, $\lambda \in \mathbb{P}_k$. Assign $\epsilon_{w_1, w_2} \in \mathbb{C}^*$ for every pair $(w_1, w_2)$ in $\mathcal{W}(\lambda)$ with $w_1 \triangleright \lambda, w_2$ in such a way that $\epsilon_{w_4, w_3} \epsilon_{w_2, w_1} + \epsilon_{w_4, w_3} \epsilon_{w_2, w_1} = 0$ for every square $(w_1, w_2, w_3, w_4)$ of $\mathcal{W}(\lambda)$ (such an assignment is possible by [28]). Set $d_{w',w} = \epsilon_{w',w} i_{w',w}^\lambda$, $d_i = \sum_{w, w' \in \mathcal{W}(\lambda)} d_{w',w}$.

The resolution of $\mathcal{B}_0(\lambda) = \bigoplus_{w, w' \in \mathcal{W}(\lambda)} M(w \circ \lambda)$, is a resolution of $L(\lambda)$.

### 6.3. Twisted BGG resolution

For $w_1, w_2, y \in \mathcal{W}(\lambda)$ with $w_1 \geq y, w_2$, set
\[ \varphi_{w_2, w}^{\lambda, y} = T_y(i_{y^{-1} w_2, y^{-1} w_1}^\lambda) : M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda). \]

A quadruple $(w_1, w_2, w_3, w_4)$ in $\mathcal{W}(\lambda)$ is called a $y$-twisted square if $w_1 \triangleright y, w_2 \triangleright y, w_4$, $w_1 \triangleright y, w_3 \triangleright y, w_4$ and $w_2 \neq w_3$.

**Theorem 6.6.** Let $k$ be an admissible number, $\lambda \in \mathbb{P}_k, y \in \mathcal{W}(\lambda)$. Assign $\epsilon_{w_1, w_2} \in \mathbb{C}^*$ for every pair $(w_1, w_2)$ with $w_1 \triangleright \lambda, y, w_2$ in $\mathcal{W}(\lambda)$ in such a way that $\epsilon_{w_4, w_3} \epsilon_{w_2, w_1} + \epsilon_{w_4, w_3} \epsilon_{w_2, w_1} = 0$ for every $y$-twisted square $(w_1, w_2, w_3, w_4)$ of $\mathcal{W}(\lambda)$.

Set $\mathcal{B}_y(\lambda) = \bigoplus_{w \in \mathcal{W}(\lambda)} M^y(w \circ \lambda)$, $d_{w, w'}^{y, y} = \epsilon_{w',w}^{y, y} \varphi_{w', w}^{\lambda, y}$, $d_i = \sum_{w, w' \in \mathcal{W}(\lambda)} d_{w',w}$.

Then $\mathbf{B}_y(\lambda) = \mathcal{B}_y(\lambda) \rightarrow \mathcal{B}_y(\lambda) \rightarrow \mathcal{B}_y(\lambda) \rightarrow \cdots \rightarrow \mathcal{B}_y(\lambda)$ is a complex of $g$-modules such that
\[ H_i(\mathbf{B}_y(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** Set $\epsilon_{y^{-1} w_1, y^{-1} w_2} = \epsilon_{w_1, w_2}^y$. Then $\{\epsilon_{w_1, w_2}^y\}$ satisfies the condition in Theorem 6.5 if and only if $\{\epsilon_{y^{-1} w_1, y^{-1} w_2}\}$ satisfies the condition in Theorem 6.3. In particular such an assignment is possible. Consider the BGG resolution $\mathbf{B}_y(\lambda)$ of $L(\lambda)$ in Theorem 6.3 associated with this assignment. We have $\mathbf{B}_y(\lambda) = T_y(\mathbf{B}_y(\lambda))[-\ell(\lambda)]$, where $[-\ell(\lambda)]$ denotes the shift of the degree. Therefore the assertion follows from Theorem 6.3.

### 6.4. System of twisted BGG resolutions

**Proposition 6.7.** Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_l}$ a reduced expression of $y \in \mathcal{W}(\lambda)$ with $\beta_i \in \Pi(\lambda)$. Set $y_i = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_i}$ for $i = 0, 1, \ldots, l$ and fix a non-zero $g$-homomorphism $\phi_{w^i}^{y_i} : M^y(w \circ \lambda) \rightarrow M^{y_i+1}(w \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$,
$i = 1, \ldots, l$. One can assign $e^{i}_{w_{2}, w_{1}} \in \mathbb{C}^{*}$ for each pair $(w_{1}, w_{2})$ with $w_{1} \triangleright_{\lambda, y} w_{2}$ for all $i = 1, \ldots, l$ in such a way that the following hold:

(i) $e_{w_{1}, w_{2}}^{i} e_{w_{2}, w_{1}}^{i} + e_{w_{1}, w_{2}}^{i} e_{w_{2}, w_{1}}^{i} = 0$ for every $y_{i}$-twisted square $(w_{1}, w_{2}, w_{3}, w_{4})$ of $W(\lambda)$.

(ii) If $w_{1} \triangleright_{\lambda, y} w_{2}$, $w_{1} \triangleright_{\lambda, y_{i-1}} w_{2}$, $\ell^{\beta}_{\lambda}(w_{1}) = \ell^{\beta}_{\lambda}(w_{1})$ and $\ell^{\gamma}(w_{2}) = \ell^{\gamma}_{\lambda}(w_{2})$, then the following diagram commutes.

$$
\begin{array}{c}
M^{\gamma-1}(w_{1} \circ \lambda) && M^{\gamma-1}(w_{2} \circ \lambda) \\
\phi^{\gamma}_{w_{1}} & \downarrow & \phi^{\gamma}_{w_{2}} \\
M^{\delta}(w_{1} \circ \lambda) && M^{\delta}(w_{2} \circ \lambda).
\end{array}
$$

\textbf{Proposition 6.8.} Let $\lambda \in \mathfrak{h}^{\ast}$ be regular dominant, $y \in W(\lambda)$, $\alpha \in \Pi(\lambda)$ such that $\ell_{\lambda}(y s_{\alpha}) = \ell_{\lambda}(y) + 1$. Set $\beta = y(\alpha)$

(i) Let $w_{1}, w_{2} \in W(\lambda)$. Suppose that $w_{1} \triangleright y w_{2}$, $w_{1} \triangleright_{y s_{\alpha}} w_{2}$ and $\ell^{\beta}_{\lambda}(w_{1}) = \ell^{\beta}_{\lambda}(w_{1})$. Then

$$\dim \text{Hom}_{g}(M^{\delta}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda)) = 1.$$ 

Moreover, either of the followings span the one-dimensional vector space $\text{Hom}_{\mathfrak{g}}(M^{\delta}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda))$:

(a) the composition $M^{\delta}(w_{1} \circ \lambda) \rightarrow M^{\delta}(w_{2} \circ \lambda) \rightarrow M^{y s_{\alpha}}(w_{2} \circ \lambda)$ of any non-trivial $\mathfrak{g}$-homomorphisms;

(b) the composition $M^{\delta}(w_{1} \circ \lambda) \rightarrow M^{y s_{\alpha}}(w_{1} \circ \lambda) \rightarrow M^{y s_{\alpha}}(w_{2} \circ \lambda)$ of any non-trivial $\mathfrak{g}$-homomorphisms.

(ii) Let $w_{1}, w_{2} \in W(\lambda)$. Suppose that $\ell^{\beta}_{\lambda}(w_{1}) = \ell^{\beta}_{\lambda}(w_{2}) + 2$ and $w_{i-1}^{-1}(\beta) \in \Delta^{\prime}_{\lambda}$ for $i = 1, 2$. Then the composition $M^{\delta}(w_{1} \circ \lambda) \rightarrow M^{\delta}(w_{2} \circ \lambda) \rightarrow M^{y s_{\alpha}}(w_{2} \circ \lambda)$ of any non-trivial homomorphisms is non-zero.

(iii) Let $w \in W(\lambda)$ and suppose that $s_{\alpha} w \triangleright_{\lambda, y} w$. Then the composition $M^{\delta}(s_{\alpha} w \circ \lambda) \rightarrow M^{\delta}(w \circ \lambda) \rightarrow M^{y s_{\alpha}}(w \circ \lambda)$ of any $\mathfrak{g}$-homomorphisms is zero.

\textbf{Proof.} (i) Since $y^{-1} w_{1} \triangleright y^{-1} w_{2}$, the Jantzen sum formula implies that

$$[M(y^{-1} w_{1} \circ \lambda) : L(y^{-1} w_{1} \circ \lambda)] = 1.$$ 

Hence $[M^{y s_{\alpha}}(y^{-1} w_{2} \circ \lambda) : L(y^{-1} w_{1} \circ \lambda)] = 1$. As

$$\text{Hom}(M^{\delta}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda)) \cong \text{Hom}(M(y^{-1} w_{1} \circ \lambda), M(y^{-1} w_{2} \circ \lambda)),$$

it follows that

$$\dim \text{Hom}(M^{\delta}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda)) \leq 1.$$ 

Now we have

$$\text{Hom}_{\mathfrak{g}}(M^{\delta}(w_{1} \circ \lambda), M^{\delta}(w_{2} \circ \lambda)) \cong \text{Hom}_{\mathfrak{g}}(M(y^{-1} w_{1} \circ \lambda), M(y^{-1} w_{2} \circ \lambda)),$$

$$\text{Hom}_{\mathfrak{g}}(M^{\delta}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{1} \circ \lambda)) \cong \text{Hom}_{\mathfrak{g}}(M(y^{-1} w_{1} \circ \lambda), M^{y s_{\alpha}}(y^{-1} w_{1} \circ \lambda)),$$

$$\text{Hom}_{\mathfrak{g}}(M^{\delta}(w_{2} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda)) \cong \text{Hom}_{\mathfrak{g}}(M(y^{-1} w_{2} \circ \lambda), M^{y s_{\alpha}}(y^{-1} w_{2} \circ \lambda)),$$

$$\text{Hom}_{\mathfrak{g}}(M^{y s_{\alpha}}(w_{1} \circ \lambda), M^{y s_{\alpha}}(w_{2} \circ \lambda)) \cong \text{Hom}_{\mathfrak{g}}(M(s_{\alpha} y^{-1} w_{1} \circ \lambda), M(s_{\alpha} y^{-1} w_{2} \circ \lambda)).$$

In particular they are all one-dimensional. Hence it remains to show that the compositions in (a) and (b) are non-trivial. This is equivalent to the non-triviality
of the compositions

\[ M(y^{-1}w_1 \circ \lambda) \rightarrow M(y^{-1}w_2 \circ \lambda) \rightarrow M^{s_{\alpha}}(y^{-1}w_2 \circ \lambda) \]

and

\[ M(y^{-1}w_1 \circ \lambda) \rightarrow M^{s_{\alpha}}(y^{-1}w_1 \circ \lambda) \rightarrow M^{s_{\alpha}}(y^{-1}w_2 \circ \lambda), \]

respectively. Therefore we may assume that \( y = 1 \).

Since \( \langle w_2(\lambda + \rho), \alpha' \rangle \in \mathbb{N} \), we have the exact sequence

\[ 0 \rightarrow M(s_{\alpha}w_2 \circ \lambda) \rightarrow M(w_2 \circ \lambda) \rightarrow M^{s_{\alpha}}(w_2 \circ \lambda) \rightarrow M^{s_{\alpha}}(s_{\alpha}w_2 \circ \lambda) \rightarrow 0 \]

by Proposition \ref{prop:exact}. On the other hand

\[ w_1 \circ \lambda \not\in_{s_{\alpha}} s_{\alpha}w_2 \circ \lambda \]

as we have the square \((s_{\alpha}w_1, w_1, s_{\alpha}w_2, w_2)\) by the assumption and \( \mathbf{(L)} \). Hence \( \mathbf{(L)} \) implies that the image of the highest weight vector of \( M(w_1 \circ \lambda) \) in \( M(w_2 \circ \lambda) \) does not lie in the kernel of the map \( M(w_2 \circ \lambda) \rightarrow M^{s_{\alpha}}(w_2 \circ \lambda) \). This proves the non-triviality of the composition map in (a) for \( y = 1 \), and thus, for all \( y \). Next we show the non-triviality of the composition in (b). Consider the exact sequence

\[ 0 \rightarrow M(s_{\alpha}w_1 \circ \lambda) \rightarrow M(s_{\alpha}w_2 \circ \lambda) \rightarrow N \rightarrow 0 \]

in the category \( \mathcal{O}_{\mathfrak{sl}(2)} \), where \( N = M(s_{\alpha}w_2 \circ \lambda) / M(s_{\alpha}w_1 \circ \lambda) \). Applying the functor \( T_{s_{\alpha}} \) we obtain the exact sequence

\[ 0 \rightarrow L_1T_{s_{\alpha}}N \rightarrow M^{s_{\alpha}}(w_1 \circ \lambda) \rightarrow M^{s_{\alpha}}(w_2 \circ \lambda) \rightarrow T_1N \rightarrow 0. \]

By Proposition \ref{prop:exact} the weights of \( L_1T_{s_{\alpha}}N \) are contained in the set of weights of \( N \), and hence of \( M(s_{\alpha}w_2 \circ \lambda) \). Therefore \( \mathbf{(L)} \) and \( \mathbf{(t)} \) imply that the image of the highest weight vector of \( M(w_1 \circ \lambda) \) in \( M^{s_{\alpha}}(w_1 \circ \lambda) \) does not belong to the kernel of the map \( M^{s_{\alpha}}(w_1 \circ \lambda) \rightarrow M^{s_{\alpha}}(w_2 \circ \lambda) \). This completes the proof of (i). (ii) Similarly as above, the problem reduces to the case \( y = 1 \). By the assumption we have \( s_{\alpha}w_1 \triangleright w_1, s_{\alpha}w_2 \triangleright w_2 \). Thus \( w_1 \not\in_{s_{\alpha}} s_{\alpha}w_2 \) because otherwise \((w_1, s_{\alpha}w_1, s_{\alpha}w_2)\) is a square. Hence \( \mathbf{(L)} \) proves the assertion by the same argument as above. (iii) Again we may assume that \( y = 1 \) and the assertion follows from \( \mathbf{(E)} \).

**Proof of Proposition \ref{prop:exact}.** We prove by induction on \( i \) that such an assignment is possible.

As we already remarked the case \( i = 0 \) is the well-known result of \( \mathbf{(E)} \). So let \( i > 0 \). Suppose that \( w_1 \triangleright w_2 \). Set \( \beta = y_{i-1}(\alpha_i) \in \Delta^+_i \). The following four cases are possible. (The case \( w_1^{-1}(\beta) \in \Delta^+ \), \( w_2^{-1}(\beta) \in \Delta^\pm \) does not happen by Lemma 11.3.)

1) \( w_1^{-1}(\beta) \in \Delta^+_i \). In this case \( w_1 \triangleright y_{i-1}(w_2), \forall \alpha_i \in \Delta^+_i \). The following four cases are possible. (The case \( w_1^{-1}(\beta) \in \Delta^+_i \), \( w_2^{-1}(\beta) \in \Delta^\pm \) does not happen by Lemma 11.3.)

II) \( w_1 = s_{\alpha}w_2 \). In this case \( w_1 \triangleright \lambda, y_{i-1}, w_1, \forall \alpha_i \in \Delta^+_i \). The following four cases are possible. (The case \( w_1^{-1}(\beta) \in \Delta^+_i \), \( w_2^{-1}(\beta) \in \Delta^\pm \) does not happen by Lemma 11.3.)

III) \( w_1^{-1}(\beta) \in \Delta^\pm \). In this case \( w_1 \triangleright \lambda, y_{i-1}, w_2, \forall \alpha_i \in \Delta^+_i \). The following four cases are possible. (The case \( w_1^{-1}(\beta) \in \Delta^+_i \), \( w_2^{-1}(\beta) \in \Delta^\pm \) does not happen by Lemma 11.3.)

Note that \( \epsilon_{s_{\alpha}w_2, w_1} \) is defined in I). and \( \epsilon_{s_{\alpha}w_2, w_1} \) are defined in II). We set

\[ \epsilon_{w_2, w_1} = \epsilon_{s_{\alpha}w_2, w_1} \frac{\epsilon_{s_{\alpha}w_1, w_2}^4}{\epsilon_{s_{\alpha}w_2, w_2}}. \]
IV) \( w_1^{-1}(\beta) \in \Delta^e_+ \), \( w_2^{-1}(\beta) \in \Delta_+^e \), \( w_2 \neq s_\gamma w_1 \). In this case there exists a unique \( w_3 \in \mathcal{W} \) such that \((s_\gamma w_1, w_1, w_3, w_2)\) is a \( y_i \)-twisted square. Note that \( w_3^{-1}(\beta) \in \Delta_+^e \) because \((w_3, w_2, s_\gamma w_3, s_\gamma w_2)\) is a \( y_i \)-twisted square by (43). Since \( \epsilon^{i}_{w_3, s_\gamma w_1}, \epsilon_{w_2, w_3} \) are defined in I) and \( \epsilon^i_{w_1, s_\gamma w_1} \) is defined in II), we can set

\[
\epsilon^i_{w_1, w_1} = - \frac{\epsilon^i_{w_3, s_\gamma w_1} \epsilon^i_{w_2, w_3}}{\epsilon^i_{w_3, s_\gamma w_1}}.
\]

Now let \((w_1, w_2, w_3, w_4)\) be a \( y_i \)-twisted square. Set

\[
A_i(w_1, w_2, w_3, w_4) = \frac{\epsilon^i_{w_4, w_2} \epsilon^i_{w_2, w_1}}{\epsilon^i_{w_4, w_3} \epsilon^i_{w_3, w_1}}.
\]

We need to show that \( A_i(w_1, w_2, w_3, w_4) = -1 \).

The following four cases are possible.

1) \( w_2 = s_\gamma w_1, w_4 = s_\gamma w_3 \). In this case the assertion follows from the definition (41).

2) \( w_2 = s_\gamma w_1, w_4 \neq s_\gamma w_3 \). In this case \((s_\gamma w)^{-1}(\beta) \in \Delta^e_+, w_4^{-1}(\beta) \in \Delta^e_+ \) because otherwise \( w_2 = s_\gamma w_4 \). Hence the assertion follows from the definition (41).

3) \( w_2 \neq s_\gamma w_1, w_4 \neq s_\gamma w_3 \). In this case \((s_\gamma w_1, w_1, s_\gamma w_2, w_2), (s_\gamma w_1, w_1, s_\gamma w_2, w_3), (s_\gamma w_2, w_2, s_\gamma w_3, w_4)\) are \( y_i \)-twisted squares:

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}
\]

We have by 1)

\[
A_i(s_\gamma w_1, w_1, s_\gamma w_2, w_2) = A_i(s_\gamma w_2, w_2, w_3, s_\gamma w_3) = -1
\]

and by 2)

\[
A_i(s_\gamma w_1, w_1, s_\gamma w_2, w_3) = -1.
\]

But

\[
A_i(w_1, w_2, w_3, s_\gamma w_3)
\]

\[
= A_i(s_\gamma w_1, w_1, s_\gamma w_2, w_2)A_i(s_\gamma w_2, w_2, w_3, s_\gamma w_3)A_i(s_\gamma w_1, s_\gamma w_2, w_1, w_3).
\]

Hence the assertion follows.

4) \( w_2 \neq s_\gamma w_1, w_4 \neq s_\gamma w_2 \). we see as in [23, p.57, c)] that \( w_4 \neq s_\beta w_2, s_\beta w_3 \), and hence as in [23, p.56, 1)] we find that \((s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)\) is also a \( y_i \)-twisted square. Hence a) \( w_i^{-1}(\beta) \in \Delta^e_+ \) for all \( i \) or b) \( w_i^{-1}(\beta) \in \Delta^e_+ \) for all \( i \).

a) The case \( w_i^{-1}(\beta) \in \Delta^e_+ \) for all \( i \): By the definition I) we have the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}
\]

\[
(48)
\]
for \( a = 2, 3 \). Since \( \epsilon^{i-1}_{w_4, w_2} \epsilon^{i-1}_{w_2, w_1} = -\epsilon^{i-1}_{w_4, w_3} \epsilon^{i-1}_{w_3, w_1} \) by the induction hypothesis the commutativity of the above diagram implies that \( \epsilon^{i}_{w_4, w_2} \epsilon^{i}_{w_2, w_1} = -\epsilon^{i}_{w_4, w_3} \epsilon^{i}_{w_3, w_1} \) by Proposition 6.10 (ii).

b) The case that \( w_i^{-1}(\beta) \in \Delta^{+,t} \) for all \( i \): We have that \((s_{\beta}w_1, w_1, s_{\beta}w_2, w_2), (s_{\beta}w_1, s_{\beta}w_2, s_{\beta}w_3, w_3), (s_{\beta}w_2, w_2, s_{\beta}w_4, w_4)\) and \((s_{\beta}w_3, w_3, s_{\beta}w_4, w_4)\) are all \( y_i \)-twisted squares. Hence the assertion follows from the equality

\[
A_i(w_1, w_2, w_3, w_4)A_i(s_{\beta}w_1, s_{\beta}w_2, w_1, w_2)A_i(s_{\beta}w_1, w_1, s_{\beta}w_3, w_3)
= A_i(s_{\beta}w_2, w_2, s_{\beta}w_4, w_4)A_i(s_{\beta}w_1, s_{\beta}w_3, w_4, w_4).
\]

\( \square \)

Let \( k \) be an admissible number, \( \lambda \in P^r_k \). Let \( y \in W(\lambda), \{ y_i \}, \{ \phi^{y_i}_{w_i} \}, \{ \epsilon^{i}_{w_2, w_1} \} \) be as in Proposition 6.10. Because \( \{ \epsilon^{i}_{w_2, w_1} \} \) satisfies the condition in Theorem 6.1 there is a corresponding twisted BGG resolution \( B^y_\bullet(\lambda) \) of \( L(\lambda) \) for \( i = 0, 1, \ldots, l = \ell(y) \). Define

\[
\phi^{y_{i+1}}_y = \bigoplus_{w \in W(\lambda)} \phi^{y_{i+1}}_{w} : B^y_\bullet(w \circ \lambda) \to B^y_\bullet(w \circ \lambda).
\]

Proposition 6.9. In the above setting \( \phi^{y_{i+1}}_y \) gives a quasi-isomorphism \( B^y_\bullet(\lambda) \sim B^y_{\bullet+1}(\lambda) \) of complexes for each \( i = 0, 1, \ldots, l - 1 \).

Lemma 6.10. Let \( \lambda \in h^*, y, y_i \) be as in Proposition 6.9, \( w_1, w_2 \in W(\lambda) \).

1. Suppose that \( w_1 \triangleright_{\lambda, y_i} w_2, \ell^{y_i}(w_1) = \ell^{y_{i+1}}(w_1) \). Then \( w_1 \triangleright_{\lambda, y_{i+1}} w_2 \).
2. Suppose that \( w_1 \triangleright_{\lambda, y_i} w_2, \ell^{y_i}(w_2) = \ell^{y_{i+1}}(w_2) \). Then either of the following two holds.
   a) \( w_2 = s_{\beta}w_1 \) and \( w_2 \triangleright_{\lambda, y_{i+1}} w_1 \).
   b) \( w_1 \triangleright_{\lambda, y_{i+1}} w_2 \).

Proof. (1) By assumption \( s_{\beta}w_1 \triangleright_{\lambda, y_i} w_2 \). Therefore \( (s_{\beta}w_1, w_1, s_{\beta}w_2, w_2) \) is a \( y_i \)-twisted square. (2) Similarly, if \( w_2 \neq s_{\beta}w_1 \) then \( (s_{\beta}w_1, w_1, s_{\beta}w_2, w_2) \) \( y_i \)-twisted square. The \( w_2 \neq s_{\beta}w_1 \) case is obvious.

Proof of Proposition 6.9. The fact that \( \phi^{y_i}_y \) defines a homomorphism of complexes follows from the commutativity of (24), Proposition 6.1 (iii), and Lemma 6.10. Since both complexes are quasi-isomorphic to \( L(\lambda) \), to show that it defines a quasi-isomorphism it suffices to check that it defines a non-trivial homomorphism between the corresponding homology spaces. This follows from the fact that \( \phi^0_y : M^{y_i}(\lambda) \to M^{y_{i+1}}(\lambda) \) sends the highest weight vector of \( M^{y_i}(\lambda) \) to the highest weight vector of \( M^{y_{i+1}}(\lambda) \).

6.5 Two-sided BGG resolutions of \( G \)-integrable admissible representations. For \( \lambda \in Pr_k \) and \( i \in \mathbb{Z} \) set

\[
\mathcal{W}^i(\lambda) = \{ w \in W(\lambda); \ell^{y_i}(w) = i \}.
\]

We note that

\[
\sharp \mathcal{W}^i(\lambda) = \begin{cases} 1 & \text{if } \hat{y} = \hat{s}_2, \\ \infty & \text{else.} \end{cases}
\]

Theorem 6.11. Let \( k \) be an admissible number, \( \lambda \in P^+_k \).
(i) The space \( \text{Hom}_Q(W(w \circ \lambda), W(w' \circ \lambda)) \) is one-dimensional for \( w, w' \in \mathcal{W}(\lambda) \) such that \( w \triangleright \lambda \triangleleft w' \).

(ii) There exists a complex

\[
C^\bullet(\lambda) : \cdots \to C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^{0}(\lambda) \xrightarrow{d_0} C^{1}(\lambda) \xrightarrow{d_1} C^{2}(\lambda) \xrightarrow{d_2} \cdots
\]

in the category \( \mathcal{O} \) of the form

\[
C^i(\lambda) = \bigoplus_{w \in \mathcal{W}^i(\lambda)} W(w \circ \lambda), \quad d_i = \sum_{w \in \mathcal{W}^i(\lambda), w' \in \mathcal{W}^{i+1}(\lambda), \triangleright} d_{w',w},
\]

where \( d_{w',w} \) is a non-trivial \( \mathfrak{g} \)-homomorphism \( W(w \circ \lambda) \to W(w' \circ \lambda) \), such that

\[
H^i(C^\bullet(\lambda)) \cong \begin{cases} 
L(\lambda) & \text{for } i = 0, \\
0 & \text{for } i \neq 0.
\end{cases}
\]

**Proof.** (ii) Let \( q \) be the denominator of \( k \) and set \( M = qQ^\gamma \) if \( (r^\gamma, q) = 1 \) and \( M = qQ \) if \( (r^\gamma, q) = r^\gamma \), so that \( \mathcal{W}(\lambda) = \mathcal{W} \times tM \). Let \( \gamma_1, \gamma_2, \ldots \), be a sequence in \( \mathcal{P}_+^\gamma \cap M \) such that \( \gamma_i - \gamma \in \mathcal{P}_+^\gamma \cap M \), \( i \to \infty \), \( \lim_\gamma (\gamma_i) = \infty \) for all \( \alpha \in \Delta^\circ_+ \).

By Proposition \( \ref{prop:inductive-system} \) there is an inductive system \( \{ \mathcal{B}^{-\gamma_i}(\lambda) \} \) of twisted BGG resolutions. Let \( \mathcal{B}^{\bullet,-\gamma_i}(\lambda) \) be the complex \( \mathcal{B}^{-\gamma_i}(\lambda) \) with the opposite homological grading. Thus it is a complex

\[
B_{\gamma_i}^\bullet(\lambda) : \cdots \to B_{\gamma_i}^{-2}(\lambda) \xrightarrow{d_{\gamma_i}^{-2}} B_{\gamma_i}^{-1}(\lambda) \xrightarrow{d_{\gamma_i}^{-1}} B_{\gamma_i}^{0}(\lambda) \xrightarrow{d_{\gamma_i}^0} B_{\gamma_i}^{1}(\lambda) \xrightarrow{d_{\gamma_i}^1} B_{\gamma_i}^{2}(\lambda) \xrightarrow{d_{\gamma_i}^2} \cdots
\]

of the form

\[
B_\gamma^p(\lambda) = \bigoplus_{w \in \mathcal{W}(\lambda), \triangleright} M^{-\gamma}(w \circ \lambda), \quad d_p = \sum_{w \in \mathcal{W}(\lambda), \triangleright} d_{w',w},
\]

\[
M^{-\gamma}(w \circ \lambda) \to M^{-\gamma}(w' \circ \lambda) \quad \text{such that} \quad H^p(B^{\bullet,-\gamma_i}(\lambda)) = \begin{cases} 
L(\lambda) & \text{if } p = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( (C^\bullet(\lambda), d_\bullet) \) be the complex obtained as the inductive limit of complex \( B^{\bullet,-\gamma_i}(\lambda) \). By Lemma \( \ref{lem:inductive-limit} \), Proposition \( \ref{prop:twisted-BGG} \) and Proposition \( \ref{prop:inductive-system} \), we have

\[
C^p(\lambda) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \lim_i M^{-\gamma_i}(w \circ \lambda) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W(w \circ \lambda) \quad \text{for } p \in \mathbb{Z},
\]

\[
H^p(C^\bullet(\lambda)) = \lim_i H^p(B^{\bullet,-\gamma_i}(\lambda)) = \begin{cases} 
L(\lambda) & \text{if } p = 0, \\
0 & \text{otherwise,}
\end{cases}
\]

and the differential \( d_p : C^p(\lambda) \to C^{p+1}(\lambda) \) has the form

\[
d_p = \sum_{w \in \mathcal{W}^p(\lambda), \triangleright} d_{w',w},
\]

where \( d_{w',w} : W(w \circ \lambda) \to W(w' \circ \lambda) \) is induced by the homomorphisms \( d_{w',w}^{-\gamma_i} : M^{-\gamma_i}(w \circ \lambda) \to M^{-\gamma_i}(w' \circ \lambda) \) with \( i = 1, 2, \ldots \). To complete the proof of (ii) it remains to show that the map \( d_{w',w} \) is nonzero for \( w \triangleright \lambda \triangleleft w' \).


Let $w', w \in \mathcal{W}(\lambda)$ such that $w \triangleright_{\lambda, i} w'$. We have the commutative diagram

$$
\begin{array}{ccc}
M^{-\gamma_i}(w' \circ \lambda) & \xrightarrow{d_{w',w}} & M^{-\gamma_i}(w \circ \lambda) \\
\phi^{-\gamma_i \circ \lambda} & \downarrow & \phi^{-w \circ \gamma_i} \\
W(w' \circ \lambda) & \xrightarrow{d_{w,w'}} & W(w \circ \lambda)
\end{array}
$$

for all $i$. By applying the functor $G_{-\gamma_i}$, we obtain the commutative diagram

$$
\begin{array}{ccc}
M(t_\gamma w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w,w'}^i)} & M(t_\gamma w \circ \lambda) \\
\downarrow G_{-\gamma_i}(\phi^{-w \circ \gamma_i}) & & \downarrow G_{-\gamma_i}(\phi^{-w \circ \gamma_i}) \\
W(t_\gamma w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w,w'}^i)} & W(t_\gamma w \circ \lambda).
\end{array}
$$

By Corollary $\square$ $d_{w,w'}^i \neq 0$ if and only if $G_{-\gamma_i}(d_{w,w'}^i) \neq 0$. Therefore it is sufficient to show that $G_{-\gamma_i}(\phi^{-w \circ \gamma_i}) \circ G_{-\gamma_i}(d_{w,w'}^i) : M(t_\gamma w' \circ \lambda) \to W(t_\gamma w \circ \lambda)$ is non-zero for a sufficiently large $i$.

Write $w' = s_\alpha w$ with $\alpha \in \Delta^\vee$. (This is possible because $s_\alpha = s_{-\alpha}$.) Then, for a sufficiently large $i, \beta := t_\gamma(\alpha) \in \Delta^\vee$ and $t_\gamma s_\alpha w = s_\beta t_\gamma w \rightarrow t_\gamma w$. The determinant formula [\cite{ars}] Proposition 2 (2)] shows that the image of the highest weight vector of $M(t_\gamma w' \circ \lambda) = M(s_\beta t_\gamma w \circ \lambda)$ in $M(t_\gamma w \circ \lambda)$ is not in the kernel of the map $G_{\gamma_i}(\phi^{-w \circ \gamma_i}) ; M(t_\gamma w \circ \lambda) \to W(t_\gamma w \circ \lambda)$. Therefore $G_{\gamma_i}(\phi^{-w \circ \gamma_i}) \circ G_{\gamma_i}(d_{w,w'}^i)$ is non-zero, and hence so is $d_{w,w'}^i$.

Finally we shall prove (i). Note that

$$\text{Hom}_F(W(w' \circ \lambda), W(w \circ \lambda)) = \lim_{\mathbb{C}} \text{Hom}_F(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$$

and that $\text{Hom}_F(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$ is at most one-dimensional by the Jantzen sum formula since $w' \triangleright_{\lambda, i} w$. It follows from (the proof of) (ii) that $\text{Hom}_F(W(w' \circ \lambda), W(w \circ \lambda))$ is spanned by $d_{w,w'}^i$. This completes the proof. \hfill $\square$

Remark 6.12. By Theorem \cite{i} (i) the resolution in Theorem \cite{ii} (ii) may be described in terms of screening operators as in \cite{iii} provided that the existence of corresponding cycles is established, see e.g. \cite{iv}.

The following assertion is an immediate consequence of Theorem \cite{iii} which generalizes \cite{iii} Theorem 4.1.

Theorem 6.13. Let $k$ be an admissible number, $\lambda \in P^+_k$, $p \in \mathbb{Z}$. We have

$$H^{\mathfrak{g}}\mathfrak{l}(a, L(\lambda)) = \bigoplus_{w \in \mathcal{W}(\lambda)} C_{w \circ \lambda} \text{ as } \mathfrak{h}\text{-modules},$$

$$H^{\mathfrak{g}}\mathfrak{l}(L_0, L(\lambda)) = \bigoplus_{w \in \mathcal{W}(\lambda)} \pi_{w \circ \lambda + h^\vee \Lambda_0} \text{ as } \mathcal{H}\text{-modules}.$$
where $C_k$ is the one-dimensional representations of $\hat{\mathfrak{g}}[t] \oplus \mathbb{C}K$ on which $\hat{\mathfrak{g}}[t]$ acts trivially and $K$ acts as the multiplication by $k$. By \cite[Proposition 5.2]{result} we have an injective homomorphism of vertex algebras

$$V^k(\hat{\mathfrak{g}}) \hookrightarrow W(k\Lambda_0)$$

for all $k \in \mathbb{C}$. Hence $V^k(\hat{\mathfrak{g}})$ may be regarded as a vertex subalgebra of $W(k\Lambda_0)$.

Note that $L(k\Lambda_0)$ is the unique simple quotient of $V^k(\hat{\mathfrak{g}})$.

**Proposition 6.14.** Let $k$ be an admissible number, $\Psi : W(\hat{s}_0 \circ k\Lambda_0) \to W(k\Lambda_0)$ a non-zero $\mathfrak{g}$-homomorphism, which exists uniquely up to a nonzero constant multiplication by Theorem \cite[(i)]{result}. Then the image of the highest weight vector of $W(\hat{s}_0 \circ k\Lambda_0)$ generates the maximal submodule of $V^k(\hat{\mathfrak{g}}) \subset W(k\Lambda_0)$.

**Proof.** By \cite{result} the maximal submodule of $V^k(\hat{\mathfrak{g}})$ is generated by a singular vector $v$ of weight $\hat{s}_0 \circ k\Lambda_0$. Consider the two-sided resolution $C^\bullet(k\Lambda_0)$ of $L(k\Lambda_0)$ in \cite{result} (ii). Because it is a resolution of $L(k\Lambda_0)$ and $V^k(\hat{\mathfrak{g}}) \subset W(k\Lambda_0)$, the vector $v$ must be in the image of $d_{1,w} : W(w \circ k\Lambda_0) \to W(k\Lambda_0)$ for some $w \in W^1(k\Lambda_0)$. Since the weight $w \circ k\Lambda_0$ is strictly smaller than $\hat{s}_0 \circ k\Lambda_0$ for $w \in W^1(k\Lambda_0) \setminus \{\hat{s}_0\}$, the only possibility is that $v$ is the image of the highest weight vector of $W(\hat{s}_0 \circ k\Lambda_0)$. \qed

### 6.7. Two-sided BGG resolutions of more general admissible representations.

Let $\lambda \in Pr_{k,y}$ with $y = \hat{y}\eta$, $\hat{y} \in \check{W}$, $\eta \in Q^\vee$. Then there exists $\lambda_1 \in Pr_k^+$ such that $\lambda = y \circ \lambda_1$. Since $y(\Delta(\lambda_1)_+ \subset \Delta^\vee_+$, $T^0 : O^\hat{\mathfrak{g}}_{[\lambda]} \to O^\hat{\mathfrak{g}}_{[\lambda]}$ is exact,

$$T^0_L(\lambda_1) \cong L(\lambda),$$

$$T^0_W(w \circ \lambda_1) \cong T^0 \lim \limits_\leftarrow i M^{-\gamma_i}(w \circ \lambda_1) \cong \lim \limits_\leftarrow i T^0 M^{-\gamma_i}(w \circ \lambda_1)$$

$$\cong \lim \limits_\leftarrow i M^{-y(\gamma_i)}(w y^{-1} \circ \lambda) \cong W^\hat{\mathfrak{g}}(w y^{-1} \circ \lambda)$$

for $w \in W(\lambda_1) = y^{-1}W(\lambda)y$ by Proposition \cite{result}, Lemmas \cite{result} and \cite{result}, where $(\gamma_1, \gamma_2, \ldots,)$ is a sequence as in proof of Theorem \cite{result}. Therefore the following assertion follows immediately from Theorem \cite{result}.

**Theorem 6.15.** Let $k$ be an admissible number, $\lambda \in Pr_{k,y}$ with $y = \hat{y}\eta$, $\hat{y} \in \check{W}$, $\eta \in \check{P}^\vee$. Then there exists a complex

$$C^\bullet(\lambda) : \cdots \xrightarrow{d_{-2}} C^{-2}(\lambda) \xrightarrow{d_{-1}} C^{-1}(\lambda) \xrightarrow{d_{0}} C^0(\lambda) \xrightarrow{d_1} C^1(\lambda) \xrightarrow{d_2} \cdots$$

in the category $O$ of the form $C^i = \bigoplus_{w \in W^\hat{\mathfrak{g}}(w \circ \lambda)} W^\hat{\mathfrak{g}}(w \circ \lambda), d_i = \sum_{w \in W^\hat{\mathfrak{g}}(w \circ \lambda)} \sum_{w' \in W^{\hat{\mathfrak{g}}+1}(w \circ \lambda)} d_{w',w}$, such that

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

**Remark** 6.16. If $\lambda \in Pr_{k,y}$ and $\hat{y} = 1$ (that is, $y \in \check{P}^\vee$), then $W^\hat{\mathfrak{g}}(w \circ \lambda) = W(w \circ \lambda)$. Hence the above is the resolution of $L(\lambda)$ in terms of (non-twisted) Wakimoto modules as conjectured in \cite{result}.\]
7. Semi-infinite restriction and induction

7.1. Feigin-Frenkel parabolic induction. Let \( \hat{\mathfrak{p}} \) be a parabolic subalgebra of \( \hat{\mathfrak{g}} \) containing \( \hat{\mathfrak{b}} \), and let \( \mathfrak{p} = \mathfrak{i} \oplus \hat{\mathfrak{m}} \) be the direct sum decomposition of \( \mathfrak{p} \) with the Levi subalgebra \( \mathfrak{i} \) containing \( \hat{\mathfrak{h}} \) and the nilpotent radical \( \hat{\mathfrak{m}} \). Denote by \( \hat{\mathfrak{m}} \subset \hat{\mathfrak{n}} \) the opposite algebra of \( \mathfrak{m} \), so that \( \hat{\mathfrak{g}} = \mathfrak{p} \oplus \hat{\mathfrak{m}} \). Let

\[
\hat{\mathfrak{i}} = \mathfrak{i}_0 \oplus \bigoplus_{i=1}^{s} \mathfrak{i}_i
\]

be the decomposition of \( \hat{\mathfrak{i}} \) into direct sum of simple Lie subalgebras \( \mathfrak{i}_i, i = 1, \ldots, s \), and its center \( \mathfrak{i}_0 \) of \( \mathfrak{i} \). Let \( \hat{\mathfrak{h}}_i = \hat{\mathfrak{i}} \cap \hat{\mathfrak{h}} \), the Cartan subalgebra of \( \hat{\mathfrak{i}}_i \), and denote by \( \hat{\Delta}_i \subset \hat{\Delta} \) the subroot system of \( \hat{\mathfrak{g}} \) corresponding to \( \hat{\mathfrak{i}}_i \). Let \( \hat{\mathfrak{h}}_i = \hat{\mathfrak{h}}_i \cap \hat{\mathfrak{h}} \) the highest root of \( \hat{\Delta}_i \), \( \theta_i \) the highest root of \( \hat{\Delta}_i \), \( \theta_{i,s} \) the highest short root of \( \hat{\Delta}_i \).

Let \( \hat{l}_i = \hat{l}[t, t^{-1}] \oplus \mathbb{C} K \subset \mathfrak{g} \) for \( i = 0, 1, \ldots, s \). Set

\[
K_i = \frac{2}{(\theta_i | \theta_i)} K,
\]

and we consider \( K_i \) as an element of \( \mathfrak{l}_i \). Thus,

\[
\mathfrak{l}_i = \hat{l}_i[t, t^{-1}] \oplus \mathbb{C} K_i,
\]

and \( \mathfrak{h}_i = \hat{\mathfrak{h}}_i \oplus \mathbb{C} K_i \) is a Cartan subalgebra of \( \hat{\mathfrak{l}}_i \).

Define

\[
\mathfrak{l} = \bigoplus_{i=0}^{s} \mathfrak{l}_i, \quad \mathfrak{t} = \bigoplus_{i=0}^{s} \mathfrak{h}_i.
\]

The grading of \( \mathfrak{l}_i \) induces the grading of \( \mathfrak{l} \).

For \( k \in \mathbb{C} \) define \( k_0, \ldots, k_s \in \mathbb{C} \) by

\[
k_0 = k + \hat{h}, \quad k_i + \hat{h} = \frac{2}{(\theta_i | \theta_i)} (k + \hat{h}) \quad \text{for } i = 1, \ldots, s.
\]

Lemma 7.1. Let \( k \) be an admissible number for \( \mathfrak{g} \). Then \( k_i, i = 1, \ldots, s \), is an admissible number for the Kac-Moody algebra \( \mathfrak{l}_i \).

Let \( \mathcal{O}^l_{(k_0, \ldots, k_s)} \) be the full subcategory of \( \mathcal{O}^l \) consisting of objects on which \( K_i \) acts as the multiplication by \( k_i, i = 0, 1, \ldots, s \). Feigin and Frenkel \cite[5.2]{FF2}, \cite[6]{Fre2} constructed a functor

\[
\text{F-ind}_l^g : \mathcal{O}^l_{(k_0, k_1, \ldots, k_s)} \to \mathcal{O}_k^g, \quad M \mapsto \text{F-ind}_l^g(M),
\]

which enjoys the property

\[
\text{F-ind}_l^g(M) \cong \mathcal{U}S(L\hat{\mathfrak{m}}) \otimes_{\mathbb{C}} M
\]

as modules over

\[
L\hat{\mathfrak{m}} = \hat{\mathfrak{m}}[t, t^{-1}] \subset \mathfrak{g},
\]

where \( L\hat{\mathfrak{m}} \) only on the first factor \( \mathcal{U}S(L\hat{\mathfrak{m}}) \). In particular \( \text{F-ind}_l^g \) is an exact functor.
Denote by $W_t(\lambda^{(i)})$ the Wakimoto module of the affine Kac-Moody algebra $l_t$ with highest weight $\lambda^{(i)} \in \mathfrak{h}_t^*$ and by $L_t(\lambda^{(i)})$ the irreducible highest weight representation of $l_t$ with highest weight $\lambda^{(i)}$ (with a convention that $W_{t_0}(\lambda^{(0)})$ is the irreducible representation of the Heisenberg algebra $l_0$ with highest weight $\lambda^{(0)}$).

For $\lambda \in \mathfrak{t}^*$ let $W_t(\lambda)$ and $L_t(\lambda)$ be the Wakimoto module and the irreducible highest weight representation of $l$ with highest weight $\lambda$:

$$W_t(\lambda) = \bigotimes_{i=0}^s W_{t_i}(\lambda|_{\mathfrak{h}_{t_i}}), \quad L_t(\lambda) = \bigotimes_{i=0}^s L_{t_i}(\lambda|_{\mathfrak{h}_{t_i}}).$$

For $\lambda \in \mathfrak{h}^*$, define $\lambda_t \in \mathfrak{t}^*$ by

$$\lambda_t|_{\mathfrak{h}_t} = \lambda|_{\mathfrak{h}_t} \quad \text{and} \quad (\lambda_t + \rho_t)(K_t) = \frac{2}{(\theta_t|_{\mathfrak{h}_t})} (\lambda + \rho)(K)$$

for $i = 0, 1, \ldots, s$.

**Proposition 7.2** (Corollary 2.5). For $\lambda \in \mathfrak{h}^*$ we have $F\text{-ind}_{\mathfrak{h}_t}^{\mathfrak{h}} W_t(\lambda_t) \cong W(\lambda)$.

**Proof.** By using the Hochschild-Serre spectral sequence for $\tilde{L} \subset \mathfrak{a}$ we see from Corollary 2.5 that

$$H^p(\mathfrak{h}, F\text{-ind}_{\mathfrak{h}_t}^{\mathfrak{h}} W_t(\lambda_t)) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the assertion follows from Theorem 2.4. \qed

**7.2. Semi-infinite restriction functors.** Let $M \in \mathcal{O}^p_k$. Then $H^p(\tilde{L}, \mathfrak{m}, M)$, $p \in \mathbb{Z}$, is naturally an $l$-module on which $K_t$ acts as the multiplication by $k_t$, see e.g. [38, Proposition 2.3]. Hence

$$S\text{-res}_l^p := H^p(\tilde{L}, \mathfrak{m}, ?)$$

defines a functor $\mathcal{O}^p_k \rightarrow \mathcal{O}^p_{(k_0, k_1, \ldots, k_s)}$. We refer to $S\text{-res}_l^p$ as the *semi-infinite restriction functor*.

The following assertion follows from Proposition 2.3.

**Proposition 7.3.** For $\lambda \in \mathfrak{h}^*$ we have $H^p(\tilde{L}, \mathfrak{m}, W(\lambda)) = 0$ for $i \neq 0$ and

$$S\text{-res}_l^p W(\lambda) \cong W_t(\lambda_t).$$

**7.3. Decomposition of integral Weyl groups.** Let $k$ be an admissible number with denominator $q$, $\lambda \in Pr_k^+$. Let $\mathcal{W}_{S_i}$ be the parabolic subgroup of $\mathcal{W}$ corresponding to $l_i$, $\mathcal{W}_S = \mathcal{W}_{S_1} \times \mathcal{W}_{S_2} \times \cdots \times \mathcal{W}_{S_s}$. Define $a_0^{(i)} \in \Delta(\lambda)$, $i = 1, \ldots, s$, by

$$a_0^{(i)} = -\theta_i + q\delta \quad \text{if } (r\mathfrak{v}, q) = 1,$n

and $a_0^{(i)} \mathfrak{v} = -\theta_i + q\delta \quad \text{if } (r\mathfrak{v}, q) = r\mathfrak{v}$. Set $s_0^{(i)} = s_{a_0^{(i)}}$.

Let $\mathcal{W}(\lambda)_{S_i}$ be the subgroup of $\mathcal{W}(\lambda)$ generated by $\mathcal{W}_{S_i}$ and $s_0^{(i)}$. Then

$$\mathcal{W}(\lambda)_S = \mathcal{W}(\lambda)_{S_1} \times \mathcal{W}(\lambda)_{S_2} \times \cdots \times \mathcal{W}(\lambda)_{S_s}$$
is the subgroup corresponding to \( \hat{\mathfrak{m}}_{\tilde{S}} \) described in \( \S 3.3 \). Let \( \mathcal{W}(\lambda)^S \subset \mathcal{W}(\lambda) \) be as in Theorem 3.4 so that

\[
(51) \quad W(\lambda) = W(\lambda)_S \times W(\lambda)^S, \quad \ell^S_\lambda(uv) = \ell^S_\lambda(u) + \ell^S_\lambda(v) \quad \text{for} \quad u \in W(\lambda)_S, \quad v \in W(\lambda)^S.
\]

Let \( w, w' \in W(\lambda)_S \subset W(\lambda) \) such that \( w \triangleright \lambda \equiv w' \). Then \( w \circ_i \lambda^{(i)} = (w \circ \lambda)^{(i)} \), where \( \circ_i \) is the dot action of \( W(\lambda)_S \) on \( \mathfrak{h}_s^* \) and \( \lambda^{(i)}_i = \lambda_i |_{\mathfrak{h}_s} \).

**Proposition 7.4.** Let \( \lambda \in \text{Pr}^+_k \), \( w, w' \in W(\lambda)_S \), with \( i \in \{1, 2, \ldots, s\} \) such that \( w \triangleright \lambda \equiv w' \). Then the correspondence \( \Phi \mapsto \text{F-ind}^g_1(\Phi) \) defines a linear isomorphism

\[
\text{Hom}(W_i((w \circ \lambda)_i), W_i((w' \circ \lambda)_i)) \rightarrow \text{Hom}_g(W(w \circ \lambda), W(w' \circ \lambda)).
\]

The inverse map is given by \( \Psi \rightarrow \text{S-res}^g_\lambda(\Psi) \).

**Proof.** By Proposition 3.4 and Theorem 4.6 (i) both \( \text{Hom}(W_i((w \circ \lambda)_i), W_i((w' \circ \lambda)_i)) \) and \( \text{Hom}_g(W(w \circ \lambda), W(w' \circ \lambda)) \) are one-dimensional. The assertion follows since the correspondence \( \Phi \mapsto \text{F-ind}^g_1(\Phi) \) is clearly injective and \( \text{S-res}^g_\lambda(\text{F-ind}^g_1(\Phi)) = \Phi \).

### 7.4. Semi-infinite restriction of admissible affine vertex algebras

Since it is defined by the semi-infinite cohomology the space \( \text{S-res}^g_\lambda(V^k(\hat{\mathfrak{g}})) \) inherits a vertex algebra structure from \( V^k(\hat{\mathfrak{g}}) \), and we have a natural vertex algebra homomorphism

\[
\bigotimes_{i=0}^s V^{k_i}(\hat{1}_i) \rightarrow \text{S-res}^g_\lambda(V^k(\hat{\mathfrak{g}})),
\]

where \( V^{k_i}(\hat{1}_i) \) denote the universal affine vertex algebra associated with \( \hat{1}_i \) at level \( k_i \). By composing with the map \( \text{S-res}^g_\lambda(V^k(\hat{\mathfrak{g}})) \rightarrow \text{S-res}^g_\lambda(L(k\Lambda_0)) \) induced by the surjection \( V^k(\hat{\mathfrak{g}}) \rightarrow L(k\Lambda_0) \) this gives rise to a vertex algebra homomorphism

\[
(52) \quad \bigotimes_{i=0}^s V^{k_i}(1) \rightarrow \text{S-res}^g_\lambda(L(k\Lambda_0)).
\]

On the other hand there is a natural surjective homomorphism

\[
\bigotimes_{i=0}^s V^{k_i}(1) = \bigotimes_{i=0}^s L_i(k_i\Lambda_0)
\]

of vertex algebras, where \( L_i(k_i\Lambda_0) \) is the unique simple quotient of \( V^{k_i}(1) \).

**Theorem 7.5.** Let \( k \) be an admissible number. The vertex algebra homomorphism \( \bigotimes_{i=0}^s V^{k_i}(1) \rightarrow \bigotimes_{i=0}^s L_i(k_i\Lambda_0) \) factors through the vertex algebra homomorphism

\[
\bigotimes_{i=0}^s L_i(k_i\Lambda_0) \rightarrow \text{S-res}^g_\lambda(L(k\Lambda_0)).
\]

**Proof.** Put \( \lambda = k\Lambda_0 \) and let \( C^\bullet(\lambda) \) be the two-sided BGG resolution of \( L(k\Lambda_0) \) in Theorem 4.6. By the vanishing assertion of Proposition 3.4 the semi-infinite cohomology \( H^\sim_{\text{S-res}}(\mathcal{M}, L(\lambda)) \) is isomorphic to the cohomology of the complex \( \text{S-res}^g_\lambda(C^\bullet(\lambda)) \) obtained from \( C^\bullet(\lambda) \) applying the functor \( \text{S-res}^g_\lambda \). Thus \( \text{S-res}^g_\lambda(L(k\Lambda_0)) \) is isomorphic to the zero-th cohomology of the complex \( \text{S-res}^g_\lambda(C^\bullet(\lambda)) \).
Consider the map $C^{-1}(\lambda) \supset W(S^{(i)}_0 \circ \lambda) \xrightarrow{d_{-1}(\lambda)} W(\lambda) \subset C^0(\lambda)$ for $i = 1, \ldots, s$. By applying the functor $S$-$\text{res}^\theta$ this induces a non-zero homomorphism

$$W_i(S^{(i)}_0 \circ \iota_1, \lambda_1) \rightarrow W_i(\lambda_1)$$

by Proposition [7.6], and the image of the highest weight vector of $W_i(S^{(i)}_0 \circ \iota_1, \lambda_1)$ generates the maximal $l_1$-submodule of $V^k_i(\lambda_1) \subset W_i(\lambda_1)$ by Proposition [7.5]. It follows that the maximal $l$-submodule of $\bigotimes_{i=0}^s V^k_i(\lambda_1) \subset W_i(\lambda)$ is in the image of $S$-$\text{res}^\theta(C^{-1}(\lambda)) \rightarrow S$-$\text{res}^\theta(C^0(\lambda))$. This completes the proof. \hfill $\square$

7.5. The case of minimal parabolic subalgebras. Consider the case that $\widehat{\mathfrak{p}}$ is generated by $\widehat{\mathfrak{b}}_+$ and $e_i$ with $i \in \widehat{I}$. Then $\widehat{l} = \widehat{l}_0 \oplus \widehat{l}_1$, $\widehat{l}_1 = \mathfrak{sl}^{(i)}_2$ and $\widehat{l}_1 = \mathfrak{sl}^{(i)}_2$.

**Theorem 7.6 (\(\widehat{\mathfrak{p}}\) minimal).** Let $k$ be an admissible number and let $M$ be a module over the vertex algebra $L(k\lambda_0)$. Then, for each $\mu \in \mathbb{Z}$, $H^{\# \mu}(\mathfrak{m}, M)$ is a direct sum of admissible representations of level $k_1$ (see [3.8]) as $\mathfrak{sl}^{(i)}_2$-modules.

**Proof.** By Theorem [3.8], $L_{l_1}(k_1\lambda_0)$ is a vertex subalgebra of $S$-$\text{res}^\theta(L(k\lambda_0)) = H^{\# \mu(0)}(\mathfrak{m}, L(k\lambda_0))$. If $M$ is a module over $L(k\lambda_0)$, then $H^{\# \mu}(\mathfrak{m}, M)$ is naturally a module over $S$-$\text{res}^\theta(L(k\lambda_0))$, and therefore, it is a module over $L_{l_1}(k_1\lambda_0)$. The assertion follows since it is known by [3.8] that any module over $L_{l_1}(k_1\lambda_0)$ in the category $O$ must be a direct sum of admissible representations of $l_1 \cong \mathfrak{sl}^{(i)}_2$. \hfill $\square$

The following assertion generalizes [3.8], Theorem 3.8 in the case that $\widehat{\mathfrak{p}}$ is minimal.

**Theorem 7.7 (\(\widehat{\mathfrak{p}}\) minimal).** Let $k$ be an admissible number, $\lambda \in Pr^+_k$. Then

$$H^{\# \mu}(\mathfrak{m}, L(\lambda)) \cong \bigoplus_{w \in \mathcal{W}(\lambda)^S} L_{l_1}(w \circ \lambda)_1$$

as $l$-modules.

**Proof.** It is known by [3.8] (see also [3.8]) that $L(\lambda)$ with $\lambda \in Pr^+_k$ is a module over $L(k\lambda_0)$. Therefore $H^{\# \mu}(\mathfrak{m}, L(\lambda))$ is a direct sum of irreducible admissible representations as $\mathfrak{sl}^{(i)}_2$-modules by Theorem [3.8]. Hence it is sufficient to determine the subspace $H^{\# \mu(\lambda)}(\mathfrak{m}, L(\lambda))$ of the singular vectors of $H^{\# \mu}(\mathfrak{m}, L(\lambda))$. Clearly, any weight of $H^{\# \mu}(\mathfrak{m}, L(\lambda))$ must be admissible for $l_1 = \mathfrak{sl}^{(i)}_2$.

As remarked in the proof of Proposition [3.8], $H^{\# \mu}(\mathfrak{m}, L(\lambda))$ is the cohomology of the complex $S$-$\text{res}^\theta(C^*(\lambda))$ and we have $S$-$\text{res}^\theta(C^*(\lambda)) = \bigoplus_{w \in \mathcal{W}(\lambda)} W_i((w \circ \lambda)_1)$ by Proposition [3.8]. Now Theorem [3.8] and Lemma [3.8] imply that

$$\{(w \circ \lambda)_1; w \in \mathcal{W}(\lambda), \ (w \circ \lambda)_1 \text{ is an admissible weight for } \mathfrak{sl}^{(i)}_2 \}$$

$$= \{(w \circ \lambda)_1; w \in \mathcal{W}(\lambda), \ (w \circ \lambda)_1 \text{ is a dominant weight for } \mathfrak{sl}^{(i)}_2 \}$$

$$= \{(w \circ \lambda)_1; w \in \mathcal{W}(\lambda)^S \}.$$
of $W_i((w \circ \lambda))$ is nonzero in $H^{\ast +}(L^\circ_\lambda, L(\lambda))$ and $\{[(w \circ \lambda)_i] | w \in W(\lambda)^S\}$ forms a basis of $H^{\ast +}(L^\circ_\lambda, L(\lambda))^{1+}$. By Theorem \(\square\) this completes the proof.

Remark 7.8. In the subsequent paper \(\square\) we prove that for an admissible number $k$ any $L(k\lambda_0)$-module in the category $O^\bullet$ must be a direct sum of admissible representations. Hence it follows from the proof that the assertion of Theorem \(\square\) is valid for any parabolic subalgebra of $\hat{\mathfrak{g}}$.

References


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