TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

TOMOYUKI ARAKAWA

Abstract. We prove the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence of two-sided BGG resolutions of G-integrable admissible representations of affine Kac-Moody algebras at fractional levels. As an application we establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras which enables an inductive study of admissible representations.

1. Introduction

Wakimoto modules are representations of non-twisted affine Kac-Moody algebras introduced by Wakimoto [Wak] in the case of $\widehat{\mathfrak{sl}}_2$ and by Feigin and Frenkel [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In these applications it is important to have a resolution of an irreducible highest weight representation $L(\lambda)$ of an affine Kac-Moody algebra $\mathfrak{g}$ in terms of Wakimoto modules, that is, a complex

$$C^{\bullet}(\lambda) :\to C^{i-1}(\lambda) \overset{d_{i-1}}{\to} C^{i}(\lambda) \overset{d_i}{\to} C^{i+1}(\lambda) \to \dots$$

with a differential d_i which is a $\mathfrak{g}$-module homomorphism such that $C^i(\lambda)$ is a direct sum of Wakimoto modules and

$$H^i(C^{\bullet}(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a resolution has been proved by Feigin and Frenkel [FF2] for any integrable representations over arbitrary $\mathfrak{g}$ and by Bernard and Felder [BF] and Feigin and Frenkel [FF2] for any admissible representation [KW2] over $\widehat{\mathfrak{sl}}_2$. In their study of W-algebras Frenkel, Kac and Wakimoto [FKW, Conjecture 3.5.1] conjectured the existence of such a resolution for any principle admissible representations over arbitrary $\mathfrak{g}$. In this paper we prove the existence of a two-sided resolution in terms of Wakimoto modules for any $\mathring{\mathfrak{g}}$-integrable admissible representations over arbitrary $\mathfrak{g}$ (Theorem 6.11), where $\mathring{\mathfrak{g}}$ is the classical part of $\mathfrak{g}$. For a general principal admissible representation of $\mathfrak{g}$ we obtain the two-sided resolution in terms of twisted Wakimoto modules (Theorem 6.15).

Let us sketch the proof of our result briefly. By Fiebig's equivalence [Fie] the block of the category $\mathcal{O}$ of $\mathfrak{g}$ containing an admissible representation $L(\lambda)$ is equivalent to the block containing an integrable representation[1]. Therefore an admissible

This work is partially supported by JSPS KAKENHI Grant Number No. 20340007 and No. 23654006.

[1] In the case $L(\lambda)$ is a non-principal G-integrable admissible representation this is a block of another Kac-Moody algebra.

representation admits a usual BGG type resolution in terms of Verma modules by the result of [GL, RCW]. Hence the idea of Arkhipov [Ark1] is applicable in our situation: One can obtain a twisted BGG resolution of $L(\lambda)$ in terms of twisted Verma modules by applying the twisting functor T_w [Ark1] to the BGG resolution of $L(\lambda)$ as we have the "Borel-Weil-Bott" vanishing property [AS]

$$\mathcal{L}_i T_w L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = \ell(w), \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the integral Weyl group of λ and $\ell : \mathcal{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ is the length function, see Theorem 5.12. It remains to show that one can construct an inductive system of twisted BGG resolutions $\{B_w^{\bullet}(\lambda)\}$ of $L(\lambda)$ such that the complex $\varinjlim_{w} B_w^{\bullet}(\lambda)$ gives the required two-sided resolution of $L(\lambda)$, see §6 for the details.

We note that by applying the (generalized) quantum Drinfeld-Sokolov reduction functor [FKW, KRW] to the (duals of the) two-sided BGG resolutions of admissible representations we obtain resolutions of some of simple modules over W-algebras in terms of free field realizations due to the vanishing of the associated BRST cohomology [A1, A2, A3, A4, A5]. In particular we obtain two-sided resolutions of all the minimal series representations [FKW, A7] of the W-algebras associated with principal nilpotent elements in terms of free bosonic realizations.

As an application of the existence of two-sided BGG resolution for admissible representations we prove a semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras (Theorem 7.7). This result is important since it enable an inductive study of admissible representations, see our subsequent paper [A6].

This paper is organized as follows. In §2 we collect and prove some basic results about semi-infinite cohomology [Feĭ] and semi-regular bimodules [Vor1] which are needed for later use. In particular we establish an important property of semi-regular bimodules in Proposition 2.1. In §2 we collect basic results on the semi-infinite Bruhat ordering (or the generic Bruhat ordering) of an affine Weyl group defined by Lusztig [Lus] and study the semi-infinite analogue of parabolic subgroups. Semi-infinite Bruhat ordering is important for us since it (conjecturally) describes the space of homomorphisms between Wakimoto modules, see Proposition 4.10 and Conjecture 4.11. The semi-infinite analogue of the minimal (or maximal) length representatives (Theorem 3.3) is important for describing the semi-infinite restriction functors studied in §7. In §4 we define Wakimoto modules and twisted Verma modules following [Vor2] and study some of their basic properties. In particular we prove the uniqueness of Wakimoto modules which was stated in [FF2] without a proof (Theorem 4.7). In §5 we generalize the Borel-Weil-Bott vanishing property of the twisting functor established in [AS] to the affine Kac-Moody algebra cases. In §6 we state and prove the main results of this paper. In §7 we study the semi-infinite restriction functor and establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras. This is a non-trivial fact since admissible representations are not unitarizable unless they are integrable.

Acknowledgments. Some part of this work was done while the author was visiting Weizmann Institute, Israel, in May 2011, Emmy Noether Center in Erlangen, Germany in June 2011, Isaac Newton Institute for Mathematical Sciences, UK, in 2011, The University of Manchester, University of Birmingham, The University of Edinburgh, Lancaster University, York University, UK, in November 2011, Academia Sinica, Taiwan, in December 2011. He is grateful to those institutes for their hospitality.

2. Semi-regular bimodules and semi-infinite cohomology

2.1. Some notation. For $\mathbb{Z}$-graded vector spaces $M=\bigoplus_{n\in\mathbb{Z}}M_n$, $N=\bigoplus_{n\in\mathbb{Z}}N_n$ with finite-dimensional homogeneous components let

$$\begin{aligned}\mathcal{H}om_{\mathbb{C}}(M,N)=&\bigoplus_{n\in\mathbb{Z}}\mathcal{H}om_{\mathbb{C}}(M,N)_n,\\ &\mathcal{H}om_{\mathbb{C}}(M,N)_n=\{f\in\mathrm{Hom}_{\mathbb{C}}(M,N);f(M_i)\subset N_{i+n}\},\\ \mathcal{E}nd_{\mathbb{C}}(M)=&\mathcal{H}om_{\mathbb{C}}(M,M).\end{aligned}$$

We denote by $M^*=\bigoplus_{n\in\mathbb{Z}}(M^*)_n$ the space $\mathcal{H}om_{\mathbb{C}}(M,\mathbb{C})$, where $\mathbb{C}$ is considered as a graded vector space concentrated in the degree 0 component. If M, N are module over an algebra A we denote by $\mathcal{H}om_A(M,N)$ the space of all A-homomorphisms in $\mathcal{H}om_{\mathbb{C}}(M,N)$.

2.2. Semi-infinite structure. Let $\mathfrak{g}$ be a complex Lie algebra. A *semi-infinite structure* [Vor1] of $\mathfrak{g}$ is is the following data:

(i) a $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ of $\mathfrak{g}$ with finite-dimensional homogeneous components, $\dim_{\mathbb{C}}\mathfrak{g}_n<\infty$ for all n,
(ii) a *semi-infinite* 1*-cochain* $\gamma:\mathfrak{g}\to\mathbb{C}$.

Here by a semi-infinite 1-cochain we mean the following: Decompose $\mathfrak{g}$ into the direct sum of two subalgebras

$$\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-, \tag{1}$$

$$\mathfrak{g}_+=\bigoplus_{i\geq 0}\mathfrak{g}_i,\quad \mathfrak{g}_-=\bigoplus_{i<0}\mathfrak{g}_i. \tag{2}$$

A linear map $\gamma:\mathfrak{g}\to\mathbb{C}$ is called a semi-infinite 1-cochain if γ satisfies

$$\gamma([x,y])=\mathrm{tr}((\mathrm{ad}\,x)_{+-}(\mathrm{ad}\,y)_{-+}-(\mathrm{ad}\,y)_{+-}(\mathrm{ad}\,x)_{-+})\quad\text{for }x,y\in\mathfrak{g},$$

where $(\mathrm{ad}\,x)_{\pm\mp}$ denotes the composition $\mathfrak{g}_{\mp}\overset{\mathrm{ad}\,x}{\to}\mathfrak{g}\overset{\text{projection}}{\longrightarrow}\mathfrak{g}_{\pm}$.

In the rest of this section we assume that $\mathfrak{g}$ is equipped with a semi-infinite structure such that $\gamma(\sum_{i\neq 0}\mathfrak{g})=0$.

We denote by U, U_-, U_+, the enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}_+$, $\mathfrak{g}_-$ by respectively. These algebras inherit a $\mathbb{Z}$-grading from the corresponding Lie algebras.

Let $\tilde{\mathcal{O}}^{\mathfrak{g}}$ be the category of $\mathbb{Z}$-graded $\mathfrak{g}$-modules $M=\bigoplus_{n\in\mathbb{Z}}M_n$ with $\dim M_n<\infty$ for all m on which $\bigoplus_{j>0}\mathfrak{g}_+$ acts locally nilpotently and $\mathfrak{g}_0$ acts locally finitely.

2.3. Semi-infinite cohomology. Choose a basis $\{x_i;i\in\mathbb{Z}\}$ of $\mathfrak{g}$ such that $\{x_i;i\geq 0\}$ and $\{x_i;i<0\}$ are bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively, and let $\{c_{ij}^k\}$ be the structure constant: $[x_i,x_j]=\sum_k c_{ij}^k x_k$.

Denote by $\mathcal{C}l(\mathfrak{g})$ the Clifford algebra associated with $\mathfrak{g}\oplus\mathfrak{g}^*$, which has the following generators and relations:

$$\begin{aligned}&\text{generators: } \psi_i,\psi_i^* \quad \text{for } i\in\mathbb{Z},\\ &\text{relations: } \{\psi_i,\psi_j^*\}=\delta_{i,j},\ \{\psi_i,\psi_j\}=\{\psi_i^*,\psi_j^*\}=0.\end{aligned}$$

Here $\{X,Y\}=XY+YX$. The *space of the semi-infinite forms* $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ of $\mathfrak{g}$ is by definition the irreducible representation of $\mathcal{C}l(\mathfrak{g})$ generated by the vector $\mathbf{1}$ satisfying

$$\psi_i\mathbf{1}=0 \quad \text{for } i\geq 0, \quad \psi_i^*\mathbf{1}=0 \quad \text{for } i>0.$$

It is graded by $\deg\mathbf{1}=0$, $\deg\psi_i^*=1$ and $\deg\psi_i=-1$: $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})=\bigoplus_{p\in\mathbb{Z}}\bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})$.

For $A\in\mathcal{E}nd_{\mathbb{C}}(\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ of degree n set

$$:\psi_kA:=\begin{cases}\psi_kA & \text{if } k<0,\\ (-1)^nA\psi_k & \text{if } k\geq 0,\end{cases} \qquad :\psi_k^*A:=\begin{cases}\psi_k^*A & \text{if } k\leq 0,\\ (-1)^nA\psi_k^* & \text{if } k>0.\end{cases} \tag{3}$$

The following defines a $\mathfrak{g}$-module structure on $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$:

$$x_i\mapsto :\mathrm{ad}(x_i):+\gamma(x_i), \tag{4}$$

where

$$:\mathrm{ad}\,x_i:=\sum_{j,k}c_{ij}^k:\psi_k\psi_j^*:.$$

For $M\in\tilde{\mathcal{O}}^{\mathfrak{g}}$, define $d\in\mathrm{End}(M\otimes\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ by

$$d=\sum_{i\in\mathbb{Z}}x_i\otimes\psi_i^*-1\otimes\frac{1}{2}\sum_{i,j,k\in\mathbb{Z}}c_{ij}^k:\psi_i^*(:\psi_j^*\psi_k:):+1\otimes\sum_{i\in\mathbb{Z}}\gamma(x_i)\psi_i^*$$

Then

$$d^2=0,\quad d(M\otimes\bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g}))\subset M\otimes\bigwedge^{\frac{\infty}{2}+p+1}(\mathfrak{g}).$$

The cohomology of the complex $(M\otimes\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}),d)$ is called the *semi-infinite $\mathfrak{g}$-cohomology* with coefficients in M and denoted by $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},M)$ ([Feĭ, Vor1]).

2.4. **Semi-regular bimodules.** We consider the full dual space $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ of U as a U-bimodule by $(Xf)(u)=f(uX)$, $(fX)(u)=f(Xu)$ for $X\in\mathfrak{g}$, $f\in\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C})$, $u\in U$. The graded duals $U_\pm^*$ of $U_\pm$ are $\mathfrak{g}_\pm$-submodule of $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$. By abuse of notation we denote by U^* the image of the embedding $U_+^*\otimes_{\mathbb{C}}U_-^*\hookrightarrow\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$, $f_+\otimes f_-\mapsto(u_-u_+\mapsto f_+(u_+)f_-(u_-))$, $f_\pm\in U_\pm^*$, $u_\pm\in U$. Then U^* is a U-bisubmodule of $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ and coincides with the image of the embedding $U_-^*\otimes_{\mathbb{C}}U_+^*\hookrightarrow\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$, $f_-\otimes f_+\mapsto(u_+u_-\mapsto f_+(u_+)f_-(u_-))$.

Following [Vor2] define

$$US(\mathfrak{g})=H^{\frac{\infty}{2}+0}(\mathfrak{g},U^*\otimes_{\mathbb{C}}U),$$

where $\mathfrak{g}$ is given the opposite semi-infinite structure and the semi-infinite $\mathfrak{g}$-cohomology is taken with respect to the diagonal left action on $U^*\otimes_{\mathbb{C}}U$. Here by the opposite semi-infinite structure we mean the one obtained by replacing $\mathfrak{g}_\pm$ with $\mathfrak{g}_\mp$ and γ

with $-\gamma$. The space $US(\mathfrak{g})$ inherits the U-bimodule structure from $U^*\otimes U$ defined by

$$X(f\otimes u)=-(fX)\otimes u,\quad (f\otimes u)X=f\otimes(uX)$$

for $X\in\mathfrak{g}$, $\in U^*$, $u\in U$. The U-bimodule $US(\mathfrak{g})$ is called the *semi-regular bimodule* of $\mathfrak{g}$. One has

$$US(\mathfrak{g})\cong U_+^*\otimes_{U_+}U \tag{5}$$

as left $\mathfrak{g}_+$-modules and right $\mathfrak{g}$-modules, and the left $\mathfrak{g}$-module structure of $US(\mathfrak{g})$ is defined through the isomorphism

$$U_+\otimes_{U_-}U\cong\mathcal{H}om_{\mathbb{C}}(U_+,U)\cong\mathcal{H}om_{U_-}(U,U_-\otimes_{\mathbb{C}}\mathbb{C}_{-\gamma}) \tag{6}$$

([Vor1, Soe2, Vor2]).

Let M be a $\mathfrak{g}$-module and consider the following four left $\mathfrak{g}$-module structures on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$:

$$\pi_1(X)(s\otimes m)=-(sX)\otimes m+s\otimes Xm,,\quad \pi_2(X)(s\otimes m)=(Xs)\otimes m, \tag{7}$$

$$\pi_1'(X)(s\otimes m)=-(sX)\otimes m,\quad \pi_2'(X)(s\otimes m)=(Xs)\otimes m+s\otimes(Xm), \tag{8}$$

for $X\in\mathfrak{g}$, $s\in US(\mathfrak{g})$, $m\in M$. Clearly, the two actions π_1 and π_2 (resp. π_1' and π_2') commute.

Proposition 2.1 (cf. [AG, 6.4])**.** *For $M\in\tilde{\mathcal{O}}^{\mathfrak{g}}$ the two U-bimodule structures (π_1,π_2) and (π_1',π_2') on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ are equivalent. Namely there exists a linear isomorphism $\Phi: US(\mathfrak{g})\otimes_{\mathbb{C}}M\xrightarrow{\sim}US(\mathfrak{g})\otimes_{\mathbb{C}}M$ such that $\Phi\circ\pi_i'(X)=\pi_i(X)\circ\Phi$ for $i=1,2$, $X\in\mathfrak{g}$.*

Proof. Define the linear isomorphism

$$\tilde{\Phi}_1: U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M\xrightarrow{\sim}U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M$$

by $\tilde{\Phi}_1(f\otimes u\otimes m)=f\otimes(\Delta(u)(1\otimes m))$ for $f\in U^*$, $u\in U$, $m\in M$, where $\Delta: U\to U\otimes_{\mathbb{C}}U$ is the coproduct. We have

$$\tilde{\Phi}_1\circ\pi_{2,L}(X)=(\pi_{2,L}(X)+\pi_{3,L}(X))\circ\tilde{\Phi}_1$$
$$\tilde{\Phi}_1\circ(\pi_{2,R}(X)+\pi_{3,R}(X))=\pi_{2,R}(X)\circ\tilde{\Phi}_1,$$

where $\pi_{i,L}$ (resp. $\pi_{i,R}$) denotes the left action (resp. the right action) of $\mathfrak{g}$ on the i-th factor of $U^*\otimes U\otimes M$, and M is considered as a right $\mathfrak{g}$-module by the action $mx=-xm$ for $m\in M$, $x\in\mathfrak{g}$.

Next consider the graded dual $M^*=\bigoplus_n(M^*)_n$ as a right module by the action $(fX)(m)=f(Xm)$. Let

$$\Psi: U^*\otimes_{\mathbb{C}}M\xrightarrow{\sim}U^*\otimes_{\mathbb{C}}M$$

be the linear isomorphism defined by $\Psi(f\otimes m)(u\otimes g)=(f\otimes m)((1\otimes g)\Delta(u))$ for $f\in U^*$, $m\in M$, $u\in U$, $g\in M^*$, where M is identified with $(M^*)^*$. Extend this to the linear isomorphism

$$\tilde{\Phi}_2: U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M\xrightarrow{\sim}U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M$$

by setting $\tilde{\Phi}_2(f\otimes u\otimes m) = \sum_i f_i\otimes u\otimes m_i$ if $\Psi(f\otimes m) = \sum_i f_i\otimes m_i$ with $f_i \in U^*$, $m_i \in M$. Then

$$\tilde{\Phi}_2 \circ \pi_{1,R}(X) = (\pi_{1,R}(X) + \pi_{3,R}(X)) \circ \tilde{\Phi}_2,$$
$$\tilde{\Phi}_2 \circ (\pi_{1,L}(X) + \pi_{3,L}(X)) = \pi_{1,L}(X) \circ \tilde{\Phi}_2.$$

Set

$$\tilde{\Phi} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1 : U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M \xrightarrow{\sim} U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M.$$

Then

$$\begin{aligned} \tilde{\Phi} \circ (\pi_{1,L}(X) + \pi_{2,L}(X)) &= \tilde{\Phi}_2 \circ (\pi_{1,L}(X) + \pi_{2,L}(X) + \pi_{3,L}(X)) \circ \tilde{\Phi}_1 \\ &= (\pi_{1,L}(X) + \pi_{2,L}(X)) \circ \tilde{\Phi}, \end{aligned} \tag{9}$$

$$\tilde{\Phi} \circ (\pi_{2,R}(X) + \pi_{3,R}(X)) = \tilde{\Phi}_2 \circ \pi_{2,R}(X) \circ \tilde{\Phi}_1 = \pi_{2,R}(X) \circ \tilde{\Phi}, \tag{10}$$

$$\tilde{\Phi} \circ \pi_{1,R}(X) = \tilde{\Phi}_2 \circ \pi_{1,R}(X) \circ \tilde{\Phi}_1 = (\pi_{1,R}(X) + \pi_{3,R}(X)) \circ \tilde{\Phi}. \tag{11}$$

By (9) and the definition of $US(\mathfrak{g})$, $\tilde{\Phi}$ gives rise to a linear isomorphism

$$\Phi : US(\mathfrak{g})\otimes_{\mathbb{C}}M \xrightarrow{\sim} US(\mathfrak{g})\otimes_{\mathbb{C}}M.$$

Moreover Φ satisfies the required properties by (10) and (11). □

2.5. **Semi-infinite induction.** Let $\mathfrak{h} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_n$ be a graded Lie subalgebra of $\mathfrak{g}$ such that $\gamma|_{\mathfrak{h}}$ is a semi-infinite 1-cochain of $\mathfrak{h}$. Following [Vor2] we define the *semi-induced* $\mathfrak{g}$*-module* $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ as

$$\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M := H^{\frac{\infty}{2}+0}(\mathfrak{h}, US(\mathfrak{g})\otimes_{\mathbb{C}}M),$$

where $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ is considered as an $\mathfrak{h}$-module by the action π_1 defined in (7). The space $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ inherits the structure of a $\mathfrak{g}$-module from the action π_2 defined in (7).

Lemma 2.2. *The assignment* $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} : M \mapsto \text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ *defines an exact functor from* $\tilde{\mathcal{O}}^{\mathfrak{h}}$ *to* $\tilde{\mathcal{O}}^{\mathfrak{g}}$.

Proof. Clearly $\text{S-ind}\, M$ is an object of $\tilde{\mathcal{O}}^{\mathfrak{g}}$ since $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ is. By Proposition 2.1 we may replace the actions of π_1 and π_2 on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ with π_1' and π_2', simultaneously. It follows that

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{h}, US(\mathfrak{g})\otimes_{\mathbb{C}}M) \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{h}, US(\mathfrak{g}))\otimes_{\mathbb{C}}M. \tag{12}$$

Since $US(\mathfrak{g})$ is free over $\mathfrak{h}_-$ and cofree over $\mathfrak{h}_+$, $H^{\frac{\infty}{2}+i}(\mathfrak{h}, US(\mathfrak{g})) = 0$ for $i \neq 0$ by [Vor1, Theorem 2.1]. (Note that the spectral sequence on [Vor1] converges since the complex $US(\mathfrak{g})\otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{h})$ is a direct sum of finite-dimensional subcomplexes consisting of homogeneous vectors.) We have shown that the functor $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}$ is exact. □

In the case that $\mathfrak{h} = \mathfrak{g}$ and $\gamma_0 = \gamma$, we have the following assertion.

Proposition 2.3 ([Vor2, (1.9)]). *The functor* $\text{S-ind}_{\mathfrak{g}}^{\mathfrak{g}} : \tilde{\mathcal{O}}^{\mathfrak{g}} \to \tilde{\mathcal{O}}^{\mathfrak{g}}$ *is isomorphic to the identify functor.*

Proof. As $H^{\frac{\infty}{2}+0}(\mathfrak{g}, US(\mathfrak{g}))$ is isomorphic to the trivial representation $\mathbb{C}$ of $\mathfrak{g}$ ([Vor1, Theorem 2.1]), (12) gives the $\mathfrak{g}$-module isomorphism $\text{S-ind}_{\mathfrak{g}}^{\mathfrak{g}} M \cong M$ as required. □

2.6. Suppose that $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$$

with graded subalgebras $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ such that the restrictions $\gamma|_{\mathfrak{a}}$ and $\gamma|_{\bar{\mathfrak{a}}}$ of γ are semi-infinite 1-cochains of $\mathfrak{a}$ and $\bar{\mathfrak{a}}$, respectively.

Lemma 2.4. *$US(\mathfrak{g}) \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}})$ as left $\mathfrak{a}$-modules and right $\bar{\mathfrak{a}}$-modules.*

Proof. We have $U_+^* \cong U(\mathfrak{a}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+)^*$ as left $\mathfrak{a}_+$-modules and right $\bar{\mathfrak{a}}_+$-modules. Consider the composition

$$\begin{aligned} US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) &\xrightarrow{\sim} (U(\mathfrak{a}_-) \otimes_{\mathbb{C}} U(\mathfrak{a}_+)^*) \otimes_{\mathbb{C}} (U(\bar{\mathfrak{a}}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_-)) \\ &\xrightarrow{\sim} U(\mathfrak{a}_+) \otimes_{\mathbb{C}} U_+^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+) \to US(\mathfrak{g}), \end{aligned}$$

where the last map is the multiplication map. From the description (5), (6) of the $\mathfrak{g}$-bimodule structure of semi-regular bimodules one sees that the image of the above map is stable under the left and the right action of $\mathfrak{g}$ on $US(\mathfrak{g})$. Hence the image must coincides with $US(\mathfrak{g})$ since it contains U_+^*. By the equality of the graded dimensions it follows that above map is an isomorphism. □

Lemma 2.5. *For $M \in \tilde{\mathcal{O}}^{\bar{\mathfrak{a}}}$, $\text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M$ as $\mathfrak{a}$-modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ of $US(\mathfrak{a}) \otimes_{\mathbb{C}} M$.*

Proof. We have

$$\begin{aligned} \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}}(M) &\cong H^{\frac{\infty}{2}+0}(\bar{\mathfrak{a}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) \otimes_{\mathbb{C}} M) \\ &\cong US(\mathfrak{a}) \otimes_{\mathbb{C}} \text{S-ind}_{\bar{\mathfrak{a}}}^{\bar{\mathfrak{a}}}(M) \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M \end{aligned}$$

by Lemmas 2.3 and 2.4. □

3. Semi-infinite Bruhat ordering

3.1. **Affine Kac-Moody algebras and affine Weyl groups.** We first fix some notation which are used for the rest of the paper.

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, and fix a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. Let $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ be the set of roots of $\mathring{\mathfrak{g}}$. Choose a subset $\Delta_+ \subset \mathring{\mathfrak{h}}^*$ of positive roots and the set $\mathring{\Pi} = \{\alpha_i; i \in \mathring{I}\} \subset \Delta_+$, $\mathring{I} = \{1, 2, \dots l\}$, of simple roots. Let θ be the highest root, θ_s the highest short root, $\Delta_- = -\Delta_+$, $\mathring{\rho} = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha$. Let $\mathring{Q} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha \subset \mathring{\mathfrak{h}}^*$, the root lattice of $\mathring{\mathfrak{g}}$, Set $\mathring{\mathfrak{n}} = \bigoplus_{\alpha \in \mathring{\Delta}_+} \mathring{\mathfrak{g}}_\alpha$, $\mathring{\mathfrak{n}}_- = \bigoplus_{\alpha \in \mathring{\Delta}_-} \mathring{\mathfrak{g}}_\alpha$, where $\mathring{\mathfrak{g}}_\alpha$ is the root space of $\mathring{\mathfrak{g}}$ with root α. We have the triangular decomposition

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}.$$

Let $(\,|\,)$ be the normalized invariant bilinear form of $\mathring{\mathfrak{g}}$. We identify $\mathring{\mathfrak{h}}$ with $\mathring{\mathfrak{h}}^*$ using $(\,|\,)$. Let $\mathring{\Delta}^\vee = \{\alpha^\vee; \alpha \in \mathring{\Delta}\}$, the set of coroots , $\mathring{Q}^\vee = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha^\vee \subset \mathring{\mathfrak{h}} = \mathring{\mathfrak{h}}^*$, the coroot lattice of $\mathring{\mathfrak{g}}$, $\mathring{\rho}^\vee = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha^\vee$, where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$.

Let $\overset{\circ}{\mathcal{W}} \subset GL(\overset{\circ}{\mathfrak{h}}{}^*)$ be the Weyl group of $\overset{\circ}{\mathfrak{g}}$, $s_\alpha \in \overset{\circ}{\mathcal{W}}$ be the reflection corresponding to $\alpha \in \Delta$: $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha$.

Let $\mathfrak{g}$ be the affine Kac-Moody algebra associated with $\overset{\circ}{\mathfrak{g}}$:

$$\mathfrak{g} = \overset{\circ}{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The commutation relations of $\mathfrak{g}$ are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \mathfrak{g}] = 0, \quad [D, xt^n] = nxt^n.$$

We consider $\overset{\circ}{\mathfrak{g}}$ as a subalgebra of $\mathfrak{g}$ by the natural embedding $\overset{\circ}{\mathfrak{g}} \hookrightarrow \mathfrak{g}$, $x \mapsto xt^0$. Let

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

the Cartan subalgebra of $\mathfrak{g}$. The bilinear form $(\ |\ )$ from $\overset{\circ}{\mathfrak{h}}$ to $\mathfrak{h}$ by letting $(K|\overset{\circ}{\mathfrak{h}}) = (D|\overset{\circ}{\mathfrak{h}}) = (K|K) = (D|D) = 0$ and $(D|K) = 1$. We identify $\overset{\circ}{\mathfrak{h}}{}^*$ with the subspace of $\mathfrak{h}^*$ consisting of elements which vanishes on $\mathbb{C}K \oplus \mathbb{C}D$. Thus,

$$\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}{}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

where Λ_0 and δ are defined by $\Lambda_0(K) = \delta(D) = 1$, $\Lambda_0(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}\delta) = \delta(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K) = 0$. The number $\langle \lambda, K\rangle$ is called the *level* of λ.

Let $\Delta^{re}_+ = \overset{\circ}{\Delta}_+ \sqcup \{\alpha + n\delta; \alpha \in \overset{\circ}{\Delta},\ n \in \mathbb{N}\}$, the set of positive real roots of $\mathfrak{g}$, $\Delta^{re}_- = -\Delta^{re}_+$, $\Delta^{re} = \Delta^{re}_+ \sqcup \Delta^{re}_-$ the set of real roots, $\Pi = \{\alpha_0 = -\theta + \delta, \alpha_1, \dots, \alpha_\ell\}$ the set of simple roots.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$, or the *affine Weyl group* of $\overset{\circ}{\mathcal{W}}$. We have

$$\mathcal{W} = \overset{\circ}{\mathcal{W}} \ltimes \overset{\circ}{Q}{}^\vee.$$

The *extended affine Weyl group* $\mathcal{W}^e$ of $\mathfrak{g}$ is the semidirect product

$$\mathcal{W}^e = \overset{\circ}{\mathcal{W}} \ltimes P^\vee$$

where $\overset{\circ}{P}{}^\vee = \{\lambda \in \overset{\circ}{\mathfrak{h}}; \langle \alpha, \lambda\rangle \in \mathbb{Z} \text{ for all } \alpha \in \overset{\circ}{\Delta}\}$, the coweight lattice of $\overset{\circ}{\mathfrak{g}}$. We have

$$\mathcal{W}^e = \mathcal{W}^e_+ \ltimes \mathcal{W},$$

where $\mathcal{W}^e_+$ subgroup of $\mathcal{W}^e$ consisting of elements which fix the set Π.

We denote by t_α or simply by α for the element of $\mathcal{W}^e$ corresponding to $\alpha \in \overset{\circ}{P}{}^\vee$. The reflection s_α corresponding $\alpha = \bar{\alpha} + n\delta \in \Delta^{re}$ is given by $s_\alpha = t_{-n\bar{\alpha}^\vee} s_{\bar{\alpha}}$. We set $s_i = s_{\alpha_i}$ for $i \in I := \{0, 1, \dots, l\}$, so that $\mathcal{W} = \langle s_i; i \in I\rangle$. The action of $\overset{\circ}{\mathcal{W}}$ on $\overset{\circ}{\mathfrak{h}}{}^*$ is extended to the action of $\mathcal{W}^e$ on $\mathfrak{h}^*$ by

$$\begin{aligned}
&w(\Lambda_0) = \Lambda_0,\ w(\delta) = \delta \quad w \in \overset{\circ}{\mathcal{W}},\\
&t_\alpha(\lambda) = \lambda + \langle \Lambda, K\rangle\alpha - (\langle \lambda, \alpha\rangle + \frac{(\alpha|\alpha)}{2}\langle \lambda, K\rangle)\delta, \quad \lambda \in \mathfrak{h}^*.
\end{aligned}$$

For $\lambda \in \mathfrak{h}^*$ let $\bar{\lambda} \in \overset{\circ}{\mathfrak{h}}{}^*$ be its restriction to $\overset{\circ}{\mathfrak{h}}$.

3.2. **Twisted Bruhat ordering.** Let $\ell : \mathcal{W}^e \to \mathbb{Z}_{\geq 0}$ the length function: $\ell(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}))$. We have

$$\ell(t_\mu y) = \sum_{\alpha \in \Delta_+ \cap y(\Delta_+)} |(\alpha|\mu)| + \sum_{\alpha \in \Delta_+ \cap y(\Delta_-)} |1 - (\alpha|\mu)| \tag{13}$$

for $\mu \in \mathring{P}^\vee$, $y \in \mathring{\mathcal{W}}$.

The *twisted length function* [Ark1] $\ell^y : \mathcal{W}^e \to \mathbb{Z}$ with the twist $y \in \mathcal{W}^e$ is defined by

$$\ell^y(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_+^{re})) - \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_-^{re})).$$

Lemma 3.1. *Let $w, y \in \mathcal{W}^e$.*

(i) *Suppose that $\ell(ys_i) = \ell(y) + 1$ for $i \in I$. Then*

$$\ell^{ys_i}(w) = \begin{cases} \ell^y(w) & \text{if } w^{-1}y(\alpha_i) \in \Delta_+^{re}, \\ \ell^y(w) - 2 & \text{if } w^{-1}y(\alpha_i) \in \Delta_-^{re}. \end{cases}$$

(ii) $\ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1})$.

Proof. (i) The assertion follows from the definition and the fact that

$$\Delta_+^{re} \cap ys_i(\Delta_-^{re}) = \Delta_+^{re} \cap y(\Delta_-^{re}) \sqcup \{y(\alpha_i)\} \quad \text{if } \ell(ys_i) = \ell(y) + 1.$$

(ii) We prove by induction on $\ell(y)$. If $\ell(y) = 0$ then $\ell^y(w) = \ell(w) = \ell(y^{-1}w)$. Suppose that $\ell(ys_i) = \ell(y) + 1$. If $w^{-1}y(\alpha_i) \in \Delta_+^{re}$ then $\ell(s_i y^{-1} w) = \ell(y^{-1}w) + 1$. Hence by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}) = \ell(s_i y^{-1} w) - \ell(s_i y^{-1}).$$

If $w^{-1}y(\alpha_i) \in \Delta_-^{re}$ then $\ell(s_i y^{-1} w) = \ell(y^{-1}w) - 1$. Again by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) - 2 = \ell(y^{-1}w) - 2 - \ell(y^{-1}) = \ell(s_i y^{-1} w) - \ell(s_i y^{-1}).$$

This completes the proof. $\square$

For $w_1, w_2, y \in \mathcal{W}$ and $\gamma \in \Delta^{re}$, write $w_1 \xrightarrow[y]{\gamma} w_2$ if $w_1 = s_\gamma w_2$ and $\ell^y(w_1) > \ell^y(w_2)$. Below, we shall often omit the symbol γ above the arrow. Also, we shall omit the symbol y under the arrow if $y = 1$. By Lemma 3.1 (ii) we have

$$w_1 \xrightarrow[y]{y(\gamma)} w_2 \iff y^{-1}w_1 \xrightarrow{\gamma} y^{-1}w_2. \tag{14}$$

In particular for $\beta = y(\alpha_i) \in \Delta_+^{re}$, $\alpha_i \in \Pi$, and $w_1, w_2 \in \mathcal{W}$ such that $\ell^y(w_2) - \ell^y(w_1) = 1$ we have the equivalence

$$\begin{tikzcd} & s_\beta w_1 \\ w_1 \arrow[ur, "y"'] \arrow[dr, "y"'] & \\ & w_2 \end{tikzcd} \iff \begin{tikzcd} s_\beta w_1 \arrow[dr, "y"'] & \\ & s_\beta w_2 \\ w_2 \arrow[ur, "y"'] & \end{tikzcd} \tag{15}$$

by [BGG, Lemma 11.3].

Define $w \succeq_y w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \xrightarrow[y]{} w_1 \xrightarrow[y]{} w_2 \xrightarrow[y]{} \dots \xrightarrow[y]{} w_k \xrightarrow[y]{} w'.$$

Note that

$$w \succeq_y w' \iff y^{-1}w \succeq y^{-1}w', \tag{16}$$

by (14), where $\succeq = \succeq_1$, the usual Bruhat ordering of $\mathcal{W}$. It follows that $\succeq_y$ defines a partial ordering of $\mathcal{W}$.

We will use the symbol $w \rhd_y w'$ to denote a covering in the twisted Bruhat order $\succeq_y$. Thus $w \rhd_y w'$ means that $w \succeq_y w'$ and $\ell^y(w) = \ell^y(w') + 1$.

3.3. **Semi-infinite Bruhat ordering.** Define the *semi-infinite length* [FF2] $\ell^{\frac{\infty}{2}}(w)$ of $w \in \mathcal{W}^e$ by

$$\ell^{\frac{\infty}{2}}(w) = \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \mathring{\Delta}_+\} - \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \mathring{\Delta}_-\}.$$

We have

$$\ell^{\frac{\infty}{2}}(t_\lambda y) = \ell(y) - 2(\mathring{\rho}|\lambda) \tag{17}$$

for $\lambda \in \mathring{P}^\vee$, $w \in \mathring{\mathcal{W}}$.

Set

$$\mathring{P}_+^\vee = \{\lambda \in \mathring{P}^\vee; \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \mathring{\Delta}_+\},$$

We say that $\lambda \in \mathring{P}_+^\vee$ is sufficiently large if $\alpha(\lambda)$ if sufficiently large for all $\alpha \in \mathring{\Delta}_+$.

By (13) and (17) we have the following assertion.

Lemma 3.2. $\ell^{\frac{\infty}{2}}(w) = \ell^\lambda(w) = -\ell^{-\lambda}(w)$ *for a sufficiently large* $\lambda \in \mathring{P}_+^\vee$.

We write

$$w_1 \xrightarrow[\frac{\infty}{2}]{\gamma} w_2$$

for $w_1, w_2 \in \mathcal{W}$ and $\gamma \in \Delta^{re}$ if $w_1 = w_2 s_\gamma$ and $\ell^{\frac{\infty}{2}}(w_1) < \ell^{\frac{\infty}{2}}(w_2)$. (We shall often omit the symbol γ above the arrow.) Define $w \succeq_{\frac{\infty}{2}} w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \xrightarrow[\frac{\infty}{2}]{} w_1 \xrightarrow[\frac{\infty}{2}]{} w_2 \xrightarrow[\frac{\infty}{2}]{} \dots \xrightarrow[\frac{\infty}{2}]{} w_k \xrightarrow[\frac{\infty}{2}]{} w'.$$

By Lemma 3.2

$$\begin{aligned} w \succeq_{\frac{\infty}{2}} w' &\iff w' \succeq_{t_\lambda} w \quad \text{for a sufficiently large } \lambda \in \mathring{P}_+^\vee, \\ &\iff w \succeq_{t_{-\lambda}} w' \quad \text{for a sufficiently large } \lambda \in \mathring{P}_+^\vee. \end{aligned}$$

It follows that $\succeq_{\frac{\infty}{2}}$ defines a partial ordering of $\mathcal{W}$. Following Arkhipov [Ark1], we call it the *semi-infinite Bruhat ordering* on $\mathcal{W}$. By [Soe1, Claim 4.14] the semi-infinite Bruhat ordering coincides with the *generic Bruhat ordering* defined by Lusztig [Lus].

We will use the symbol $w \rhd_{\frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\frac{\infty}{2}}$. Thus $w \rhd_{\frac{\infty}{2}} w'$ means that $w \succeq_{\frac{\infty}{2}} w'$ and $\ell^{\frac{\infty}{2}}(w) = \ell^{\frac{\infty}{2}}(w') - 1$.

3.4. **Semi-infinite analogue of parabolic subgroups and minimal (maximal) length representatives.** Let S be a subset of $\mathring{\Pi}$, $\mathring{\Delta}_S$ the subroot system of $\mathring{\Delta}$ generated by $\alpha_i \in S$, $\mathring{\Delta}_S = \bigsqcup_{i=1}^r \mathring{\Delta}_{S,i}$ the decomposition into the simple subroot systems $\mathring{\Delta}_{1,S}, \dots, \mathring{\Delta}_{r,S}$. Let θ_i be the longest root of $\mathring{\Delta}_{S,i}$.

Set

$$\Delta_S = \{\alpha + n\delta \in \Delta^{re}; \alpha \in \mathring{\Delta}_S, n \in \mathbb{Z}\}, \quad \mathcal{W}_S = \langle s_\alpha; \alpha \in \Delta_S \rangle \subset \mathcal{W}.$$

Then Δ_S is a subroot system of Δ^{re} isomorphic to the affine root system associated with $\mathring{\Delta}_S$. Put $\Delta_{S,+} = \Delta_S \cap \Delta_+^{re}$, the set of positive root of Δ_S. Then $\Pi_S = S \sqcup \{-\theta_1 + \delta, \dots, -\theta_s + \delta\}$ is a set of simple roots of Δ_S. We have $\mathcal{W}_S = \mathring{\mathcal{W}}_S \ltimes t_{\mathring{Q}^\vee{}_S}$, where $\mathring{Q}_S^\vee = \sum_{\alpha \in \mathring{\Delta}_S} \mathbb{Z}\alpha^\vee$. By (17), the restriction of the semi-infinite length function to $\mathcal{W}_S$ coincides with the semi-infinite length function of the affine Weyl group $\mathcal{W}_S$.

Define

$$\mathcal{W}^S = \{w \in \mathcal{W}; w^{-1}(\Delta_{S,+}) \subset \Delta_+^{re}\}.$$

Theorem 3.3 ([Pet]). *The multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$, $(u,v) \mapsto uv$, is a bijection. Moreover, we have*

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v) \quad \textit{for } u \in \mathcal{W}_S,\ v \in \mathcal{W}^S.$$

Proof. First, we show the injectivity of the multiplication map. Suppose that $u_1 v_1 = u_2 v_2$ with $u_i \in \mathcal{W}_S$, $v_i \in \mathcal{W}^S$. Then $v_1 = u v_2$ with $u = u_1^{-1} u_2 \in \mathcal{W}_S$. If $u \neq 1$ then there exists $\alpha \in \Delta_{S,+}$ such that $u^{-1}(\alpha) \in -\Delta_{S,+}$. But then $v_2 \in \mathcal{W}^S$ implies that $v_1^{-1}(\alpha) = v_2^{-1} u^{-1}(\alpha) \in \Delta_-^{re}$, and this contradicts that $v_1 \in \mathcal{W}^S$. Hence $u_1 = u_2$, and so $v_1 = v_2$.

Second, we show that the multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$ is surjective. We will prove by induction on $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re})$ that there exists $u \in \mathcal{W}_S$ such that $u^{-1}w \in \mathcal{W}^S$. If $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) = 0$, $w \in \mathcal{W}^S$ there is nothing to show. Suppose that $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) > 0$. Then there exists $\beta \in \Pi_S$ such that $w^{-1}(\beta) \in \Delta_-^{re}$. Indeed, any element $\alpha \in \Delta_{S,+}$ is expressed as $\alpha = \sum_{\beta \in \Pi_S} n_\beta \beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Thus $w^{-1}(\alpha) = \sum_{\beta \in \Pi_S} n_\beta w^{-1}(\beta) \in \Delta_-^{re}$ implies that one of $w^{-1}(\beta)$ must belong to Δ_-^{re}. Now because $(s_\beta w)^{-1}(\Delta_{S,+}) = w^{-1} s_\beta(\Delta_{S,+}) = w^{-1}(\Delta_{S,+} \backslash \{\beta\} \sqcup \{-\beta\}) = w^{-1}(\Delta_{S,+}) \backslash \{w^{-1}(\beta)\} \sqcup \{-w^{-1}(\beta)\}$,

$$(s_\beta w)^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} = w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} \backslash \{w^{-1}(\beta)\}.$$

Hence by applying the induction hypothesis to $s_\beta w$ we find an element $u \in \mathcal{W}_S$ such that $u^{-1} s_\beta w \in \mathcal{W}^S$.

Finally, we prove the equality of the semi-infinite length. By (17), we have $\ell^{\frac{\infty}{2}}(t_\mu w) = \ell^{\frac{\infty}{2}}(t_\mu) + \ell^{\frac{\infty}{2}}(w)$ for any $\mu \in \mathring{Q}^\vee$. Hence we may assume that $u \in \mathring{\mathcal{W}}_S$. We will prove by induction on the length $\ell(u)$ of $u \in \mathring{\mathcal{W}}_S$ that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ for any $v \in \mathcal{W}^S$. Suppose that $\ell(u) = 1$, so that $u = s_i$ for some $\alpha_i \in S$. Let

$v=t_\mu y\in\mathcal{W}^S$ with $\mu\in\mathring{Q}^\vee$, $y\in\mathring{\mathcal{W}}$. Note that $v\in\mathcal{W}^S$ is equivalent to that

$$\text{if } \alpha\in\mathring{\Delta}_{S,+} \text{ then } \quad \alpha(\mu)=\begin{cases}0 & \text{if } y^{-1}(\alpha)\in\mathring{\Delta}_+,\\ 1 & \text{if } y^{-1}(\alpha)\in\mathring{\Delta}_-.\end{cases} \tag{18}$$

Since

$$\ell^{\frac{\infty}{2}}(s_i t_\mu y)=\ell(t_{s_i(\mu)}s_i y)=\ell(s_i y)-2(\rho|\mu-\alpha_i(\mu)\alpha_i^\vee)=\ell(s_i y)-2(\rho|\mu)+2\alpha_i(\mu),$$

(18) implies that $\ell^{\frac{\infty}{2}}(s_i v)=\ell^{\frac{\infty}{2}}(v)+1$. Next let $u=s_i u_1\in\mathring{\mathcal{W}}_S$ with $u_1\in\mathring{\mathcal{W}}_S$, $\alpha_i\in S$, $\ell(u)=\ell(u_1)+1$, so that $u_1^{-1}(\alpha_i)\in\mathring{\Delta}_+$. Let $v=t_\mu y\in\mathcal{W}^S$ as above. We have

$$\ell^{\frac{\infty}{2}}(uv)=\ell^{\frac{\infty}{2}}(t_{s_i u_1(\mu)}s_i u_1 y)=\ell(s_i u_1 y)-2(\rho|s_i u_1(\mu)).$$

If $\ell(s_i u_1 y)=\ell(u_1 y)+1$, then $\mathring{\Delta}_+\ni(u_1 y)^{-1}(\alpha_i)=y^{-1}(u_1^{-1}(\alpha_i))$. Hence $(\mu|u_1^{-1}(\alpha_i))=0$ by (18), which means $s_i u_1(\mu)=u_1(\mu)$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv)=\ell^{\frac{\infty}{2}}(u)+\ell^{\frac{\infty}{2}}(v)$. If $\ell(s_i u_1 y)=\ell(u_1 y)-1$, then $\mathring{\Delta}_-\ni(u_1 y)^{-1}(\alpha_i)=y^{-1}(u_1^{-1}(\alpha_i))$. So (18) gives $(\mu|u_1^{-1}(\alpha_i))=1$, which means $s_i u_1(\mu)=u_1(\mu)-\alpha_i^\vee$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv)=\ell^{\frac{\infty}{2}}(u)+\ell^{\frac{\infty}{2}}(v)$ as required. □

4. Wakimoto modules and twisted Verma modules

4.1. The category $\mathcal{O}$ of $\mathfrak{g}$. For any $\mathfrak{h}$-module M we set $M_\mu=\{m\in M; hm=\mu(h)m \text{ for all } h\in\mathfrak{h}\}$.

Let $\mathcal{O}^{\mathfrak{g}}$ be the full subcategory of $\tilde{\mathcal{O}}^{\mathfrak{g}}$ consisting of modules on which $\mathfrak{h}$ acts semisimply. The formal character of $M\in\mathcal{O}^{\mathfrak{g}}$ is defined by

$$\operatorname{ch} M=\sum_{\mu\in\mathfrak{h}^*}(\dim_{\mathbb{C}} M_\mu)e^\mu.$$

Let $\mathcal{O}_k^{\mathfrak{g}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects of level k, where a $\mathfrak{g}$-module M is said to be of level k if K acts as the multiplication by k.

4.2. Twisting functors and twisted Verma modules. By abuse of notation we denote also by w a Tits lifting of $w\in\mathcal{W}^e$ to $\operatorname{Aut}(\mathfrak{g})$.

For each $w\in\mathcal{W}$ the twisting functor $T_w:\mathcal{O}^{\mathfrak{g}}\to\mathcal{O}^{\mathfrak{g}}$ is defined as follows ([Ark1]): Let $\mathfrak{n}_w=\mathfrak{n}_-\cap w^{-1}(\mathfrak{n}_+)$ and set $N_w=U(\mathfrak{n}_w)$. Put

$$S_w=U\otimes_{N_w}N_w^*.$$

The space S_w has a U-bimodule structure, which is described as follows: Let $f\in\mathfrak{n}_-\backslash\{0\}$, and set $U_{(f)}=U\otimes_{\mathbb{C}[f]}\mathbb{C}[f,f^{-1}]$. Then $U_{(f)}$ is an associative algebra which contains U as a subalgebra. We set $S_f=U_{(f)}/U$. Choose a filtration $\mathfrak{n}_w=F^0\supset F^1\supset\cdots\supset F^r\supset 0$, $r=\ell(w)$, consisting of ideals $F^p\subset\mathfrak{n}_w$ of codimension p. If $f_p\in F^{p-1}\backslash F^p$ we have an isomorphism of U-bimodules

$$S_w=S_{f_1}\otimes_U S_{f_2}\otimes_U\ldots\otimes_U S_{f_r}. \tag{19}$$

We have

$$S_w\cong N_w^*\otimes_{N_w}U \tag{20}$$

as right U-modules and left N_w-modules. Put

$$\mathbf{1}_w^* = f_1^{-1} \otimes f_2^{-1} \otimes \ldots \otimes f_r^{-1} \in S_w.$$

For $M \in \mathcal{O}^{\mathfrak{g}}$ define

$$T_w(M) = \phi_w(S_w \otimes_{U(\mathfrak{g})} M),$$

where ϕ_w means that the action of $\mathfrak{g}$ is twisted by the automorphism w of $\mathfrak{g}$. This define a right exact functor $T_w : \mathcal{O}^{\mathfrak{g}} \to \mathcal{O}^{\mathfrak{g}}$ such that

$$T_{ws_i} \cong T_w T_i \quad \text{if } \alpha_i \in \Pi \text{ and } \ell(ws_i) = \ell(w) + 1, \tag{21}$$

where $T_i = T_{s_i}$.

The functor T_w admits a right adjoint functor G_w in the category $\mathcal{O}^{\mathfrak{g}}$ ([AS, §4]):

$$G_w(M) = \mathcal{H}om_U(S_w, \phi_w^{-1}(M)).$$

It is straightforward to extend the definition of T_w and G_w to $w \in \mathcal{W}^e$ ([A1]).

The following assertion follows in the same manner as [Soe2, Theorem 2.1].

Lemma 4.1. *Let $M \in \mathcal{O}^{\mathfrak{g}}$, $w \in \mathcal{W}^e$*

(i) *Suppose that M is free over $\mathfrak{n}_w$. Then $M \cong G_w T_w(M)$.*
(ii) *Suppose that M is cofree over $w(\mathfrak{n}_w)$. Then $M \cong T_w G_w(M)$.*

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of $\mathfrak{g}$ with highest weight λ. Set

$$M^w(\lambda) = T_w M(w^{-1} \circ \lambda).$$

The $\mathfrak{g}$-module $M^w(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ is called the *twisted Verma module* $M^w(\lambda)$ with highest weight λ and twist $w \in \mathcal{W}^e$. Note that by (20) we have

$$M^w(\lambda)_\mu \cong \phi_w(N_w^* \otimes_{N_w} U(\mathfrak{n}_-))_{\mu-\lambda} \cong (U(w(\mathfrak{n}_-) \cap \mathfrak{n}_+)^* \otimes_{\mathbb{C}} U(w(\mathfrak{n}_-) \cap \mathfrak{n}_-))_{\mu-\lambda} \tag{22}$$

as $\mathfrak{h}$-modules. Hence

$$\operatorname{ch} M^w(\lambda) = \operatorname{ch} M(\lambda).$$

In particular $M^w(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

By Lemma 4.1 (1) we have

$$M(\mu) \cong G_w M^w(w \circ \mu).$$

Hence the functor T_w gives the isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(M^w(w \circ \lambda), M^w(w \circ \mu)) \tag{23}$$

for $\lambda, \mu \in \mathfrak{h}^*$.

We have [AL, Proposition 6.3]

$$M^w(\lambda) \cong M(\lambda) \quad \text{if } \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N} \quad \text{for all } \alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}). \tag{24}$$

4.3. **Hom spaces between twisted Verma modules.** For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ and $\mathcal{W}(\lambda)$ be its *integral root system* and *integral Weyl group*, respectively:

$$\Delta(\lambda) = \{\alpha \in \Delta^{re}; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\},$$
$$\mathcal{W}(\lambda) = \langle s_\alpha; \alpha \in \Delta(\lambda)\rangle \subset \mathcal{W}.$$

Let $\Delta(\lambda)_+ = \Delta(\lambda) \cap \Delta_+^{re}$ the set of positive roots of $\Delta(\lambda)$, $\Pi(\lambda) \subset \Delta(\lambda)_+$ the set of simple roots of $\Delta(\lambda)$, $\ell : \mathcal{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ the length function.

For $y \in \mathcal{W}(\lambda)$ the twisted length function ℓ^y and the twisted Bruhat ordering $\succeq_{\lambda,y}$ are defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \rhd_{\lambda,y} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda,y}$.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called *regular dominant* if $\langle \lambda+\rho, \alpha^\vee\rangle \notin \{0, -1, -2, \dots\}$ for all $\alpha \in \Delta_+^{re}$. It is called *regular anti-dominant* if $\langle \lambda + \rho, \alpha^\vee\rangle \notin \{0, 1, 2, \dots\}$ for all $\alpha \in \Delta_+^{re}$.

Theorem 4.2. *Let $w, w', y \in \mathcal{W}(\lambda)$.*

(i) *If λ is regular dominant then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\lambda,y} w', \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If λ is regular anti-dominant then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\lambda,y} w', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) By (23) the assertion follows from (16) and [KT, Proposition 2.5.5 (ii)]. Proof of (ii) is similar. □

4.4. **Wakimoto modules.** Let $\mathfrak{g}$, $\mathfrak{h}$ be as in §3.1, and let us consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \bigoplus_{\alpha\in\Pi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ of root α. Let $\rho = \mathring{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$, where $h^\vee$ is the dual Coxeter number of $\mathring{\mathfrak{g}}$. Then $\langle \rho, \alpha^\vee\rangle = 1$ for all $\alpha \in \Pi$ and 2ρ define a semi-infinite 1-cochain of $\mathfrak{g}$ [Ark2].

Let $L\mathring{\mathfrak{n}}$, $L\mathring{\mathfrak{n}}_-$, $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ be graded subalgebras of $\mathfrak{g}$ defined by

$$L\mathring{\mathfrak{n}} = \mathring{\mathfrak{n}}[t, t^{-1}], \quad L\mathring{\mathfrak{n}}_- = \mathring{\mathfrak{n}}_-[t, t^{-1}],$$
$$\mathfrak{a} = L\mathring{\mathfrak{n}} \oplus \mathring{\mathfrak{h}}[t^{-1}]t^{-1}, \quad \bar{\mathfrak{a}} = L\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

Then $0 = 2\rho|_{L\mathring{\mathfrak{n}}} = 2\rho|_{L\mathring{\mathfrak{n}}_-} = 2\rho|_{\mathfrak{a}}$ gives semi-infinite 1-cochains of $L\mathring{\mathfrak{n}}$, $L\mathring{\mathfrak{n}}_-$, $\mathfrak{a}$, and $2\rho|_{\bar{\mathfrak{a}}}$ gives a semi-infinite 1-cochain of $\bar{\mathfrak{a}}$.

Following [Vor2] we define the *Wakimoto module* $W(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ by

$$W(\lambda) = \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to λ regarded as a $\bar{\mathfrak{a}}$-module by the natural projection $\bar{\mathfrak{a}} \twoheadrightarrow \mathfrak{h}$. By Lemma 2.5 we have

$$W(\lambda) \cong US(\mathfrak{a}) \text{ as } \mathfrak{a}\text{-modules}, \tag{25}$$

and hence

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, W(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i=0, \\ 0 & \text{otherwise} \end{cases} \quad \text{as } \mathfrak{h}\text{-modules,} \tag{26}$$

$$\operatorname{ch} W(\lambda) = \operatorname{ch} M(\lambda). \tag{27}$$

In particular $W(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

Theorem 4.7 below shows that the above definition of Wakimoto module coincides with that of Feigin and Frenkel [FF2, Fre2].

4.5. **Wakimoto modules as inductive limits of twisted Verma modules.** Let $y, w, u \in \mathcal{W}$ such that $w = yu$ and $\ell(w) = \ell(y) + \ell(u)$. Then $T_w = T_y T_u$ and $S_w \cong S_y \otimes_U \phi_y(S_u)$. Let

$$j_{w,y} : S_y \longrightarrow S_w$$

be the homomorphism of left U-modules which maps $s \in S_y$ to $s \otimes \mathbf{1}_u^* \in S_y \otimes_U \phi_y(S_u) = S_w$. Define $\nu_{w,y}^\lambda \in \operatorname{Hom}_{\mathfrak{g}}(M^y(\lambda), M^w(\lambda))$ by

$$\nu_{w,y}^\lambda(s \otimes v_{y^{-1}\circ\lambda}) = j_{w,y}(s) \otimes v_{w^{-1}\circ\lambda} \quad \text{for } s \in S_y,$$

where v_μ denotes the highest weight vector of $M(\mu)$ for $\mu \in \mathfrak{h}^*$. Then

$$\operatorname{Hom}_{\mathfrak{g}}(M^y(\lambda), M^w(\lambda)) = \mathbb{C}\nu_{w,y}^\lambda$$

by (23). We have

$$\nu_{w_3,w_2}^\lambda \circ \nu_{w_2,w_1}^\lambda = \nu_{w_3,w_1}^\lambda \tag{28}$$

if $w_3 = w_2 u_2$, $w_2 = w_1 u_1$ with $\ell(w_1) = \ell(w_2) + \ell(u_2)$, $\ell(w_2) = \ell(w_1) + \ell(u_1)$.

Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n\to\infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \mathring{\Delta}_+$. Then $t_{-\gamma_{i+1}} = t_{-\gamma_i} t_{-(\gamma_{i+1}-\gamma_i)}$ with $\ell(t_{-\gamma_{i+1}}) = \ell(t_{-\gamma_i}) + \ell(t_{-(\gamma_{i+1}-\gamma_i)})$ for all i. It follows that $\{M^{-\gamma_n}(\lambda) : \nu_{-\gamma_m,-\gamma_n}^\lambda\}$ forms an inductive system of $\mathfrak{g}$-modules.

Proposition 4.3 ([Ark1, Lemma 6.1.7])**.** *There is an isomorphism of $\mathfrak{g}$-modules*

$$W(\lambda) \cong \varinjlim_n M^{-\gamma_n}(\lambda).$$

Proof. For the reader's convenience we shall give a proof of Proposition 4.3 here. Set $W(\lambda)' = \varinjlim_n M^{-\gamma_n}(\lambda)$. First note that

$$t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) = t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_+ = \operatorname{span}_{\mathbb{C}}\{x_\alpha t^n ; \alpha \in \Delta_+,\ 0 \le n < \alpha(\gamma_i)\},$$
$$t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_- = (\mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}})[t^{-1}]t^{-1} \oplus \operatorname{span}_{\mathbb{C}}\{x_{-\alpha} t^{-n} ; \alpha \in \Delta_+,\ n > \alpha(\gamma_i)\},$$

where x_α is a root vector of $\mathring{\mathfrak{g}}$ of root α. Thus we have $t_{-\gamma_1}(\mathfrak{n}_{-\gamma_1}) \subset t_{-\gamma_2}(\mathfrak{n}_{-\gamma_2}) \subset \cdots \subset \mathfrak{a}_+$ and $\mathfrak{a}_+ = \bigcup_{i\ge1} t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i})$. The map $j_{-\gamma_i,-\gamma_j} : S_{-\gamma_i} \to S_{-\gamma_j}$ restricts to the embedding $j_{-\gamma_i,-\gamma_j} : N_{-\gamma_i}^* \hookrightarrow N_{-\gamma_j}^*$ for $i < j$, and we have

$$U(\mathfrak{a}_+)^* \cong \varinjlim_i \phi_{-\gamma_i}(N_{-\gamma_i}^*)$$

as left $\mathfrak{a}_+$-modules. Let $j_{-\gamma_i} : \phi_{-\gamma_i}(N_{-\gamma_i}^*) \hookrightarrow U(\mathfrak{a}_+)^*$ be the embedding of left $\phi_{-\gamma_i}(N_{-\gamma_i})$-modules under the above identification.

Since $t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) = \mathrm{span}_{\mathbb{C}}\{x_\alpha t^{-n}; \alpha \in \Delta_+,\ 0 < n \leq \alpha(\gamma_i)\} \subset \mathfrak{a}$,
$$W(\lambda) \cong T_{-\gamma_i} G_{-\gamma_i}(W(\lambda))$$
by Lemma 4.1 (ii). Hence
$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(W(\lambda))).$$
As $\mathrm{ch}\, G_{-\gamma_i}(W(\lambda)) = \mathrm{ch}\, M(t_{\gamma_i} \circ \lambda)$, there exists a unique $\mathfrak{g}$-module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \to G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to w_i, a vector of $G_{-\gamma_i}(W(\lambda))$ of weight $t_{\gamma_i} \circ \lambda$. Up to a non-zero constant multiplication, w_i equals to the the element of $G_{-\gamma_i}(W(\lambda)) = \mathcal{H}om_{N_{-\gamma_i}}(N^*_{-\gamma_i}, \phi^{-1}_{-\gamma_i}(W(\lambda)))$ which sends $f \in N^*_{t_{-\gamma_i}}$ to $j_{-\gamma_i}(f) \otimes 1_\lambda \in US(\mathfrak{a}) \otimes \mathbb{C}_\lambda = W(\lambda)$. The corresponding homomorphism $T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \to W(\lambda)$ is given by
$$T_{-\gamma_i}(\psi_i)(f \otimes v_{t_{\gamma_i} \circ \lambda}) = j_{-\gamma_i}(f) \otimes 1_\lambda \quad \text{for } f \in N^*_{-\gamma_i}\ . \tag{29}$$
It follows that $T_{-\gamma_i}(\psi_j) \circ \nu^\lambda_{\gamma_j, \gamma_i} = T_{-\gamma_i}(\psi_i)$ for $i < j$, and the sequence $\{T_{-\gamma_i}(\psi_j)\}$ yields a $\mathfrak{g}$-module homomorphism
$$\Phi : W(\lambda)' = \varinjlim_i M^{-\gamma_i}(\lambda) \longrightarrow W(\lambda).$$
Fix $\mu \in \mathfrak{h}^*$. Since $W(\lambda) \cong US(\mathfrak{a})$ as an $\mathfrak{a}$-module, it follows from (22) that $T_{-\gamma_i}$ restricts to the isomorphism $M^{-\gamma_i}(\lambda)_\mu \xrightarrow{\sim} W(\lambda)_\mu$ for a sufficiently large i. This completes the proof. □

4.6. **Endmorphisms of Wakimoto modules.**

Proposition 4.4. *Let* $\alpha \in \overset{\circ}{P}{}^\vee_+$, $\lambda \in \mathfrak{h}^*$.

(i) $T_{-\alpha} W(\lambda) \cong W(t_{-\alpha} \circ \lambda)$.
(ii) $G_{-\alpha} W(\lambda) \cong W(t_\alpha \circ \lambda)$.

Proof. (i) Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\overset{\circ}{P}{}^\vee_+$ such that $\gamma_i - \gamma_{i-1} \in \overset{\circ}{P}{}^\vee_+$ and $\lim_{n\to\infty} \beta(\gamma_n) = \infty$ for all $\beta \in \overset{\circ}{\Delta}_+$. Set $\gamma_i' = \gamma_i + \alpha$. Then the sequence $\{\gamma_1', \gamma_2', \dots\}$ satisfies the same property. Hence by Proposition 4.3 and the fact that a homology functor commutes with inductive limits we have $T_{-\alpha} W(\lambda) \cong T_{-\alpha}(\varinjlim M^{-\gamma_i}(\lambda)) = \varinjlim T_{-\alpha} M^{-\gamma_i}(\lambda) = \varinjlim T_{-\alpha} T_{-\gamma_i} M(t_{\gamma_i} \circ \lambda) = \varinjlim T_{-\gamma_i'} M(t_{\gamma_i} \circ \lambda) = \varinjlim M^{-\gamma_i'}(t_\alpha \circ \lambda) \cong W(t_\alpha \circ \lambda)$. (ii) Since $\mathfrak{n}_{t_{-\alpha}} \subset \mathfrak{a}_-$, $W(\lambda)$ is free over $\mathfrak{n}_{t_{-\alpha}}$. Hence $W(t_\alpha \circ \lambda) = G_{-\alpha} T_{-\alpha} W(t_\alpha \circ \lambda) \cong G_{-\alpha} W(\lambda)$ by Lemma 4.1 and (i). □

Corollary 4.5. *Let* $\alpha \in \overset{\circ}{P}{}^\vee_+$. *The functor* $G_{-\alpha}$ *gives the isomorphism*
$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(W(t_\alpha \circ \lambda), W(t_\alpha \circ \mu)).$$
for $\lambda, \mu \in \mathfrak{h}^*$.

Proposition 4.6. *For* $\lambda \in \mathfrak{h}^*$ *we have* $\mathrm{End}_{\mathfrak{g}}(W(\lambda)) = \mathbb{C}$.

Proof. Let $\{\gamma_1, \gamma_2, \dots,\}$ be in Subsection 4.5. Then
$$\begin{aligned} \mathrm{End}_{\mathfrak{g}}(W(\lambda)) &= \mathrm{Hom}_{\mathfrak{g}}(\varinjlim_i M^{-\gamma_i}(\lambda), W(\lambda)) \quad \text{(by Proposition 4.3)} \\ &= \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i} W(\lambda)) \\ &\cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), W(t_{\gamma_i} \circ \lambda)) \quad \text{(by Proposition 4.4)}. \end{aligned}$$

As we have seen in the proof of Proposition 4.3, the space $\mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i}\circ\lambda), W(t_{\gamma_i}\circ\lambda))$ is one-dimensional and $\nu^{\lambda}_{-\gamma_m,\gamma_n}$ induces the isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_m}(\lambda), W(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_n}(\lambda), W(\lambda)).$$

This completes the proof. □

4.7. **Uniqueness of Wakimoto modules.** A finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset M_r = M$ of a $\mathfrak{g}$-module M is called a *Wakimoto flag* if each successive quotient M_i/M_{i-1} is isomorphic to $W(\lambda_i)$ for some λ_i.

Theorem 4.7. *Suppose that k is non-critical, that is, $k \neq -h^\vee$. For an object M of $\mathcal{O}_k^{\mathfrak{g}}$ the following conditions are equivalent.*

(i) *M admits a Wakimoto flag.*
(ii) *$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ and $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional.*

If this is the case the multiplicity $(M : W(\lambda))$ of $W(\lambda)$ in a Wakimoto flag of M equals to $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)_\lambda$. In particular if

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

as $\mathfrak{h}$-modules, M is isomorphic to $W(\lambda)$.

The proof of Theorem 4.7 will be given in Subsection 4.8.

We put on record some of consequences of Theorem 4.7:

Proposition 4.8. *A tilting module in $\mathcal{O}^{\mathfrak{g}}$ at a non-critical level admits a Wakimoto flag.*

Proof. By definition a tilting module M admits both a Verma flag and a dual Verma flag. It follows that M is free over $\mathfrak{n}_-$ and cofree over $\mathfrak{n}_+$. In particular M is free over $\mathring{\mathfrak{n}}[t^{-1}]t^{-1}$ and cofree over $\mathring{\mathfrak{n}}[t]$. Hence by [Vor1, Theorem 2.1], we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$. The fact that $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional follows from the Euler-Poincaré principle. □

Proposition 4.9. *Suppose that $\langle\lambda+\rho, K\rangle \notin \mathbb{Q}_{\geq 0}$. Then $W(t_\alpha\circ\lambda) \cong M(t_\alpha\circ\lambda)$ for a sufficiently large $\alpha \in \mathring{P}^\vee_+$.*

Proof. Let α be sufficiently large. By the hypothesis $\langle t_\alpha(\lambda+\rho), \beta^\vee\rangle \notin \mathbb{N}$ for all $\beta \in \Delta^{re}_+$ such that $\bar{\beta} \in \mathring{\Delta}_+$. It follows from [A1, Theorem 3.1] that $M(t_\alpha\circ\lambda)$ is cofree over $\mathring{\mathfrak{n}}[t] = \mathfrak{a}_+$. Because $M(t_\alpha\circ\lambda)$ is obviously free over $\mathfrak{a}_-$ we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M(t_\alpha\circ\lambda)) \cong \begin{cases} \mathbb{C}_{t_\alpha\circ\lambda} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$ □

The following assertion follows from Proposition 4.9 and Corollary 4.5.

Proposition 4.10. *Let $\lambda, \mu \in \mathfrak{h}^*$ be of level k, and suppose that $k + h^\vee \notin \mathbb{Q}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_\alpha\circ\lambda), M(t_\alpha\circ\mu))$$

for a sufficiently large $\alpha \in \overset{\circ}{P}{}^\vee_+$. *In particular if* $\lambda \in \mathfrak{h}^*$ *is integral, regular anti-dominant, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\frac{\infty}{2}} y \\ 0 & \text{else} \end{cases}$$

for $w, y \in \mathcal{W}$.

Conjecture 4.11. Let $\lambda \in \mathfrak{h}^*$ be integral, regular dominant. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\frac{\infty}{2}} y \\ 0 & \text{else} \end{cases}$$

for $w, y \in \mathcal{W}$.

In Theorem 6.11 below we prove Conjecture 4.11 in the case that $w \rhd_{\frac{\infty}{2}} y$ (in a slightly more general setting).

4.8. **Proof of Theorem 4.7.** Let

$$\mathcal{H} = \overset{\circ}{\mathfrak{h}}[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g},$$

the Heisenberg subalgebra. Denote by π_λ the irreducible representation of $\mathcal{H}$ with highest weight λ. We have $\pi_\lambda \cong U(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1})$ as a module over $\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1} \subset \mathcal{H}$ provided that $\lambda(K) \neq 0$.

For $M \in \mathcal{O}_k^{\mathfrak{g}}$ one knows that $H^{\frac{\infty}{2}+\bullet}(L\overset{\circ}{\mathfrak{n}}, M)$ is naturally an $\mathcal{H}$-module of level $k + h^\vee$ ([FF2]).

Lemma 4.12. *Let* M *be an object of* $\mathcal{O}_k^{\mathfrak{g}}$ *with* $k \neq -h^\vee$. *Then the following conditions are equivalent:*

(i) $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ *for* $i \neq 0$;
(ii) $H^{\frac{\infty}{2}+i}(L\overset{\circ}{\mathfrak{n}}, M) = 0$ *for* $i \neq 0$.

Proof. The assumption that $k \neq -h^\vee$ implies that $H^{\frac{\infty}{2}+\bullet}(L\overset{\circ}{\mathfrak{n}}, M)$ is semi-simple as an $\mathcal{H}$-module and is a direct sum of π_μs. Consider the Hochschild-Serre spectral sequence for the ideal $L\overset{\circ}{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M)$. By definition, we have

$$E_2^{p,q} = \begin{cases} H_{-p}(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M)) & \text{for } p \leq 0, \\ 0 & \text{for } p > 0. \end{cases}$$

By the above mentioned fact $H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M)$ is free over $U(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1})$. Hence

$$E_2^{p,q} = \begin{cases} H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M))/\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1}(H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M))) & \text{for } p = 0. \\ 0 & \text{for } p \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_2 = E_\infty$, and $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ if and only if $H^{\frac{\infty}{2}+i}(L\overset{\circ}{\mathfrak{n}}, M) = 0$ for $i \neq 0$. This completes the proof. □

Proposition 4.13. *Let* M *be an object of* $\mathcal{O}_k$ *at a non-critical level* k *such that* $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ *for* $i \neq 0$. *Then*

$$M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$$

as $\mathfrak{a}$*-modules and* $\mathfrak{h}$*-modules, where* $\mathfrak{a}$ *acts only on the first factor* $US(\mathfrak{a})$ *and* $\mathfrak{h}$ *acts as* $h(s \otimes m) = \mathrm{ad}(h)(s) \otimes m + s \otimes hm$.

Proof. By Proposition 2.3 it suffices to show that $\operatorname{S-ind}_{\mathfrak{a}}^{\mathfrak{a}} M \cong US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. As in the proof of Lemma 4.12, we shall consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, US(\mathfrak{a})\otimes M)$. By definition we have

$$E_1^{\bullet,q} = H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)\otimes_{\mathbb{C}} \bigwedge^{\bullet}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}), \tag{30}$$

$$E_2^{p,q} = H_{-p}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)). \tag{31}$$

To compute the E_1-term set

$$F^p US(\mathfrak{a}) = \bigoplus_{\langle \mu, \mathring{\rho}^\vee\rangle \geq p} US(\mathfrak{a})_\mu,$$

where $US(\mathfrak{a})$ is considered as an $\mathfrak{h}$-module by the adjoint action. Then

$$US(\mathfrak{a}) = F^0 US(\mathfrak{a}) \supset F^1 US(\mathfrak{a}) \supset \dots, \quad \bigcap F^p US(\mathfrak{a}) = 0,$$
$$F^p US(\mathfrak{a}) \cdot L\mathring{\mathfrak{n}} \subset F^{p+1} US(\mathfrak{a}).$$

Define the filtration $F^\bullet(US(\mathfrak{a})\otimes_{\mathbb{C}} M\otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}))$ by setting

$$F^p(US(\mathfrak{a})\otimes_{\mathbb{C}} M\otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}})) = F^p US(\mathfrak{a})\otimes_{\mathbb{C}} M\otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}).$$

This defines a decreasing, weight-wise regular filtration of the complex. Consider the associated spectral sequence $E'_r \Rightarrow H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)$. Because the associated graded space $\operatorname{gr} US(\mathfrak{a})$ with respect to this filtration is a trivial $L\mathring{\mathfrak{n}}$-module the E_1-term of the spectral sequence E'_r is isomorphic to $US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, M)$. Hence by the hypothesis and Lemma 4.12 the spectral sequence E'_r collapses at $E'_1 = E'_\infty$ and we obtain the isomorphism of $\mathfrak{h}$-modules

$$H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M) \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{n}}, M) & \text{for } i=0,\\ 0 & \text{for } i\neq 0.\end{cases} \tag{32}$$

This is also an isomorphism of $\mathfrak{a}$-modules since $US(\mathfrak{a}) \cong \operatorname{gr} US(\mathfrak{a})$ as *left* $\mathfrak{a}$-modules, where $x_\alpha t^n \in \mathfrak{a}$ is considered as an operator on $\operatorname{gr} US(\mathfrak{a}) = \bigoplus_p F^p US(\mathfrak{a})/F^{p+1} US(\mathfrak{a})$ which maps $F^p US(\mathfrak{a})/F^{p+1} US(\mathfrak{a})$ to $F^{p+\alpha(\mathring{\rho}^\vee)} US(\mathfrak{a})/F^{p+\alpha(\mathring{\rho}^\vee)+1} US(\mathfrak{a})$. We have computed the E_1-term (30):

$$E_1^{\bullet,q} \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{n}}, M)\otimes_{\mathbb{C}} \bigwedge^{\bullet}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}) & \text{for } q=0,\\ 0 & \text{for } q\neq 0.\end{cases}$$

It follows that

$$E_2^{p,q} \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) & \text{for } p=q=0,\\ 0 & \text{otherwise}\end{cases} \tag{33}$$

as $\mathfrak{h}$-modules and $\mathfrak{a}$-modules, see the proof of Lemma 4.12. The spectral sequence collapses at $E_2 = E_\infty$ and we obtain the required isomorphism. □

Set

$$Q_{\frac{\infty}{2},+} = \sum_{\substack{\alpha\in\Delta^{\mathrm{re}}\\ \bar\alpha\in\mathring{\Delta}_-}} \mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\delta \subset \mathfrak{h}^*,$$

and define the partial ordering $\leq_{\frac{\infty}{2}}$ on $\mathfrak{h}^*$ by $\mu \leq_{\frac{\infty}{2}} \lambda \iff \lambda - \mu \in Q_{\frac{\infty}{2},+}$. Note that $\mu \leq_{\frac{\infty}{2}} \lambda$ if and only if $t_\alpha \circ \mu \leq t_\alpha \circ \lambda$ for a sufficiently large $\alpha \in \mathring{Q}^\vee$.

Theorem 4.7. Since The direction (i) $\Rightarrow$ (ii) in Theorem 4.7 is obvious by (26), we shall prove that (ii) implies (i). Let $\{\lambda_1, \dots, \lambda_r\}$ be the set of weights of $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ with multiplicities counted, so that

$$M \cong \bigoplus_{i=1}^{r} US(\mathfrak{a}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_i} \tag{34}$$

as $\mathfrak{a}$-modules and $\mathfrak{h}$-modules by Proposition 4.13. We may assume that if $\lambda_i \leq_{\frac{\infty}{2}} \lambda_j$ then $j < i$.

Set $\lambda = \lambda_1$. We shall show that there is a $\mathfrak{g}$-module embedding $W(\lambda) \hookrightarrow M$. Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n\to\infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \mathring{\Delta}_+$, so that $W(\lambda) = \varinjlim_n M^{-\gamma_n}(\lambda)$ by Proposition 4.3. By Lemma 4.1 (ii) we have $M \cong T_{-\gamma_i} G_{-\gamma_i}(M)$, and hence,

$$\operatorname{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), M) \cong \operatorname{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(M)).$$

By (34), $\operatorname{ch} G_{-\gamma_i}(M) = \sum_{i=1}^r \operatorname{ch} M(t_{\gamma_i} \circ \lambda)$. Let i be sufficiently large so that $t_{\gamma_i} \circ \lambda$ is maximal in $G_{-\gamma_i}(M)$. Denote by Φ_i the $\mathfrak{g}$-module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \to G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to a vector of $G_{-\gamma_i}(M)$ of weight $t_{\gamma_i} \circ \lambda$. As in the proof of Proposition 4.3 $\{T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \mapsto M\}$ yield an injective $\mathfrak{g}$-module homomorphism

$$\Phi : W(\lambda) = \varinjlim_i M^{-\gamma_i}(\lambda) \hookrightarrow M.$$

The map Φ induces the homomorphism $H^{\frac{\infty}{2}+0}(\mathfrak{a}, W(\lambda)) = \mathbb{C}_\lambda \to H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ which is certainly injective. It follows from the long exact sequence associated with the exact sequence $0 \to W(\lambda) \xrightarrow{\Phi} M \to M/W(\lambda) \to 0$ we obtain that $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M/W(\lambda)) = 0$ for $i \neq 0$ and $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M/W(\lambda)) = \dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) - 1$. Theorem 4.7 follows by the induction on $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. $\square$

4.9. **Twisted Wakimoto modules.** For $w \in \mathring{W}$ we have the decomposition $\mathfrak{g} = w(\mathfrak{a}) \oplus w(\bar{\mathfrak{a}})$, and 2ρ defines a semi-infinite 1-cochain of the graded subalgebra $w(\bar{\mathfrak{a}})$. Hence we can define the *twisted Wakimoto module* $W^w(\lambda)$ with highest weight λ and twist $w \in \mathring{W}$ by

$$W^w(\lambda) = \text{S-ind}_{w(\bar{\mathfrak{a}})}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to λ regarded as a $\bar{\mathfrak{a}}$-module by the projection $\bar{\mathfrak{a}} \to \mathfrak{h}$. We have

$$W^w(\lambda) \cong US(w(\mathfrak{a})) \text{ as } w(\mathfrak{a})\text{-modules and } \operatorname{ch} W^w(\lambda) = \operatorname{ch} M(\lambda),$$

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), W^w(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{as } \mathfrak{h}\text{-modules.}$$

Let $\{\gamma_1,\gamma_2,\dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i-\gamma_{i-1}\in \mathring{P}_+^\vee$ and $\lim_{n\to\infty}\alpha(\gamma_n)=\infty$ for all $\alpha\in\mathring{\Delta}_+$. The following assertion can be proved in the same manner as Proposition 4.3.

Proposition 4.14. *Let $\lambda\in\mathfrak{h}^*$, $w\in\mathring{\mathcal{W}}$. There is an isomorphism of $\mathfrak{g}$-modules*

$$W^w(\lambda)\cong \varinjlim_n M^{-w(\gamma_n)}(\lambda).$$

The following assertion can be proved in the same manner as Theorem 4.7.

Theorem 4.15. *Let $\lambda\in\mathfrak{h}^*$ be non-critical, $w\in\mathring{\mathcal{W}}$. Let M be an object of $\mathcal{O}^{\mathfrak{g}}$ such that*

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}),M)\cong\begin{cases}\mathbb{C}_\lambda & \text{if } i=0,\\ 0 & \text{otherwise},\end{cases}$$

as $\mathfrak{h}$-modules. Then M is isomorphic to $W^w(\lambda)$.

5. Borel-Weil-Bott vanishing property of Twisting functors

5.1. **Left derived functors of twisting functors.** The functor T_w, $w\in\mathcal{W}^e$, admits the left derived functor $\mathcal{L}_\bullet T_w$ in the category $\mathcal{O}^{\mathfrak{g}}$ since it is a Lie algebra homology functor:

$$\mathcal{L}_iT_w(M)=\phi_w(H_i(\mathfrak{g},S_w\otimes_{\mathbb{C}}M)),$$

where $\mathfrak{g}$ acts on $N_w^*\otimes_{\mathbb{C}}M$ by $X(f\otimes m)=-fX\otimes m+f\otimes Xm$. Because

$$\mathcal{L}_iT_w(M)\cong\phi_w(H_i(\mathfrak{n}_w,N_w^*\otimes_{\mathbb{C}}M)) \tag{35}$$

as $w(\mathfrak{n}_w)$-modules, we have the following assertion.

Lemma 5.1. *Suppose $M\in\mathcal{O}^{\mathfrak{g}}$ is free over $\mathfrak{n}_w$. Then $\mathcal{L}_iT_w(M)=0$ for $i\geq 1$.*

Let $\{e_i,h_i,f_i;i\in I\}$, $e_i\in\mathfrak{g}_{\alpha_i}$, $f_i\in\mathfrak{g}_{-\alpha_i}$, be the Chevalley generators of $\mathfrak{g}$. For $i\in I$, let $\mathfrak{sl}_2^{(i)}$ denote the copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ spanned by $\{e_i,h_i,f_i\}$

Proposition 5.2. *Let $M\in\mathcal{O}^{\mathfrak{g}}$, $i\in I$. Denote by N the largest $\mathfrak{sl}_2^{(i)}$-integrable submodule of M. Then $T_i(M)\cong T_i(M/N)$, $\operatorname{ch}\mathcal{L}_1T_i(M)\cong\operatorname{ch}N$ and $\mathcal{L}_pT_i(M)=0$ for $p\geq 2$.*

Proof. Let $T_i^{(i)}$ denote the twisting functor for $\mathfrak{sl}_2^{(i)}$ corresponding to the reflection s_{α_i}. Because $T_i(M)\cong T_i^{(i)}(M)$ as $\mathfrak{sl}_2^{(i)}$-modules and $\mathfrak{h}$-modules, we have

$$\mathcal{L}_pT_i(M)\cong\mathcal{L}_pT_i^{(i)}(M)\quad\text{as } \mathfrak{sl}_2^{(i)}\text{-modules and } \mathfrak{h}\text{-modules.} \tag{36}$$

In particular $\mathcal{L}_pT_i(M)=0$ for $p\geq 2$. It follows that the exact sequence

$$0\to N\to M\to M/N\to 0$$

yields the long exact sequence

$$\begin{aligned}0\to\mathcal{L}_1T_i(N)\to\mathcal{L}_1T_i(M)\to\mathcal{L}_1T_i(M/N)\\ \to T_i(N)\to T_i(M)\to T_i(M/N)\to 0.\end{aligned}$$

Since M/N is free as $\mathbb{C}[f_i]$-module $\mathcal{L}_1T_i(M/N)=0$ by Lemma 5.1. Also, $T_i(N)=0$ and $\mathcal{L}_1T_i(N)\cong N$ as $\mathfrak{h}$-modules by [AS, Theorem 6.1] and (36). This completes the proof. ∎

Let $L(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ be the irreducible highest weight representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$.

Theorem 5.3 ([AS, Theorem 6.1])**.** *Let $\lambda \in \mathfrak{h}^*$ and suppose that $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ with $i \in I$. Then*

$$\mathcal{L}_p T_i(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 1. \end{cases}$$

Proof. The hypothesis implies that $L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$-integrable. Therefore $\mathcal{L}_p T_i(L(\lambda)) = 0$ for $p \neq 1$ and $\operatorname{ch} \mathcal{L}_1 T_i(L(\lambda)) = \operatorname{ch} L(\lambda)$ by Proposition 5.2. $\square$

5.2. **Twisting functors associated with integral Weyl group.**

Lemma 5.4. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$. There exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_\alpha = x s_i x^{-1}$, $\ell(s_\alpha) = 2\ell(x) + 1$ and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$.*

Proof. Let $s_\alpha = s_{j_l} s_{j_{l-1}} \dots s_{j_1}$ be a reduced expression of s_α in $\mathcal{W}$. Then

$$\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) = \{\alpha_1, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} \dots s_{j_{l-1}}(\alpha_{j_l})\}$$

Since $\ell_\lambda(\alpha) = 1$, $\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$. Thus there exists r such that $\alpha = s_{j_1} \dots s_{j_{r-1}}(\alpha_{j_r})$. Set $x = s_{j_1} \dots s_{j_{r-1}}$, $i = j_r$. Then $s_\alpha = s_{x(\alpha_i)} = x s_i x^{-1}$. It follows that $s_{j_l} \dots s_{j_{r+1}} = x$ and $\ell(s_\alpha) = 2\ell(x) + 1$. Also $\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$ implies that $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. $\square$

Note that if λ, α, α_i, x are as in Lemma 5.4 then

$$T_\alpha = T_x \circ T_i \circ T_{x^{-1}}.$$

Let $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ be the block of $\mathcal{O}^{\mathfrak{g}}$ corresponding to λ, that is, the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects M such that $[M : L(\mu)] \neq 0 \Rightarrow \mu \in \mathcal{W}(\lambda) \circ \mu$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of M.

Lemma 5.5. *Let $\lambda \in \mathfrak{h}^*$, $y \in \mathcal{W}$, and suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta_+^{re} \cap y^{-1}(\Delta_-^{re})$. Then $T_y M(w \circ \lambda) \cong M(yw \circ \lambda)$, $T_y L(w \circ \lambda) \cong L(yw \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$. Moreover T_w gives an equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[w \circ \lambda]}^{\mathfrak{g}}$. The same is true for G_w.*

Proof. First note that the assumption implies that $\mathcal{W}(y \circ \lambda) = y \mathcal{W}(\lambda) y^{-1}$.

We prove by induction on $\ell(y)$. Let $\ell(y) = 1$, so that $y = s_i$ for $i \in I$. Then the fact that $T_i M(w\lambda) \cong M(s_i w \circ \lambda)$ with $w \in \mathcal{W}(\lambda)$ follow from (24). By [A1, Theorems 3.1, 3.2] any object of $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ and $\mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ is free over $\mathbb{C}[f_i]$ and cofree over $\mathbb{C}[e_i]$. Hence by Lemma 4.1 T_i gives an equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ with a quasi-inverse G_i. It follows that $T_i L(\lambda)$ is a simple $\mathfrak{g}$-module which is a quotient of $T_i M(\lambda) = M(s_i \circ \lambda)$, and hence is isomorphic to $L(s_i \circ \lambda)$. Next let $y = s_i z$ with $z \in \mathcal{W}$, $\ell(y) = \ell(z) + 1$. Then $\Delta_+^{re} \cap y^{-1}(\Delta_-^{re}) = \{z^{-1}(\alpha_i)\} \sqcup (\Delta_+^{re} \cap z^{-1} \Delta_-^{re})$. The assertion follows from the induction hypothesis. $\square$

Corollary 5.6. *Let λ, α, α_i, x be as in Lemma 5.4. Then T_x give an equivalence of categories $\mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ such that $T_x M(\mu) \cong M(x \circ \mu)$, $T_x L(\mu) \cong M(x \circ \mu)$ for $\mu \in \mathcal{W}(x^{-1} \circ \lambda) \circ x^{-1}\lambda = x^{-1} \mathcal{W}(\lambda) \circ \lambda$.*

Lemma 5.7. *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$. Then $T_i M^w(\lambda) \cong M^{s_i w s_i}(s_i \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$.*

Proof. By Lemma 5.5, $T_i M^w(\lambda) \cong T_i T_w M(w^{-1}\circ\lambda) \cong T_i T_w T_i M(s_i w^{-1}\circ\lambda) \cong T^{s_i w s_i} M(s_i w^{-1} s_i s_i \circ \lambda)$. □

Lemma 5.8. *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda+\rho, \alpha_i^\vee\rangle \notin \mathbb{Z}$. Then $T_i^2 : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is isomorphic to the identity functor, and so is $G_i^2 : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$.*

Proof. By Lemma 5.5 T_i^2 induces an auto-equivalence of the category $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ such that $T_i^2 M(w\circ\lambda) \cong M(w\circ\lambda)$ and $T_i^2(L(w\circ\lambda)) \cong L(w\circ\lambda)$ for all $w \in \mathcal{W}(\lambda)$. The standard argument shows that such a functor must be isomorphic to the identify functor. □

Corollary 5.9. *Let $\lambda \in \mathfrak{h}^*$, $w = s_\alpha y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$, $y \in \mathcal{W}(\lambda)$, $\ell_\lambda(w) = \ell_\lambda(y)+1$. Then $T_w : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[w\circ\lambda]}$ is isomorphic to the functor $T_{s_\alpha}\circ T_y : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[w\circ\lambda]}$.*

Proposition 5.10. *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ and suppose that $\langle w(\lambda+\rho), \alpha^\vee\rangle \notin \mathbb{N}$. Then the following sequence is exact:*

$$0 \to M(s_\alpha w\circ\lambda) \xrightarrow{\varphi_1} M(w\circ\lambda) \xrightarrow{\varphi_2} M^{s_\alpha}(w\circ\lambda) \xrightarrow{\varphi_3} M^{s_\alpha}(s_\alpha w\circ\lambda) \to 0,$$

where $\varphi_1, \varphi_2, \varphi_3$ are any non-trivial $\mathfrak{g}$-homomorphisms.

Proof. First observe that $\mathrm{Hom}_{\mathfrak{g}}(M(s_\alpha w\circ\lambda), M(w\circ\lambda))$, $\mathrm{Hom}_{\mathfrak{g}}(M(w\circ\lambda), M^{s_\alpha}(w\circ\lambda))$ and $\mathrm{Hom}_{\mathfrak{g}}(M^{s_\alpha}(w\circ\lambda), M^{s_\alpha}(s_\alpha w\circ\lambda))$ are all one-dimensional. (The first and the third are one-dimensional by Theorem 4.2.) By Lemma 5.4 there exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_\alpha = x s_i x^{-1}$, $\ell(s_\alpha) = 2\ell(x)+1$, and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. We have

$$M(y\circ\lambda) \cong T_x M(x^{-1}y\circ\lambda),$$
$$M^{s_\alpha}(y\circ\lambda) = T_x T_i T_{x^{-1}} M(x s_i x^{-1} y\circ\lambda) \cong T_x T_i M(s_i x^{-1} y\circ\lambda) \cong T_x M^{s_i}(x^{-1}y\circ\lambda)$$

for $y \in \mathcal{W}(\lambda)$ by Lemma 5.5. Since $\langle x^{-1}w(\lambda+\rho), \alpha_i^\vee\rangle = \langle w(\lambda+\rho), \alpha^\vee\rangle \in \mathbb{N}$ there is an exact sequence

$$0 \to M(s_i x^{-1} w\circ\lambda) \to M(x^{-1}w\circ\lambda) \to M^{s_i}(x^{-1}w\circ\lambda) \to M^{s_i}(s_i x^{-1} w\circ\lambda) \to 0$$

by [AL, Propostion 6.2]. The required exact sequence is obtained by applying the exact functor $T_x : \mathcal{O}^{\mathfrak{g}}_{[x^{-1}\circ\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ to the above. □

Proposition 5.11. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$, $M \in \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$. Take $\alpha_i \in \Pi$, $x \in \mathcal{W}$ such that $\alpha = x(\alpha_i)$ and $x^{-1}\Delta(\lambda)_+ \subset \Delta_+^{re}$ as in Lemma 5.4. Let N' be the largest $\mathfrak{sl}_2^{(i)}$-integrable submodule of $T_{x^{-1}}(M)$ and set $N = T_x(N') \subset M$. Then $T_\alpha(M) \cong T_{s_\alpha}(M/N)$, $\mathrm{ch}\,\mathcal{L}_1 T_{s_\alpha}(M) = \mathrm{ch}\,N$ and $\mathcal{L}_p T_{s_\alpha}(M) = 0$ for $p \geq 2$.*

Proof. We have $T_\alpha = T_x T_i T_{x^{-1}}$ and $T_{x^{-1}} : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[x^{-1}\circ\lambda]}$, $T_x : \mathcal{O}^{\mathfrak{g}}_{x^{-1}\circ\lambda} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ are exact functors by Corollary 5.6. Therefore

$$\mathcal{L}_p T_{s_\alpha}(M) = T_x(\mathcal{L}_p T_i(T_{x^{-1}}M)). \tag{37}$$

Hence Proposition 5.2 gives that

$$T_{s_\alpha}(M) = T_x T_i T_{x^{-1}}(M) \cong T_x T_i(T_{x^{-1}}(M)/N') \cong T_x T_i T_{x^{-1}}(M/N) = T_{s_\alpha}(M/N),$$
$$\mathrm{ch}\,\mathcal{L}_1 T_{s_\alpha}(M) = \mathrm{ch}\,T_x T_{x^{-1}}(N) = \mathrm{ch}\,N,$$
$$\mathcal{L}_p T_{s_\alpha}(M) = 0 \quad \text{for } p \geq 0.$$

This completes the proof. □

Theorem 5.12. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant weight, $w \in \mathcal{W}(\lambda)$. Then*
$$\mathcal{L}_p T_w(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\alpha \in \Pi(\lambda)$. Since $T_{x^{-1}} L(\lambda) = L(x^{-1} \circ \lambda)$ and $\langle x^{-1} \circ \lambda + \rho, \alpha_i^\vee \rangle = \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$, $T_{x^{-1}} L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$-integrable. Thus,
$$\mathcal{L}_p T_i T_{x^{-1}} L(\lambda) \cong \begin{cases} T_{x^{-1}} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 0 \end{cases}$$
by Theorem 5.3. It follows from (37) that
$$\mathcal{L}_p T_{s_\alpha}(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$
Finally the assertion follows in the same manner as in [AS, Corollary 6.2] by Corollary 5.9. ☐

6. Two-sided BGG resolutions of admissible representations

6.1. **Admissible representations.** A weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if it is regular dominant and
$$\mathbb{Q}\Delta(\lambda) = \mathbb{Q}\Delta^{re}.$$
The irreducible representation $L(\lambda)$ is called admissible if λ is admissible. A complex number k is called an *admissible number* for $\mathfrak{g}$ if the weight $k\Lambda_0$ is admissible.

Let $r^\vee$ be the lacing number of $\mathring{\mathfrak{g}}$, that is, the maximal number of the edges of the Dynkin digram of $\mathring{\mathfrak{g}}$. Also, let h be the Coxeter number of $\mathring{\mathfrak{g}}$.

Proposition 6.1 ([KW2, KW3])**.** *A complex number k is admissible if and only if*
$$k + h^\vee = \frac{p}{q} \quad \textit{with } p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \textit{if } (r^\vee, q) = 1, \\ h & \textit{if } (r^\vee, q) = r^\vee. \end{cases} \tag{39}$$

A complex number k of the form (39) is called an *admissible number with denominator q*. For an an admissible number k with denominator q, we have
$$\Delta(k\Lambda_0) = \{\alpha + nq\delta; \alpha \in \Delta,\ n \in \mathbb{Z}\} \cong \Delta^{re} \text{ and } \mathcal{W}(k\Lambda_0) \cong \mathcal{W} \text{ if } (r^\vee, q) = 1,$$
$$\Delta(k\Lambda_0)^\vee = \{\alpha^\vee + nq\delta; \alpha \in \Delta,\ n \in \mathbb{Z}\} \cong {}^L\Delta^{re} \text{ and } \mathcal{W}(k\Lambda_0) \cong {}^L\mathcal{W} \text{ if } (r^\vee, q) = r^\vee,$$
where $\Delta(\lambda)^\vee = \{\alpha^\vee; \alpha \in \Delta(\lambda)\}$ and ${}^L\Delta^{re}$ and ${}^L\mathcal{W}$ are the real root system and the Weyl group of the non-twisted affine Kac-Moody algebra ${}^L\mathfrak{g}$ associated with the Langlands dual ${}^L\mathring{\mathfrak{g}}$ of $\mathring{\mathfrak{g}}$, respectively. Set
$$\dot{\alpha}_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1, \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$
Then $\Pi(k\Lambda_0) = \{\alpha_1, \dots, \alpha_\ell, \dot{\alpha}_0\}$. Put $\dot{s}_0 = s_{\dot{\alpha}_0} \in \mathcal{W}(k\Lambda_0)$, so that $\mathcal{W}(k\Lambda_0) = \langle s_1, \dots, s_\ell, \dot{s}_0 \rangle$.

For an admissible number k let Pr_k^+ be the set of admissible weights λ of level k such that $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \mathring{\Delta}_+$. Then $\{L(\lambda); \lambda \in Pr_k^+\}$ is the set of irreducible admissible representations of level k which are integrable over $\mathring{\mathfrak{g}} \subset \mathfrak{g}$. We have $\Delta(\lambda) = \Delta(k\Lambda_0)$ for $\lambda \in Pr_k^+$.

For an admissible number k denote by Pr_k the set of admissible weights λ of level k such that $\Delta(\lambda) \cong \Delta(k\Lambda_0)$ as root systems. Then [KW2]

$$Pr_k = \bigcup_{\substack{y \in \mathcal{W}^e \\ y(\Delta(k\Lambda_0) \subset \Delta_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} = y \circ Pr_k^+. \tag{40}$$

Note that

$$\mathcal{W}(\lambda) = y\mathcal{W}(k\Lambda_0)y^{-1} \quad \text{for } \lambda \in Pr_{k,y}. \tag{41}$$

For $\lambda \in Pr_k$, let $\ell_\lambda^{\frac{\infty}{2}}(?)$ be the semi-infinite length function of the affine Weyl group $\mathcal{W}(\lambda)$. The semi-infinite Bruhat ordering $\preceq_{\lambda,\frac{\infty}{2}}$ are also defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \rhd_{\lambda,\frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda,\frac{\infty}{2}}$.

Remark 6.2. The admissible weight $\lambda \in Pr_k$ is called the *principal admissible weight* [KW2] if $\Delta(\lambda) \cong \Delta^{re}$, that is, if the denominator q of k is prime to $r^\vee$.

6.2. **Fiebig's equivalence and BGG resolution of admissible representations.** The following theorem is the special case of a result of Fiebig [Fie, Theorem 11].

Theorem 6.3 ([Fie])**.** *Let λ be regular dominant. Suppose that there exists a symmetrizable Kac-Moody algebra $\mathfrak{g}'$ whose Weyl group $\mathcal{W}'$ is isomorphic to $\mathcal{W}(\lambda)$. Let λ' be an integral dominant weight of $\mathfrak{g}'$, $\mathcal{O}^{\mathfrak{g}'}_{[\lambda']}$ the block of $\mathcal{O}^{\mathfrak{g}'}$ containing the irreducible highest weight representation $L^{\mathfrak{g}'}(\lambda')$ of $\mathfrak{g}$ with highest weight λ'. Then there is an equivalence of categories*

$$\mathcal{O}^{\mathfrak{g}}_{[\lambda]} \cong \mathcal{O}^{\mathfrak{g}'}_{[\lambda']}$$

which maps $M(w\circ\lambda)$ and $L(w\circ\lambda)$, $w \in \mathcal{W}(\lambda)$, to $M^{\mathfrak{g}'}(\phi(w)\circ\lambda')$ and $L^{\mathfrak{g}}(\phi(w)\circ\lambda')$, respectively. Here $M^{\mathfrak{g}'}(\lambda')$ is the Verma module of $\mathfrak{g}'$ with highest weight λ' and $\phi : \mathcal{W}(\lambda) \xrightarrow{\sim} \mathcal{W}'$ is the isomorphism.

Let k be an admissible number with denominator q, $\lambda \in Pr_k$. By Theorem 6.3 the block $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is equivalent to a block of the category $\mathcal{O}$ of $\mathfrak{g}$ or ${}^L\mathfrak{g}$ containing an integrable representation. In particular the existence of a BGG resolution of an integrable representation of an affine Kac-Moody algebra [GL, RCW] implies the existence of a BGG resolution for $L(\lambda)$:

Theorem 6.4. *Let k be an admissible number, $\lambda \in Pr_k$. Then there exists a complex*

$$\mathcal{B}_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0$$

of the form $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i}} M(w\circ\lambda)$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i,\ w \rhd_\lambda w'}} d_{w',w}$, $d_{w',w} \in \mathrm{Hom}_{\mathfrak{g}}(M(w\circ \lambda), M(w' \circ \lambda))$, such that

$$H_i(\mathcal{B}_\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The resolution of $L(\lambda)$ in Theorem 6.4 can be combinatorially constructed as follows [BGG]: Fix a $\mathfrak{g}$-homomorphisms

$$i^{\lambda}_{w',w} : M(w\circ\lambda) \to M(w'\circ\lambda)$$

for $w, w' \in \mathcal{W}(\lambda)$ with $w \succeq_\lambda w'$ in such a way that $i^{\lambda}_{w'',w'} \circ i^{\lambda}_{w',w} = i^{\lambda}_{w'',w}$ if $w \succeq_\lambda w' \succeq_\lambda w$.

A quadruple (w_1, w_2, w_3, w_4) in $\mathcal{W}(\lambda)$ is called a *square* if $w_1 \rhd_\lambda w_2 \rhd_\lambda w_4$, $w_1 \rhd_\lambda w_3 \rhd_\lambda w_4$ and $w_2 \neq w_3$.

Theorem 6.5. *Let k be an admissible number, $\lambda \in Pr_k$. Assign $\epsilon_{w_2,w_1} \in \mathbb{C}^*$ for every pair (w_1, w_2) in $\mathcal{W}(\lambda)$ with $w_1 \rhd_\lambda w_2$ in such a way that $\epsilon_{w_4,w_2}\epsilon_{w_2,w_1} + \epsilon_{w_4,w_3}\epsilon_{w_3,w_1} = 0$ for every square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$ (such an assignment is possible by [BGG]). Set $d_{w',w} = \epsilon_{w',w} i^{\lambda}_{w',w}$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i,\ w \rhd_\lambda w'}} d_{w',w}$. Then*

$$\mathcal{B}_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0,$$

where $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i}} M(w\circ\lambda)$, is a resolution of $L(\lambda)$.

6.3. **Twisted BGG resolution.** For $w_1, w_2, y \in \mathcal{W}(\lambda)$ with $w_1 \succeq_y w_2$, set

$$\varphi^{\lambda,y}_{w_2,w_1} = T_y(i^{\lambda}_{y^{-1}w_2, y^{-1}w_1}) : M^y(w_1\circ\lambda) \to M^y(w_2\circ\lambda).$$

A quadruple (w_1, w_2, w_3, w_4) in $\mathcal{W}(\lambda)$ is called a *y-twisted square* if $w_1 \rhd_y w_2 \rhd_y w_4$, $w_1 \rhd_y w_3 \rhd_y w_4$ and $w_2 \neq w_3$.

Theorem 6.6. *Let k be an admissible number, $\lambda \in Pr_k$, $y \in \mathcal{W}(\lambda)$. Assign $\epsilon^y_{w_2,w_1} \in \mathbb{C}^*$ for every pair (w_1, w_2) with $w_1 \rhd_{\lambda,y} w_2$ in $\mathcal{W}(\lambda)$ in such a way that $\epsilon^y_{w_4,w_2}\epsilon^y_{w_2,w_1} + \epsilon^y_{w_4,w_3}\epsilon^y_{w_3,w_1} = 0$ for every y-twisted square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$. Set $\mathcal{B}^y_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell^y_\lambda(w)=i}} M^y(w\circ\lambda)$, $d^y_{w',w} = \epsilon^y_{w',w}\varphi^{\lambda,y}_{w',w}$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell^y_\lambda(w)=i,\ w \rhd_{\lambda,y} w'}} d_{w',w} : \mathcal{B}^y_i(\lambda) \to \mathcal{B}^y_{i-1}(\lambda)$. Then*

$$\mathcal{B}^y_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}^y_2(\lambda) \xrightarrow{d_2} \mathcal{B}^y_1(\lambda) \xrightarrow{d_1} \mathcal{B}^y_0(\lambda) \xrightarrow{d_0} \mathcal{B}^y_{-1}(\lambda) \to \ldots \to \mathcal{B}^y_{-\ell(y)}(\lambda) \to 0$$

is a complex of $\mathfrak{g}$-modules such that

$$H_i(B^y_\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $\epsilon_{y^{-1}w_1, y^{-1}w_2} = \epsilon^y_{w_1,w_2}$. Then $\{\epsilon^y_{w_1,w_2}\}$ satisfies the condition in Theorem 6.6 if and only if $\{\epsilon_{y^{-1}w_1, y^{-1}w_2}\}$ satisfies the condition in Theorem 6.4. In particular such an assignment is possible. Consider the BGG resolution $\mathcal{B}_\bullet(\lambda)$ of $L(\lambda)$ in Theorem 6.5 associated with this assignment. We have $\mathcal{B}^y_\bullet(\lambda) = T_y(B_\bullet(\lambda))[-\ell(y)]$, where $[-\ell(y)]$ denotes the shift of the degree. Therefore the assertion follows from Theorem 5.12. □

6.4. **System of twisted BGG resolutions.**

Proposition 6.7. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y = s_{\beta_1}s_{\beta_2}\ldots s_{\beta_l}$ a reduced expression of $y \in \mathcal{W}(\lambda)$ with $\beta_i \in \Pi(\lambda)$. Set $y_i = s_{\beta_1}s_{\beta_2}\ldots s_{\beta_i}$ for $i = 0, 1, \ldots, l$ and fix a non-zero $\mathfrak{g}$-homomorphism $\phi^{y_i}_w : M^{y_i}(w\circ\lambda) \to M^{y_{i+1}}(w\circ\lambda)$ for $w \in \mathcal{W}(\lambda)$,*

$i = 1, \dots, l$. *One can assign* $\epsilon^i_{w_2,w_1} \in \mathbb{C}^*$ *for each pair* (w_1, w_2) *with* $w_1 \rhd_{\lambda,y_i} w_2$ *for all* $i = 1, \dots, l$ *in such a way that the following hold:*

(i) $\epsilon^i_{w_4,w_2}\epsilon^i_{w_2,w_1} + \epsilon^i_{w_4,w_3}\epsilon^i_{w_3,w_1} = 0$ *for every* y_i*-twisted square* (w_1, w_2, w_3, w_4) *of* $\mathcal{W}(\lambda)$,

(ii) *If* $w_1 \rhd_{\lambda,y_i} w_2$, $w_1 \rhd_{\lambda,y_{i-1}} w_2$, $\ell^{y_i}_\lambda(w_1) = \ell^{y_{i-1}}_\lambda(w_1)$ *and* $\ell^{y_i}_\lambda(w_2) = \ell^{y_{i-1}}_\lambda(w_2)$, *then the the following diagram commutes.*

$$\begin{array}{ccc} M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i-1}_{w_2,w_1}\varphi^{\lambda,y_{i-1}}_{w_2,w_1}} & M^{y_{i-1}}(w_2 \circ \lambda) \\ {\scriptstyle \phi^{y_{i-1}}_{w_1}}\big\downarrow & & \big\downarrow{\scriptstyle \phi^{y_{i-1}}_{w_2}} \\ M^{y}(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i}_{w_2,w_1}\varphi^{\lambda,y_{i}}_{w_2,w_1}} & M^{y}(w_2 \circ \lambda). \end{array} \tag{42}$$

Proposition 6.8. *Let* $\lambda \in \mathfrak{h}^*$ *be regular dominant,* $y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ *such that* $\ell_\lambda(ys_\alpha) = \ell_\lambda(y) + 1$. *Set* $\beta = y(\alpha)$

(i) *Let* $w_1, w_2 \in \mathcal{W}(\lambda)$. *Suppose that* $w_1 \rhd_y w_2$, $w_1 \rhd_{ys_\alpha} w_2$ *and* $\ell^y_\lambda(w_1) = \ell^{ys_\alpha}_\lambda(w_1)$. *Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) = 1.$$

Moreover, either of the followings span the one-dimensional vector space $\mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda))$:

(a) *the composition* $M^y(w_1 \circ \lambda) \to M^y(w_2 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial* $\mathfrak{g}$*-homomorphisms;*

(b) *the composition* $M^y(w_1 \circ \lambda) \to M^{ys_\alpha}(w_1 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial* $\mathfrak{g}$*-homomorphisms.*

(ii) *Let* $w_1, w_2 \in \mathcal{W}(\lambda)$. *Suppose that* $\ell^y_\lambda(w_1) = \ell^y_\lambda(w_2) + 2$ *and* $w_i^{-1}(\beta) \in \Delta^{re}_+$ *for* $i = 1, 2$. *Then the composition* $M^y(w_1 \circ \lambda) \to M^y(w_2 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial homomorphisms is non-zero.*

(iii) *Let* $w \in \mathcal{W}(\lambda)$ *and suppose that* $s_\alpha w \rhd_{\lambda,y} w$. *Then the composition* $M^y(s_\alpha w \circ \lambda) \to M^y(w \circ \lambda) \to M^{ys_\alpha}(w \circ \lambda)$ *of any* $\mathfrak{g}$*-homomorphisms is zero.*

Proof. (i) Since $y^{-1}w_1 \rhd y^{-1}w_2$, the Jantzen sum formula implies that

$$[M(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1.$$

Hence $[M^{s_\alpha}(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1$. As

$$\mathrm{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \mathrm{Hom}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)),$$

it follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \leq 1$$

Now we have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^y(w_2 \circ \lambda)) &\cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M(y^{-1}w_2 \circ \lambda)), \\ \mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_1 \circ \lambda)) &\cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_1 \circ \lambda)), \\ \mathrm{Hom}_{\mathfrak{g}}(M^y(w_2 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_2 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)), \\ \mathrm{Hom}_{\mathfrak{g}}(M^{ys_\alpha}(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \mathrm{Hom}_{\mathfrak{g}}(M(s_\alpha y^{-1}w_1 \circ \lambda), M(s_\alpha y^{-1}w_2 \circ \lambda)). \end{aligned}$$

In particular they are all one-dimensional. Hence it remains to show that the compositions in (a) and (b) are non-trivial. This is equivalent to the non-triviality of the compositions

$$M(y^{-1}w_1\circ\lambda)\to M(y^{-1}w_2\circ\lambda)\to M^{s_\alpha}(y^{-1}w_2\circ\lambda)$$
$$\text{and } M(y^{-1}w_1\circ\lambda)\to M^{s_\alpha}(y^{-1}w_1\circ\lambda)\to M^{s_\alpha}(y^{-1}w_2\circ\lambda),$$

respectively. Therefore we may assume that $y=1$.

Since $\langle w_2(\lambda+\rho),\alpha^\vee\rangle\in\mathbb{N}$, we have the exact sequence

$$0\to M(s_\alpha w_2\circ\lambda)\to M(w_2\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)\to M^{s_\alpha}(s_\alpha w_2\circ\lambda)\to 0 \tag{43}$$

by Proposition 5.10. On the other hand

$$w_1\circ\lambda\not\trianglelefteq_\lambda s_\alpha w_2\circ\lambda \tag{44}$$

as we have the square $(s_\alpha w_1, w_1, s_\alpha w_2, w_2)$ by the assumption and (15). Hence (43) implies that the image of the highest weight vector of $M(w_1\circ\lambda)$ in $M(w_2\circ\lambda)$ does not lie in the kernel of the map $M(w_2\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)$. This proves the non-triviality of the composition map in (a) for $y=1$, and thus, for all y. Next we show the non-triviality of the composition in (b). Consider the exact sequence

$$0\to M(s_\alpha w_1\circ\lambda)\to M(s_\alpha w_2\circ\lambda)\to N\to 0$$

in the category $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$, where $N=M(s_\alpha w_2\circ\lambda)/M(s_\alpha w_1\circ\lambda)$. Applying the functor T_{s_α} we obtain the exact sequence

$$0\to\mathcal{L}_1T_{s_\alpha}N\to M^{s_\alpha}(w_1\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)\to T_iN\to 0. \tag{45}$$

By Proposition 5.11, the weights of $\mathcal{L}_1T_{s_\alpha}N$ are contained in the set of weights of N, and hence of $M(s_\alpha w_2\circ\lambda)$. Therefore (44) and (45) imply that the image of the highest weight vector of $M(w_1\circ\lambda)$ in $M^{s_\alpha}(w_1\circ\lambda)$ does not belong to the kernel of the map $M^{s_\alpha}(w_1\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)$. This competes the proof of (i). (ii) Similarly as above, the problem reduces to the case $y=1$. By the assumption we have $s_\beta w_1\triangleright_\lambda w_1$, $s_\beta w_2\triangleright_\lambda w_2$. Thus $w_1\not\trianglelefteq_\lambda s_\beta w_2$ because otherwise $(w_1,s_\beta w_1,s_\beta w_1,w_2)$ is a square. Hence (43) proves the assertion by the same argument as above. (iii) Again we may assume that $y=1$ and the assertion follows from (43). □

Proof of Proposition 6.7. We prove by induction on i that such an assignment is possible.

As we already remarked the case $i=0$ is the well-known result of [BGG]. So let $i>0$. Suppose that $w_1\triangleright_{\lambda,y_i}w_2$. Set $\beta=y_{i-1}(\alpha_i)\in\Delta_+^{re}$. The following four cases are possible. (The case $w_1^{-1}(\beta)\in\Delta_+^{re}$, $w_2^{-1}(\beta)\in\Delta_-^{re}$ does not happen by [BGG, Lemma 11.3].)

I) $w_1^{-1}(\beta),w_2^{-1}(\beta)\in\Delta_+^{re}$. In this case $w_1\triangleright_{\lambda,y_{i-1}}w_2$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)$ and $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)$. By Proposition 6.8 there exists a unique $\epsilon^i_{w_2,w_1}$ which makes the diagram (42) commutes.

II) $w_1=s_\beta w_2$. In this case $w_2\triangleright_{\lambda,y_{i-1}}w_1$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)-2$ and $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)$. We set $\epsilon^i_{w_2,w_1}=\epsilon^{i-1}_{w_1,w_2}$.

III) $w_1^{-1}(\beta),w_2^{-1}(\beta)\in\Delta_-^{re}$. In this case $w_1\triangleright_{\lambda,y_{i-1}}w_2$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)-2$ $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)-2$, and we have the y_i-twisted square $(w_1,s_\beta w_1,w_2,s_\beta w_2)$. Note that $\epsilon^i_{s_\beta w_2,s_\beta w_1}$ is defined in I), and $\epsilon^i_{s_\beta w_1,w_1}$, $\epsilon^i_{s_\beta w_2,w_2}$ are defined in II). We set

$$\epsilon^i_{w_2,w_1}=-\frac{\epsilon^i_{s_\beta w_1,w_1}\epsilon^i_{s_\beta w_2,s_\beta w_1}}{\epsilon^i_{s_\beta w_2,w_2}}. \tag{46}$$

IV) $w_1^{-1}(\beta) \in \Delta_-^{re}$, $w_2^{-1}(\beta) \in \Delta_+^{re}$, $w_2 \neq s_\beta w_1$. In this case there exists a unique $w_3 \in \mathcal{W}$ such that $(s_\beta w_1, w_1, w_3, w_2)$ is a y_i-twisted square. Note that $w_3^{-1}(\beta) \in \Delta_+^{re}$ because $(w_3, w_2, s_\beta w_3, s_\beta w_2)$ is a y_i-twisted square by (15). Since $\epsilon^i_{w_3, s_\beta w_1}$, $\epsilon^i_{w_2,w_3}$ are defined in I) and $\epsilon^i_{w_1, s_\beta w_1}$ is defined in II), we can set

$$\epsilon^i_{w_1,w_1} = -\frac{\epsilon^i_{w_3,s_\beta w_1}\epsilon^i_{w_2,w_3}}{\epsilon^i_{w_1,s_\beta w_1}}. \tag{47}$$

Now let (w_1, w_2, w_3, w_4) be a y_i-twisted square. Set

$$A_i(w_1, w_2, w_3, w_4) = \frac{\epsilon^i_{w_4,w_2}\epsilon^i_{w_2,w_1}}{\epsilon^i_{w_4,w_3}\epsilon^i_{w_3,w_1}}.$$

We need to show that $A_i(w_1, w_2, w_3, w_4) = -1$.

The following four cases are possible.

1) $w_2 = s_\beta w_1$, $w_4 = s_\beta w_3$. In this case the assertion follows from the definition (46).

2) $w_2 = s_\beta w_1$, $w_4 \neq s_\beta w_3$. In this case $(s_\beta w)^{-1}(\beta) \in \Delta_-^{re}$, and $w_4^{-1}(\beta) \in \Delta_+^{re}$ because otherwise $w_3 = s_\beta w_4$. Hence the assertion follows from the definition (47).

3) $w_2 \neq s_\beta w_1$, $w_4 = s_\beta w_3$. In this case $(s_\beta w_1, w_1, s_\beta w_2 . w_2)$, $(s_\beta w_1, w_1, s_\beta w_2, w_3)$, $(s_\beta w_2, w_2, s_3, w_4)$ are y_i-twisted squares:

$$\begin{tikzcd}
s_\beta w_1 \arrow[r, "y_i"'] \arrow[dr, "y_i"'] & w_1 \arrow[r, "y_i"'] \arrow[dr] & w_2 \arrow[dr, "y_i"] & \\
& s_\beta w_2 \arrow[r, "y_i"'] \arrow[ur] & w_3 \arrow[r, "y_i"'] & s_\beta w_3
\end{tikzcd}$$

We have by 1)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) = A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) = -1$$

and by 2)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_3) = -1.$$

But

$$\begin{aligned} &A_i(w_1, w_2, w_3, s_\beta w_3) \\ &= A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) A_i(s_\beta w_1, s_\beta w_2, w_1, w_3). \end{aligned}$$

Hence the assertion follows.

4) $w_2 \neq s_\beta w_1$, $w_4 \neq s_\beta w_2$. we see as in [BGG, p.57, c)] that $w_4 \neq s_\beta w_2, s_\beta w_3$, and hence as in [BGG, p.56, 1)] we find that $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$ is also a y_i-twisted square. Hence a) $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i or b) $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all i.

a) The case $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i: By the definition I) we have the commutative diagram

$$\begin{tikzcd}
M^{y_{i-1}}(w_1 \circ \lambda) \arrow[r, "\epsilon^{i-1}_{w_4,w_a}\epsilon^{i-1}_{w_a,w_1}\varphi^{\lambda,y_{i-1}}_{w_4,w_1}"] \arrow[d, "\phi^{y_{i-1}}_{w_1}"'] & M^{y_{i-1}}(w_4 \circ \lambda) \arrow[d, "\phi^{y_{i-1}}_{w_4}"] \\
M^{y}(w_1 \circ \lambda) \arrow[r, "\epsilon^{i}_{w_4,w_a}\epsilon^{i}_{w_a,w_1}\varphi^{\lambda,y_i}_{w_4,w_1}"] & M^{y}(w_4 \circ \lambda)
\end{tikzcd} \tag{48}$$

for $a=2,3$. Since $\epsilon^{i-1}_{w_4,w_2}\epsilon^{i-1}_{w_2,w_1}=-\epsilon^{i-1}_{w_4,w_3}\epsilon^{i-1}_{w_3,w_1}$ by the induction hypothesis the commutativity of the above diagram implies that $\epsilon^{i}_{w_4,w_2}\epsilon^{i}_{w_2,w_1}=-\epsilon^{i}_{w_4,w_3}\epsilon^{i}_{w_3,w_1}$ by Proposition 6.8 (ii).

b) The case that $w_i^{-1}(\beta)\in\Delta^{re}_-$ for all i: We have that $(s_\beta w_1,w_1,s_\beta w_2,w_2)$, $(s_\beta w_1,w_1,s_\beta w_3,w_3)$, $(s_\beta w_1,s_\beta w_2,s_\beta w_3,s_\beta w_4)$, $(s_\beta w_2,w_2,s_\beta w_4,w_4)$ and $(s_\beta w_3,w_3,s_\beta w_4,w_4)$ are all y_i-twisted squares. Hence the assertion follows from the equality

$$\begin{aligned}&A_i(w_1,w_2,w_3,w_4)A_i(s_\beta w_1,s_\beta w_2,w_1,w_2)A_i(s_\beta w_1,w_1,s_\beta w_3,w_3)\\ =\ &A_i(s_\beta w_1,s_\beta w_2,s_\beta w_3,s_\beta w_4)A_i(s_\beta w_2,w_2,s_\beta w_4,w_4)A_i(s_\beta w_3,s_\beta w_4,w_3,w_4).\end{aligned}$$

□

Let k be an admissible number, $\lambda\in Pr_k$. Let $y\in\mathcal{W}(\lambda)$, $\{y_i\}$, $\{\phi^{y_i}_w\}$, $\{\epsilon^i_{w_2,w_1}\}$ be as in Proposition 6.7. Because $\{\epsilon^i_{w_2,w_1}\}$ satisfies the condition in Theorem 6.6 there is a corresponding twisted BGG resolution $\mathcal{B}^{y_i}_\bullet(\lambda)$ of $L(\lambda)$ for $i=0,1,\dots,l=\ell(y)$. Define

$$\Phi^{y_{i+1},y_i}_p=\bigoplus_{\substack{w\in\mathcal{W}(\lambda)\\ \ell^{y_i}_\lambda(w)=\ell^{y_{i+1}}_\lambda(w)=p}}\phi^{y_{i+1},y_i}_w:\mathcal{B}^{y_i}_p(w\circ\lambda)\to\mathcal{B}^{y_{i+1}}_p(w\circ\lambda).$$

Proposition 6.9. *In the above setting $\Phi^{y_{i+1},y_i}_\bullet$ gives a quasi-isomorphism $\mathcal{B}^{y_i}_\bullet(\lambda)\sim\mathcal{B}^{y_{i+1}}_\bullet(\lambda)$ of complexes for each $i=0,1,\dots,l-1$.*

Lemma 6.10. *Let $\lambda\in\mathfrak{h}^*$, y, y_i be as in Proposition 6.7, $w_1,w_2\in\mathcal{W}(\lambda)$.*

(i) *Suppose that $w_1\rhd_{\lambda,y_i}w_2$, $\ell^{y_i}(w_1)=\ell^{y_{i+1}}(w_1)$. Then $w_1\rhd_{\lambda,y_{i+1}}w_2$.*
(ii) *Suppose that $w_1\rhd_{\lambda,y_i}w_2$, $\ell^{y_i}(w_2)=\ell^{y_{i+1}}(w_2)$. Then either of the following two holds.*
 (a) *$w_2=s_\beta w_1$ and $w_2\rhd_{\lambda,y_{i+1}}w_1$.*
 (b) *$w_1\rhd_{\lambda,y_{i+1}}w_2$.*

Proof. (1) By assumption $s_\beta w_1\rhd_{\lambda,y_i}w_2$. Therefore $(s_\beta w_1,w_1,s_\beta w_2,w_2)$ is a y_i-twisted square. (2) Similarly, if $w_2\neq s_\beta w_1$ then $(s_\beta w_1,w_1,s_\beta w_2,w_2)$ y_i-twisted square. The $w_2\neq s_\beta w_1$ case is obvious. □

Proof of Proposition 6.9. The fact that $\Phi^{y_i}_\bullet$ defines a homomorphism of complexes follows from the commutativity of (42), Proposition 6.8 (iii), and Lemma 6.10. Since both complexes are quasi-isomorphic to $L(\lambda)$, to show that it defines a quasi-isomorphism it suffices to check that it defines a non-trivial homomorphism between the corresponding homology spaces. This follows from the fact that $\phi^{y_i}_1:M^{y_i}(\lambda)\to M^{y_{i+1}}(\lambda)$ sends the highest weight vector of $M^{y_i}(\lambda)$ to the highest weight vector of $M^{y_{i+1}}(\lambda)$. □

6.5. **Two-sided BGG resolutions of G-integrable admissible representations.** For $\lambda\in Pr_k$ and $i\in\mathbb{Z}$ set

$$\mathcal{W}^i(\lambda)=\{w\in\mathcal{W}(\lambda);\ell^{\frac{\infty}{2}}_\lambda(w)=i\}.$$

We note that

$$\sharp\mathcal{W}^i(\lambda)=\begin{cases}1 & \text{if } \mathring{\mathfrak{g}}=\mathfrak{sl}_2,\\ \infty & \text{else.}\end{cases}$$

Theorem 6.11. *Let k be an admissible number, $\lambda\in Pr^+_k$*

(i) *The space* $\mathrm{Hom}_{\mathfrak{g}}(W(w\circ\lambda), W(w'\circ\lambda))$ *is one-dimensional for* $w, w' \in \mathcal{W}(\lambda)$ *such that* $w \rhd_{\lambda,\frac{\infty}{2}} w'$.

(ii) *There exists a complex*

$$C^\bullet(\lambda) : \cdots \to C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \cdots$$

in the category $\mathcal{O}$ *of the form*

$$C^i(\lambda) = \bigoplus_{w\in\mathcal{W}^i(\lambda)} W(w\circ\lambda), \quad d_i = \sum_{\substack{w\in\mathcal{W}^i(\lambda),\ w'\in\mathcal{W}^{i+1}(\lambda)\\ w\rhd_{\lambda,\frac{\infty}{2}} w'}} d_{w',w},$$

where $d_{w',w}$ *is a non-trivial* $\mathfrak{g}$*-homomorphism* $W(w\circ\lambda) \to W(w'\circ\lambda)$*, such that*

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \textit{for } i = 0,\\ 0 & \textit{for } i \neq 0.\end{cases}$$

Proof. (ii) Let q be the denominator of k and set $M = q\mathring{Q}^\vee$ if $(r^\vee, q) = 1$ and $M = q\mathring{Q}$ if $(r^\vee, q) = r^\vee$, so that $\mathcal{W}(\lambda) = \mathring{\mathcal{W}} \ltimes t_M$. Let $\gamma_1, \gamma_2, \ldots,$ be a sequence in $\mathring{P}^\vee_+ \cap M$ such that $\gamma_i - \gamma \in \mathring{P}^\vee_+ \cap M$, $\lim_{i\to\infty} \alpha(\gamma_i) = \infty$ for all $\alpha \in \mathring{\Delta}_+$.

By Proposition 6.9 there is an inductive system $\{\mathcal{B}^{-\gamma_i}_\bullet(\lambda)\}$ of twisted BGG resolutions. Let $\mathcal{B}^\bullet_{-\gamma_i}(\lambda)$ be the complex $\mathcal{B}^{-\gamma_i}_\bullet(\lambda)$ with the opposite homological grading. Thus it is a complex

$$B^\bullet_{-\gamma_i}(\lambda) : \cdots \xrightarrow{d_{-2}} \mathcal{B}^{-1}_{\gamma_i}(\lambda) \xrightarrow{d_{-1}} \mathcal{B}^0_{-\gamma_i}(\lambda) \xrightarrow{d_0} B^1_{\gamma_i}(\lambda) \xrightarrow{d_1} \cdots$$

of the form $\mathcal{B}^p_{\gamma_i}(\lambda) = \bigoplus_{\substack{w\in\mathcal{W}(\lambda)\\ \ell^{-\gamma_i}_\lambda(w)=-p}} M^{-\gamma_i}(w\circ\lambda)$, $d_p = \sum_{\substack{w,w'\in\mathcal{W}(\lambda)\\ \ell^{-\gamma_i}_\lambda(w)=-p,\, w\rhd_{\lambda,t_{-\gamma_i}} w'}} d^{\gamma_i}_{w',w}$, $d^{\gamma_i}_{w',w}$: $M^{-\gamma_i}(w\circ\lambda) \to M^{-\gamma_i}(w'\circ\lambda)$ such that $H^p(B^\bullet_{-\gamma_i}(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0,\\ 0 & \text{otherwise.}\end{cases}$

Let $(C^\bullet(\lambda), d_\bullet)$ be the complex obtained as the inductive limit of complex $\mathcal{B}^\bullet_{-\gamma_i}(\lambda)$. By Lemma 3.2, Proposition 4.3 and Proposition 6.9 we have

$$C^p(\lambda) = \bigoplus_{w\in\mathcal{W}^p(\lambda)} \varinjlim_i M^{-\gamma_i}(w\circ\lambda) = \bigoplus_{w\in\mathcal{W}^p(\lambda)} W(w\circ\lambda) \quad \text{for } p\in\mathbb{Z},$$

$$H^p(C^\bullet(\lambda)) = \varinjlim_i H^p(B^\bullet_{-\gamma_i}(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0,\\ 0 & \text{otherwise,}\end{cases}$$

and the differential $d_p : C^p(\lambda) \to C^{p+1}(\lambda)$ has the form

$$d_p = \sum_{\substack{w\in\mathcal{W}^p(\lambda),\ w'\in\mathcal{W}^{p+1}(\lambda)\\ w\rhd_{\lambda,\frac{\infty}{2}} w'}} d_{w',w},$$

where $d_{w',w} : W(w\circ\lambda) \to W(w'\circ\lambda)$ is induced by the homomorphisms $d^{-\gamma_i}_{w',w}$: $M^{-\gamma_i}(w\circ\lambda) \to M^{-\gamma_i}(w'\circ\lambda)$ with $i = 1, 2, \ldots,$. To complete the proof of (ii) it remains to show that the map $d_{w',w}$ is nonzero for $w \rhd_{\lambda,\frac{\infty}{2}} w'$.

Let $w', w \in \mathcal{W}(\lambda)$ such that $w \rhd_{\lambda, \frac{\infty}{2}} w'$. We have the commutative diagram

$$\begin{array}{ccc} M^{-\gamma_i}(w' \circ \lambda) & \xrightarrow{d^{-\gamma_i}_{w,w'}} & M^{-\gamma_i}(w \circ \lambda) \\ \downarrow \phi^{w' \circ \lambda}_{-\gamma_i} & & \downarrow \phi^{w \circ \lambda}_{-\gamma_i} \\ W(w' \circ \lambda) & \xrightarrow{d_{w,w'}} & W(w \circ \lambda) \end{array}$$

for all i. By applying the functor $G_{-\gamma_i}$ we obtain the commutative diagram

$$\begin{array}{ccc} M(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d^{-\gamma_i}_{w,w'})} & M(t_{\gamma_i} w \circ \lambda) \\ \downarrow G_{-\gamma_i}(\phi^{w' \circ \lambda}_{-\gamma_i}) & & \downarrow G_{-\gamma_i}(\phi^{w' \circ \lambda}_{-\gamma_i}) \\ W(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w,w'})} & W(t_{\gamma_i} w \circ \lambda). \end{array}$$

By Corollary 4.5 $d_{w,w'} \neq 0$ if and only if $G_{-\gamma_i}(d_{w,w'}) \neq 0$. Therefore it is sufficient to show that $G_{-\gamma_i}(\phi^{w' \circ \lambda}_{-\gamma_i}) \circ G_{-\gamma_i}(d^{-\gamma_i}_{w,w'}) : M(t_{\gamma_i} w' \circ \lambda) \to W(t_{\gamma_i} w \circ \lambda)$ is non-zero for a sufficiently large i.

Write $w' = s_\alpha w$ with $\alpha \in \Delta^{re}$, $\bar{\alpha} \in \mathring{\Delta}_-$. (This is possible because $s_\alpha = s_{-\alpha}$.) Then, for a sufficiently large i, $\beta := t_{\gamma_i}(\alpha) \in \Delta^{re}_+$ and $t_{\gamma_i} s_\alpha w = s_\beta t_{\gamma_i} w \to t_{\gamma_i} w$. The determinant formula [Fre1, Proposition 2 (2)] shows that the image of the highest weight vector of $M(t_{\gamma_i} w' \circ \lambda) = M(s_\beta t_{\gamma_i} w \circ \lambda)$ in $M(t_{\gamma_i} w \circ \lambda)$ is not in the kernel of the map $G_{\gamma_i}(\phi^{w',\lambda}_{\gamma_i}); M(t_{\gamma_i} w \circ \lambda) \to W(t_{\gamma_i} w \circ \lambda)$. Therefore $G_{\gamma_i}(\phi^{w',\lambda}_{\gamma_i}) \circ G_{\gamma_i}(d^{\gamma_i}_{w,w'})$ is non-zero, and hence so is $d_{w,w'}$.

Finally we shall prove (i). Note that

$$\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda)) = \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$$

and that $\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$ is at most one-dimensional by the Jantzen sum formula since $w' \rhd_\lambda w$. It follows from (the proof of) (ii) that $\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda))$ is spanned by $d_{w,w'}$. This completes the proof. □

Remark 6.12. By Theorem 6.11 (i) the resolution in Theorem 6.11 (ii) may be described in terms of screening operators as in [BF] provided that the existence of corresponding cycles is established, see e.g. [TK].

The following assertion is an immediate consequence of Theorem 6.11 which generalizes [FF2, Theorem 4.1].

Theorem 6.13. *Let k be an admissible number, $\lambda \in Pr^+_k$, $p \in \mathbb{Z}$. We have*

$$H^{\frac{\infty}{2}+p}(\mathfrak{a}, L(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \mathbb{C}_{w \circ \lambda} \quad \textit{as } \mathfrak{h}\textit{-modules},$$

$$H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{n}}, L(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \pi_{w \circ \lambda + h^\vee \Lambda_0} \quad \textit{as } \mathcal{H}\textit{-modules}.$$

6.6. **A description of vacuum admissible representation.** Let $V^k(\mathring{\mathfrak{g}})$ be the universal affine vertex algebra associated with $\mathring{\mathfrak{g}}$ at level k:

$$V^k(\mathring{\mathfrak{g}}) = U(\mathfrak{g}) \otimes_{U(\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representations of $\overset{\circ}{\mathfrak{g}}[t] \oplus \mathbb{C}K$ on which $\overset{\circ}{\mathfrak{g}}[t]$ acts trivially and K acts as the multiplication by k. By [Fre2, Proposition 5.2] we have an injective homomorphism of vertex algebras

$$V^k(\overset{\circ}{\mathfrak{g}}) \hookrightarrow W(k\Lambda_0)$$

for all $k \in \mathbb{C}$. Hence $V^k(\overset{\circ}{\mathfrak{g}})$ may be regarded as a vertex subalgebra of $W(k\Lambda_0)$.

Note that $L(k\Lambda_0)$ is the unique simple quotient of $V^k(\overset{\circ}{\mathfrak{g}})$.

Proposition 6.14. *Let k be an admissible number, $\Psi : W(\dot{s}_0 \circ k\Lambda_0) \to W(k\Lambda_0)$ a non-zero $\mathfrak{g}$-homomorphism, which exists uniquely up to a nonzero constant multiplication by Theorem 6.11 (i). Then the image of the highest weight vector of $W(\dot{s}_0 \circ k\Lambda_0)$ generates the maximal submodule of $V^k(\overset{\circ}{\mathfrak{g}}) \subset W(k\Lambda_0)$.*

Proof. By [KW1] the maximal submodule of $V^k(\overset{\circ}{\mathfrak{g}})$ is generated by a singular vector v of weight $\dot{s}_0 \circ k\Lambda_0$. Consider the two-sided resolution $C^\bullet(k\Lambda_0)$ of $L(k\Lambda_0)$ in Theorem 6.11 (ii). Because it is a resolution of $L(k\Lambda_0)$ and $V^k(\overset{\circ}{\mathfrak{g}}) \subset W(k\lambda_0)$, the vector v must be in the image of $d_{1,w} : W(w \circ k\Lambda_0) \to W(k\lambda_0)$ for some $w \in \mathcal{W}^{-1}(k\Lambda_0)$. Since the weight $w \circ k\Lambda_0$ is strictly smaller than $\dot{s}_0 \circ k\Lambda_0$ for $w \in \mathcal{W}^{-1}(k\Lambda_0)\backslash\{\dot{s}_0\}$, the only possibility is that v is the image of the highest weight vector of $W(\dot{s}_0 \circ k\Lambda_0)$. $\square$

6.7. **Two-sided BGG resolutions of more general admissible representations.** Let $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{Q}{}^\vee$. Then there exists $\lambda_1 \in Pr_k^+$ such that $\lambda = y \circ \lambda_1$. Since $y(\Delta(\lambda_1)_+) \subset \Delta_+^{re}$, $T_y : \mathcal{O}^{\mathfrak{g}}_{[\lambda_1]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is exact,

$$\begin{aligned}
&T_y L(\lambda_1) \cong L(\lambda),\\
&T_y W(w \circ \lambda_1) \cong T_y \varinjlim_i M^{-\gamma_i}(w \circ \lambda_1) \cong \varinjlim_i T_y M^{-\gamma_i}(w \circ \lambda_1)\\
&\cong \varinjlim_i M^{-y(\gamma_i)}(ywy^{-1} \circ \lambda) \cong W^{\bar{y}}(ywy^{-1} \circ \lambda)
\end{aligned}$$

for $w \in \mathcal{W}(\lambda_1) = y^{-1}\mathcal{W}(\lambda)y$ by Proposition 4.14, Lemmas 5.5 and 5.7, where $(\gamma_1, \gamma_2, \dots,)$ is a sequence as in proof of Theorem 6.11. Therefore the following assertion follows immediately from Theorem 6.6.

Theorem 6.15. . *Let k be an admissible number, $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{P}{}^\vee$. Then there exists a complex*

$$C^\bullet(\lambda) : \cdots \xrightarrow{d_{-3}} C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \cdots$$

in the category $\mathcal{O}$ of the form $C^i = \bigoplus_{w \in \mathcal{W}^i(\lambda)} W^{\bar{y}}(w \circ \lambda)$, $d_i = \sum_{\substack{w \in \mathcal{W}^i(\lambda),\ w' \in \mathcal{W}^{i+1}(\lambda) \\ w \rhd_{\lambda, \frac{\infty}{2}} w'}} d_{w',w}$.

such that

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Remark 6.16. If $\lambda \in Pr_{k,y}$ and $\bar{y} = 1$ (that is, $y \in \overset{\circ}{P}{}^\vee$), then $W^{\bar{y}}(w \circ \lambda) = W(w \circ \lambda)$. Hence the above is the resolution of $L(\lambda)$ in terms of (non-twisted) Wakimoto modules as conjectured in [FKW].

7. Semi-infinite restriction and induction

7.1. **Feigin-Frenkel parabolic induction.** Let $\mathring{\mathfrak{p}}$ be a parabolic subalgebra of $\mathring{\mathfrak{g}}$ containing $\mathring{\mathfrak{b}}_-$, and let $\mathring{\mathfrak{p}} = \mathring{\mathfrak{l}} \oplus \mathring{\mathfrak{m}}_-$ be the direct sum decomposition of $\mathring{\mathfrak{p}}$ with the Levi subalgebra $\mathring{\mathfrak{l}}$ containing $\mathring{\mathfrak{h}}$ and the nilpotent radical $\mathring{\mathfrak{m}}_-$. Denote by $\mathring{\mathfrak{m}} \subset \mathring{\mathfrak{n}}$ the opposite algebra of $\mathring{\mathfrak{m}}_-$, so that $\mathring{\mathfrak{g}} = \mathring{\mathfrak{p}} \oplus \mathring{\mathfrak{m}}$. Let

$$\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \bigoplus_{i=1}^{s} \mathring{\mathfrak{l}}_i$$

be the decomposition of $\mathring{\mathfrak{l}}$ into direct sum of simple Lie subalgebras $\mathring{\mathfrak{l}}_i$, $i = 1, \dots, s$, and its center $\mathring{\mathfrak{l}}_0$ of $\mathring{\mathfrak{l}}$. Let $\mathring{\mathfrak{h}}_i = \mathring{\mathfrak{l}} \cap \mathring{\mathfrak{h}}$, the Cartan subalgebra of $\mathring{\mathfrak{l}}_i$, and denote by $\mathring{\Delta}_i \subset \mathring{\Delta}$ the subroot system of $\mathring{\mathfrak{g}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\Pi}_i = \mathring{\Pi} \cap \mathring{\Delta}_i$. Let $h_i^\vee$ be the dual Coxeter number of $\mathring{\mathfrak{l}}_i$ (with a convention $h_0^\vee = 0$), θ_i the highest root of $\mathring{\Delta}_i$, $\theta_{i,s}$ the highest short roof of $\mathring{\Delta}_i$.

Let $\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g}$ for $i = 0, 1, \dots, s$. Set

$$K_i = \frac{2}{(\theta_i|\theta_i)} K,$$

and we consider K_i as an element of $\mathfrak{l}_i$. Thus,

$$\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K_i,$$

and $\mathfrak{h}_i := \mathring{\mathfrak{h}}_i \oplus \mathbb{C}K_i$ is a Cartan subalgebra of $\mathring{\mathfrak{l}}_i$.

Define

$$\mathfrak{l} = \bigoplus_{i=0}^{s} \mathfrak{l}_i. \quad \mathfrak{t} = \bigoplus_{i=0}^{s} \mathfrak{h}_i.$$

The grading of $\mathfrak{l}_i$ induces the grading of $\mathfrak{l}$.

For $k \in \mathbb{C}$ define $k_0, \dots, k_s \in \mathbb{C}$ by

$$k_0 = k + h^\vee, \quad k_i + h_i^\vee = \frac{2}{(\theta_i|\theta_i)}(k + h^\vee) \quad \text{for } i = 1, \dots, s. \tag{49}$$

Lemma 7.1. *Let k be an admissible number for $\mathfrak{g}$. Then k_i, $i = 1, \dots, s$, is an admissible number for the Kac-Moody algebra $\mathfrak{l}_i$.*

Let $\mathcal{O}^{\mathfrak{l}}_{(k_0,\dots,k_s)}$ be the full subcategory of $\mathcal{O}^{\mathfrak{l}}$ consisting of objects on which K_i acts as the multiplication by k_i, $i = 0, 1, \dots, s$. Feigin and Frenkel [FF2, 5.2], [Fre2, §6] constructed a functor

$$\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}} : \mathcal{O}^{\mathfrak{l}}_{(k_0,k_1,\dots,k_s)} \to \mathcal{O}_k^{\mathfrak{g}}, \quad M \to \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M),$$

which enjoys the property

$$\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M) \cong US(L\mathring{\mathfrak{m}}) \otimes_{\mathbb{C}} M \tag{50}$$

as modules over

$$L\mathring{\mathfrak{m}} = \mathring{\mathfrak{m}}[t, t^{-1}] \subset \mathfrak{g},$$

where $L\mathring{\mathfrak{m}}$ only on the first factor $US(L\mathring{\mathfrak{m}})$. In particular $\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}$ is an exact functor.

Denote by $W_{\mathfrak{l}_i}(\lambda^{(i)})$ the Wakimoto module of the affine Kac-Moody algebra $\mathfrak{l}_i$ with highest weight $\lambda^{(i)} \in \mathfrak{h}_i{}^*$ and by $L_{\mathfrak{l}}(\lambda^{(i)})$ the irreducible highest weight representation of $\mathfrak{l}_i$ with highest weight $\lambda^{(i)}$ (with a convention that $W_{\mathfrak{l}_0}(\lambda^{(0)})$ is the irreducible representation of the Heisenberg algebra $\mathfrak{l}_0$ with highest weight $\lambda^{(0)}$). For $\lambda \in \mathfrak{t}^*$ let $W_{\mathfrak{l}}(\lambda)$ and $L_{\mathfrak{l}}(\lambda)$ be the Wakimoto module and the irreducible highest weight representation of $\mathfrak{l}$ with highest weight λ:

$$W_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^{s} W_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}), \quad L_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}).$$

For $\lambda \in \mathfrak{h}^*$, define $\lambda_{\mathfrak{l}} \in \mathfrak{t}^*$ by

$$\lambda_{\mathfrak{l}}|_{\mathring{\mathfrak{h}}_i} = \lambda|_{\mathring{\mathfrak{h}}_i} \text{ and } (\lambda_{\mathfrak{l}} + \rho_i)(K_i) = \frac{2}{(\theta_i|\theta_i)}(\lambda+\rho)(K)$$

for $i = 0, 1, \dots, s$.

Proposition 7.2 ([FF2])**.** *For $\lambda \in \mathfrak{h}^*$ we have* $\text{F-ind}_{\mathfrak{p}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}) \cong W(\lambda)$.

Proof. By using the Hochschild-Serre spectral sequence for $L\mathring{\mathfrak{m}} \subset \mathfrak{a}$ we see from (50) that

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the assertion follows from Theorem 4.7. □

7.2. **Semi-infinite restriction functors.** Let $M \in \mathcal{O}_k^{\mathfrak{g}}$. Then $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$, $p \in \mathbb{Z}$, is naturally an $\mathfrak{l}$-module on which K_i acts as the multiplication by k_i, see e.g. [HT, Proposition 2.3]. Hence

$$\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}} := H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{m}}, ?)$$

defines a functor $\mathcal{O}_k^{\mathfrak{g}} \to \mathcal{O}_{(k_0,k_1,\dots,k_s)}^{\mathfrak{l}}$. We refer to $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$ as the *semi-infinite restriction functor*.

The following assertion follows from Proposition 7.2.

Proposition 7.3. *For $\lambda \in \mathfrak{h}^*$ we have $H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{m}}, W(\lambda)) = 0$ for $i \neq 0$ and*

$$\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}} W(\lambda) \cong W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}).$$

7.3. **Decomposition of integral Weyl groups.** Let k be an admissible number with denominator q, $\lambda \in Pr_k^+$. Let $\mathring{\mathcal{W}}_{S_i}$ be the parabolic subgroup of $\mathring{\mathcal{W}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\mathcal{W}}_S = \mathring{\mathcal{W}}_{S_1} \times \mathring{\mathcal{W}}_{S_2} \times \cdots \times \mathring{\mathcal{W}}_{S_s}$. Define $\dot{\alpha}_0^{(i)} \in \Delta(\lambda)$, $i = 1, \dots, s$, by

$$\dot{\alpha}_0^{(i)} = -\theta_i + q\delta \quad \text{if } (r^\vee, q) = 1,$$
$$\text{and } (\dot{\alpha}_0^{(i)})^\vee = -\theta_{i,s}^\vee + q\delta \quad \text{if } (r^\vee, q) = r^\vee.$$

Set $\dot{s}_0^{(i)} = s_{\dot{\alpha}_0^{(i)}}$.

Let $\mathcal{W}(\lambda)_{S_i}$ be the subgroup of $\mathcal{W}(\lambda)$ generated by $\mathring{\mathcal{W}}_{S_i}$ and $\dot{s}_0^{(i)}$. Then

$$\mathcal{W}(\lambda)_S = \mathcal{W}(\lambda)_{S_1} \times \mathcal{W}(\lambda)_{S_2} \times \cdots \times \mathcal{W}(\lambda)_{S_s}$$

is the subgroup corresponding to $\mathring{\mathcal{W}}_S$ described in §3.4. Let $\mathcal{W}(\lambda)^S \subset \mathcal{W}(\lambda)$ be as in Theorem 3.3 so that

(51)
$$\mathcal{W}(\lambda) = \mathcal{W}(\lambda)_S \times \mathcal{W}(\lambda)^S, \quad \ell_\lambda^{\frac{\infty}{2}}(uv) = \ell_\lambda^{\frac{\infty}{2}}(u) + \ell_\lambda^{\frac{\infty}{2}}(v) \text{ for } u \in \mathcal{W}(\lambda)_S,\ v \in \mathcal{W}(\lambda)^S.$$

Let $w, w' \in \mathcal{W}(\lambda)_{S_i} \subset \mathcal{W}(\lambda)$ such that $w \rhd_{\lambda, \frac{\infty}{2}} w'$. Then $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)} = (w \circ \lambda)_{\mathfrak{l}}^{(i)}$, where $\circ_{\mathfrak{l}_i}$ is the dot action of $\mathcal{W}(\lambda)_{S_i}$ on $\mathfrak{h}_i^*$ and $\lambda_{\mathfrak{l}_i}^{(i)} = \lambda_{\mathfrak{l}}|_{\mathfrak{h}_i}$.

Proposition 7.4. *Let $\lambda \in Pr_k^+$, $w, w' \in \mathcal{W}(\lambda)_{S_i}$ with $i \in \{1, 2, \dots, s\}$ such that $w \rhd_{\lambda, \frac{\infty}{2}} w'$. Then the correspondence $\Phi \mapsto \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\Phi)$ defines a linear isomorphism*

$$\mathrm{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda)).$$

The inverse map is given by $\Psi \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(\Psi)$.

Proof. By Proposition 4.6 and Theorem 6.11 (i) both $\mathrm{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}}))$ and $\mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ are one-dimensional. The assertion follows since the correspondence $\Phi \mapsto \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\tilde{\Phi})$ is clearly injective and $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\Phi)) = \Phi$. □

7.4. **Semi-infinite restriction of admissible affine vertex algebras.** Since it is defined by the semi-infinite cohomology the space $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}}))$ inherits a vertex algebra structure from $V^k(\mathring{\mathfrak{g}})$, and we have a natural vertex algebra homomorphism

$$\bigotimes_{i=0}^{s} V^{k_i}(\mathring{\mathfrak{l}}_i) \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}})),$$

where $V^{k_i}(\mathring{\mathfrak{l}}_i)$ denote the universal affine vertex algebra associated with $\mathring{\mathfrak{l}}_i$ at level k_i. By composing with the map $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}})) \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ induced by the surjection $V^k(\mathring{\mathfrak{g}}) \twoheadrightarrow L(k\Lambda_0)$ this gives rise to a vertex algebra homomorphism

$$\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)). \tag{52}$$

On the other hand there is a natural surjective homomorphism

$$\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \twoheadrightarrow \bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(k_i\Lambda_0)$$

of vertex algebras, where $L_{\mathfrak{l}_i}(k_i\Lambda_0)$ is the unique simple quotient of $V^{k_i}(\mathfrak{l}_i)$.

Theorem 7.5. *Let k be an admissible number. The vertex algebra homomorphism (52) factors through the vertex algebra homomorphism*

$$\bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(k_i\Lambda_0) \hookrightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)).$$

Proof. Put $\lambda = k\Lambda_0$ and let $C^\bullet(\lambda)$ be the two-sided BGG resolution of $L(k\Lambda_0)$ in Theorem 6.11. By the vanishing assertion of Proposition 7.3 the semi-infinite cohomology $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is isomorphic to the cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$ obtained from $C^\bullet(\lambda)$ applying the functor $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$. Thus $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ is isomorphic to the zero-th cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$.

Consider the map $C^{-1}(\lambda) \supset W(\dot{s}_0^{(i)} \circ \lambda) \overset{d_{1,\dot{s}_0^{(i)}}}{\to} W(\lambda) \subset C^0(\lambda)$ for $i = 1, \dots, s$. By applying the functor S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}$ this induces a non-zero homomorphism

$$W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}) \to W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$$

by Proposition 7.4, and the image of the highest weight vector of $W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}})$ generates the maximal $\mathfrak{l}_i$-submodule of $V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$ by Proposition 6.14. It follows that the maximal $\mathfrak{l}$-submodule of $\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda)$ is in the image of S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(d_{-1})$: S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^{-1}(\lambda)) \to$ S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^0(\lambda))$. This completes the proof. □

7.5. **The case of minimal parabolic subalgebras.** Consider the case that $\mathring{\mathfrak{p}}$ is generated by $\mathring{\mathfrak{b}}_-$ and e_i with $i \in \mathring{I}$. Then $\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \mathring{\mathfrak{l}}_1$, $\mathring{\mathfrak{l}}_1 = \mathfrak{sl}_2^{(i)}$ and $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

Theorem 7.6 ($\mathring{\mathfrak{p}}$ minimal)**.** *Let k be an admissible number and let M be a module over the vertex algebra $L(k\Lambda_0)$. Then, for each $p \in \mathbb{Z}$, $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is a direct sum of admissible representations of level k_1 (see (49)) as $\widehat{\mathfrak{sl}}_2^{(i)}$-modules.*

Proof. By Theorem 7.5, $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ is a vertex subalgebra of S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(L(k\Lambda_0)) = H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{m}}, L(k\Lambda_0))$. If M is a module over $L(k\Lambda_0)$ then $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is naturally a module over S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(L(k\Lambda_0))$, and therefore, it is a module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$. The assertion follows since it is known by [AM] that any module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ in the category $\mathcal{O}^{\mathfrak{l}_1}$ must be a direct sum of admissible representations of $\mathfrak{l}_1 \cong \widehat{\mathfrak{sl}}_2$. □

The following assertion generalizes [HT, Theorem 3.8] in the case that $\mathring{\mathfrak{p}}$ is minimal.

Theorem 7.7 ($\mathring{\mathfrak{p}}$ minimal)**.** *Let k be an admissible number, $\lambda \in Pr_k^+$. Then*

$$H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, L(\lambda)) \cong \bigoplus_{\substack{w \in \mathcal{W}(\lambda)^S \\ \ell^{\frac{\infty}{2}}(w)=p}} L_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$$

as $\mathfrak{l}$-modules.

Proof. It is known by [MF] (see also [FM]) that $L(\lambda)$ with $\lambda \in Pr_k^+$ is a module over $L(k\Lambda_0)$. Therefore $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is a direct sum of irreducible admissible representations as $\widehat{\mathfrak{sl}}_2^{(i)}$-modules by Theorem 7.6. Hence it is sufficient to determine the subspace $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ of the singular vectors of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$. Clearly, any weight of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ must be admissible for $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

As is remarked in the proof of Proposition 7.5, $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is the cohomology of the complex S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^\bullet(\lambda))$ and we have S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^p(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ by Proposition 7.3. Now Theorem 3.3 and Lemma 7.1 imply that

$$\begin{aligned}
&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda),\ (w \circ \lambda)_{\mathfrak{l}} \text{ is an admissible weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\
=&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda),\ (w \circ \lambda)_{\mathfrak{l}} \text{ is a dominant weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\
=&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda)^S\}.
\end{aligned}$$

It follows that if a weight μ of $W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ is admissible for $\widehat{\mathfrak{sl}}_2^{(i)}$ then $w \in \mathcal{W}(\lambda)^S$ and $\mu = (w \circ \lambda)_{\mathfrak{l}}$. Therefore the image $[|(w \circ \lambda)_{\mathfrak{l}}\rangle]$ of the highest weight vector $|(w \circ \lambda)_{\mathfrak{l}}\rangle$

of $W_{\mathfrak{l}}((w\circ\lambda)_{\mathfrak{l}})$ is nonzero in $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ and $\{[|(w\circ\lambda)_{\mathfrak{l}}\rangle]; w\in\mathcal{W}(\lambda)^S\}$ forms a basis of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$. By Theorem 3.3, this completes the proof. □

Remark 7.8. In the subsequent paper [A6] we prove that for an admissible number k any $L(k\Lambda_0)$-module in the category $\mathcal{O}^{\mathfrak{g}}$ must be a direct sum of admissible representations. Hence it follows from the proof that the assertion of Theorem 7.7 is valid for any parabolic subalgebra of $\mathring{\mathfrak{g}}$.

References

[AG] S. Arkhipov and D. Gaitsgory. Differential operators on the loop group via chiral algebras. *Int. Math. Res. Not.*, (4):165–210, 2002.

[AL] H. H. Andersen and N. Lauritzen. Twisted Verma modules. In *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, volume 210 of *Progr. Math.*, pages 1–26. Birkhäuser Boston, Boston, MA, 2003.

[AM] Dražen Adamović and Antun Milas. Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$. *Math. Res. Lett.*, 2(5):563–575, 1995.

[A1] Tomoyuki Arakawa. Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction. *Int. Math. Res. Not.*, (15):730–767, 2004.

[A2] Tomoyuki Arakawa. Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture. *Duke Math. J.*, 130(3):435–478, 2005.

[A3] Tomoyuki Arakawa. Representation theory of W-algebras. *Invent. Math.*, 169(2):219–320, 2007.

[A4] Tomoyuki Arakawa. Representation theory of W-algebras, II. In *Exploring new structures and natural constructions in mathematical physics*, volume 61 of *Adv. Stud. Pure Math.*, pages 51–90. Math. Soc. Japan, Tokyo, 2011.

[A5] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras. arXiv:1004.1554[math.QA].

[A6] T. Arakawa. Rationality of admissible affine vertex algebras in the category $\mathcal{O}$. arXiv:1207.4857[math.QA].

[A7] Tomoyuki Arakawa. Rationality of W-algebras; principal nilpotent cases. arXiv:1211.7124[math.QA].

[Ark1] Sergey Arkhipov. A new construction of the semi-infinite BGG resolution. *preprint*, 1996. math.QA/9605043.

[Ark2] S. M. Arkhipov. Semi-infinite cohomology of associative algebras and bar duality. *Internat. Math. Res. Notices*, (17):833–863, 1997.

[AS] Henning Haahr Andersen and Catharina Stroppel. Twisting functors on $\mathcal{O}$. *Represent. Theory*, 7:681–699 (electronic), 2003.

[BF] D. Bernard and G. Felder. Fock representations and BRST cohomology in SL(2) current algebra. *Comm. Math. Phys.*, 127(1):145–168, 1990.

[BGG] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64. Halsted, New York, 1975.

[Feĭ] B. L. Feĭgin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. *Uspekhi Mat. Nauk*, 39(2(236)):195–196, 1984.

[FF1] B. L. Feĭgin and È. V. Frenkel′. A family of representations of affine Lie algebras. *Uspekhi Mat. Nauk*, 43(5(263)):227–228, 1988.

[FF2] Boris L. Feĭgin and Edward V. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. *Comm. Math. Phys.*, 128(1):161–189, 1990.

[Fie] Peter Fiebig. The combinatorics of category $\mathcal{O}$ over symmetrizable Kac-Moody algebras. *Transform. Groups*, 11(1):29–49, 2006.

[FKW] Edward Frenkel, Victor Kac, and Minoru Wakimoto. Characters and fusion rules for W-algebras via quantized Drinfel′d-Sokolov reduction. *Comm. Math. Phys.*, 147(2):295–328, 1992.

[FM] Igor Frenkel and Fyodor Malikov. Kazhdan-Lusztig tensoring and Harish-Chandra categories. *preprint*, 1997. arXiv:q-alg/9703010.

[Fre1] Edward Frenkel. Determinant formulas for the free field representations of the Virasoro and Kac-Moody algebras. *Phys. Lett. B*, 286(1-2):71–77, 1992.

[Fre2] Edward Frenkel. Wakimoto modules, opers and the center at the critical level. *Adv. Math.*, 195(2):297–404, 2005.

[GL] Howard Garland and James Lepowsky. Lie algebra homology and the Macdonald-Kac formulas. *Invent. Math.*, 34(1):37–76, 1976.

[HT] Shinobu Hosono and Akihiro Tsuchiya. Lie algebra cohomology and $N = 2$ SCFT based on the GKO construction. *Comm. Math. Phys.*, 136(3):451–486, 1991.

[Kos] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

[KRW] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.

[KT] Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case. *Asian J. Math.*, 2(4):779–832, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century.

[KW1] Victor G. Kac and Minoru Wakimoto. Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. *Proc. Nat. Acad. Sci. U.S.A.*, 85(14):4956–4960, 1988.

[KW2] V. G. Kac and M. Wakimoto. Classification of modular invariant representations of affine algebras. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, volume 7 of *Adv. Ser. Math. Phys.*, pages 138–177. World Sci. Publ., Teaneck, NJ, 1989.

[KW3] Victor G. Kac and Minoru Wakimoto. On rationality of W-algebras. *Transform. Groups*, 13(3-4):671–713, 2008.

[Lus] George Lusztig. Hecke algebras and Jantzen's generic decomposition patterns. *Adv. in Math.*, 37(2):121–164, 1980.

[MF] F. G. Malikov and I. B. Frenkel′. Annihilating ideals and tilting functors. *Funktsional. Anal. i Prilozhen.*, 33(2):31–42, 95, 1999.

[Pet] D. Peterson. Quantum cohomology of G/P. Lecture Notes, Cambridge, MA, Spring: Massachusetts Institute of Technology, 1997.

[RCW] Alvany Rocha-Caridi and Nolan R. Wallach. Projective modules over graded Lie algebras. I. *Math. Z.*, 180(2):151–177, 1982.

[Soe1] Wolfgang Soergel. Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997.

[Soe2] Wolfgang Soergel. Character formulas for tilting modules over Kac-Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998.

[TK] Akihiro Tsuchiya and Yukihiro Kanie. Fock space representations of the Virasoro algebra. Intertwining operators. *Publ. Res. Inst. Math. Sci.*, 22(2):259–327, 1986.

[Vor1] Alexander A. Voronov. Semi-infinite homological algebra. *Invent. Math.*, 113(1):103–146, 1993.

[Vor2] Alexander A. Voronov. Semi-infinite induction and Wakimoto modules. *Amer. J. Math.*, 121(5):1079–1094, 1999.

[Wak] Minoru Wakimoto. Fock representations of the affine Lie algebra $A_1^{(1)}$. *Comm. Math. Phys.*, 104(4):605–609, 1986.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

E-mail address: arakawa@kurims.kyoto-u.ac.jp