L_∞-Induced Norm Analysis of Sampled-Data Systems via Piecewise Constant and Linear Approximations ⋆

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Abstract

This paper deals with the L_1 analysis of linear sampled-data systems, by which we mean the computation of the L_∞-induced norm of linear sampled-data systems. Two computation methods based on piecewise constant and piecewise linear approximations are provided through fast-lifting, by which the sampling interval $[0, h)$ is divided into M subintervals with an equal width. Even though the central part of the method with the former approximation essentially coincides with a conventional method via fast-sample/fast-hold (FSFH) approximation after all, we show that both methods successfully lead to upper and lower bounds of the L_∞-induced norm, whose gap converges to 0 at the rate of $1/M$ in the former approximation and $1/M^2$ in the latter extended approximation. Such achievements are in sharp contrast with an existing result on the former (i.e., FSFH) approximation, which only shows the convergence rate of the error in the resulting estimate of the L_∞-induced norm, without providing any readily computable upper and lower bounds. A numerical example is given to illustrate the effectiveness of these methods.

Key words: Sampled-data control; L_1 optimal control; L_∞-induced norm; Operator approximation; Numerical methods.

1 Introduction

The L_∞-induced norm (or l_∞-induced norm) of control systems is the maximum magnitude of the regulated output for the worst persistent exogenous input with a unit magnitude. Because this norm corresponds to the L_1 (or l_1) norm of the impulse response of the system in the linear continuous-time (or discrete-time) case, the study associated with the treatment of the L_∞-induced norm (or l_∞-induced norm) has been called the L_1 (or l_1) problem. There have been a number of studies on the L_1 (or l_1) problem for linear systems [5]–[8],[14],[18],[22] since evaluating the maximum magnitude of the regulated output is very important in many control systems and this problem is pertinent to bounded persistent disturbances such as steps and sinusoids, which are often encountered in control systems.

Some special cases of the L_1 problem were discussed in [22]. Regarding a more general situation, the continuous-time case was dealt with in [5],[6],[18] while the discrete-time case (i.e., the l_1 problem) was discussed in [7],[8],[14]. Stimulated by the success in the studies of the L_1 and l_1 problems for continuous-time and discrete-time systems, extension of the L_1 problem to linear sampled-data systems (with inter-sample behavior taken into account) has been addressed in [2],[9],[19]. However, in contrast to the cases of the H_2 [3],[11],[16],[17] and H_∞ [4],[12],[13],[16],[17],[20],[21],[23] problems of sampled-data systems (where the study in [10] plays an important role in the latter problem), no precise solution has been obtained even for the *analysis* of the L_∞-induced norm, for which only approximate methods have been provided. More precisely, in [2],[9],[19], a sampled-data system is "approximated" by a discrete-time system through the fast-sample/fast-hold (FSFH) approximation technique [1], and it is shown that the l_∞-induced norm of the approximating discrete-time system converges to the L_∞-induced norm of the original sampled-data system as the FSFH approximation parameter M tends to infinity. A drawback of these studies is that they are not pertinent to evaluating how close the l_∞-induced norm for a given M is to the exact value of the L_∞-induced norm. More precisely, no readily computable upper and lower bounds have been derived in [2],[9],[19] for the L_∞-induced norm of sampled-data systems.

As a significant advance over the existing result, this

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paper develops two methods for computing upper and lower bounds of the L_∞-induced norm of sampled-data systems by using ideas of piecewise constant and piecewise linear approximations. This direction of the arguments is stimulated by the success of employing these ideas in [15] in computing the $L_\infty[0,h)$-induced norm of compression operators, which are infinite-rank operators that inevitably arise in the lifting approach to sampled-data systems (as well as time-delay systems). The technique called fast-lifting [12] plays an important role in introducing both approximation approaches, which divides the sampling interval $[0,h)$ into M subintervals with an equal width (without applying sampling of signals). Even though the central part of the method with the former approximation essentially coincides with a conventional method via FSFH approximation after all, we show that our new arguments supported by the application of fast-lifting not only successfully allow us to develop the extended piecewise linear approximation approach but also lead to upper and lower bounds of the L_∞-induced norm, whose gap converges to 0 at the rate of $1/M$ in the piecewise constant approximation and $1/M^2$ in the extended (i.e., piecewise linear) approximation. Furthermore, we examine effectiveness of these methods through a numerical study, and we show that the latter approximation method works far more effectively than the former (equivalently conventional) approximation method.

The organization of this paper is as follows. We first review the lifting approach [4],[21],[23] to sampled-data systems in Section 2. We next develop preliminary arguments for the computation of the L_∞-induced norm of sampled-data systems in Section 3. We give our main results in Section 4, by which we can compute explicit upper and lower bounds of the L_∞-induced norm of sampled-data systems. More precisely, we apply the ideas of piecewise constant approximation and piecewise linear approximation to this problem through fast-lifting [12] to arrive at two computation methods. In each approximation approach, we show that the L_∞-induced norm of sampled-data systems is approximated by the ∞-norm of a suitably constructed matrix and that an upper bound and a lower bound of the L_∞-induced norm can be computed easily. We further show that the gap between these bounds is ensured to converge to 0 in the order of $1/M$ and $1/M^2$ for the piecewise constant and piecewise linear approximations, respectively. We also provide a guideline for taking the parameters employed in the approximation processes. We finally demonstrate the effectiveness of these computation methods through a numerical example in Section 5.

In the following, we use the notations $\mathbb{N}$ and $\mathbb{R}^n$ to denote the set of positive integers and the Banach space of n-dimensional real vectors equipped with vector ∞-norm, respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N}\cup\{0\}$. $L_\infty[0,h)$ denotes the set of essentially bounded functions on $[0,h)$, and $(L_\infty[0,h))^\nu$ is denoted by $\mathcal{K}_\nu$, for simplicity. However, we sometimes drop ν and simply write $\mathcal{K}$, and slightly abuse a term especially when we refer to the induced norm of an operator; for an operator $T: X \to Y$ with X and Y being Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, we call $\|T\| := \sup_{x\in X\setminus\{0\}} \|Tx\|_Y/\|x\|_X$ the $L_\infty[0,h)$-induced norm of T if both $X=\mathcal{K}$ and $Y=\mathcal{K}$. A similar convention applies when $L_\infty[0,h)$ is replaced by $L_\infty[0,h/M)$ or L_∞. The notation $\|\cdot\|$ is used to mean either the $L_\infty[0,h)$ norm of a vector function, i.e.,

$$\|f(\cdot)\| := \max_i \operatorname*{ess\,sup}_{0\le t<h} |f_i(t)| \tag{1}$$

(or that with h replaced by h/M or ∞), the $L_\infty[0,h)$-induced norm (or that with h/M or ∞ instead of h) of an operator in the above sense, as well as the ∞-norm of a matrix or a vector, whose distinction will be clear from the context.

2 Lifted Representation of Sampled-Data Systems

This paper is concerned with the sampled-data system $\Sigma_{\rm SD}$ shown in Fig. 1, where P denotes the continuous-time linear time-invariant (LTI) system, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time signals and discrete-time signals, respectively. Suppose that P and Ψ are described respectively by

$$P: \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x \end{cases} \tag{2}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{3}$$

where $x(t)\in\mathbb{R}^n$, $w(t)\in\mathbb{R}^{n_w}$, $u(t)\in\mathbb{R}^{n_u}$, $z(t)\in\mathbb{R}^{n_z}$, $y(t)\in\mathbb{R}^{n_y}$, $\psi_k\in\mathbb{R}^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh\le t<(k+1)h)$.

Given $f(t)\in(L_\infty[0,\infty))^\nu$, its lifting $\{\widehat{f}_k\}_{k=0}^\infty$ with $\widehat{f}_k(\cdot)\in\mathcal{K}_\nu$ $(k\in\mathbb{N}_0)$ (with sampling period h) is defined as follows [4],[21],[23]:

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0\le\theta<h) \tag{4}$$

By applying lifting to $w(t)$ and $z(t)$, the lifted representation of the sampled-data system $\Sigma_{\rm SD}$ is described by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{5}$$

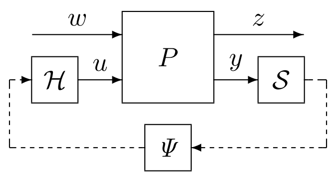

Fig. 1. Sampled-data system $\Sigma_{\rm SD}$.

with $\xi_k := [x_k^T \ \psi_k^T]^T$ $(x_k := x(kh))$, the matrix $\mathcal{A}$ and the operators $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ defined as

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d} D_\Psi C_{2d} & B_{2d} C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi} \tag{6}$$

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : \mathcal{K}_{n_w} \to \mathbb{R}^{n+n_\Psi} \tag{7}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to \mathcal{K}_{n_z} \tag{8}$$

$$\mathcal{D} = \mathbf{D}_{11} : \mathcal{K}_{n_w} \to \mathcal{K}_{n_z} \tag{9}$$

where

$$A_d := \exp(Ah), \ B_{2d} := \int_0^h \exp(A\theta) B_2 d\theta, \ C_{2d} := C_2 \tag{10}$$

$$J_\Sigma = \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi)\times n}, \quad C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{11}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta)) B_1 w(\theta) d\theta \tag{12}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix} \right)(\theta) = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \theta \right) \begin{bmatrix} x \\ u \end{bmatrix} \tag{13}$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 \exp(A(\theta - \tau)) B_1 w(\tau) d\tau + D_{11} w(\theta) \tag{14}$$

In the following, we assume that the sampled-data system $\Sigma_{\rm SD}$ is internally asymptotically stable, i.e., $\mathcal{A}$ has all its eigenvalues in the open unit disc.

3 Preliminaries for the Computation of the L_∞-induced Norm of Sampled-Data Systems

In this section, we give preliminaries for the arguments in this paper, i.e., the Toeplitz structure of the input/output relation of $\Sigma_{\rm SD}$ and its fast-lifting treatment.

3.1 Toeplitz Structure of Input/Output Relation and Truncation

To compute the L_∞-induced norm of $\Sigma_{\rm SD}$, we first note (5) and describe the relation between $\widehat{w}_k$ and $\widehat{z}_k$ $(k \in \mathbb{N}_0)$ as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \\ \vdots \end{bmatrix} \tag{15}$$

Since the above operator has a Toeplitz structure and since $\|f\| = \sup_{k\in\mathbb{N}_0} \|\widehat{f}_k\|$ for $f \in L_\infty$, it follows readily from the properties of L_∞ that the L_∞-induced norm of $\Sigma_{\rm SD}$ coincides with the $L_\infty[0,h)$-induced norm of

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{16}$$

Remark 1 *Essentially the same assertion can be found in [19], but noting the Toeplitz structure leads to a concise statement (as above) as well as an obvious proof of the assertion.*

Remark 2 *Implicitly assumed in (5) (and thus (16)) is the assumption that $t = 0$ is a sampling instant. One might argue that if an intersample instant is taken as $t = 0$, the corresponding L_∞-induced norm might become different from the present one. Since the input-output mapping of $\Sigma_{\rm SD}$ between w and z is h-periodic, however, this is not the case as an immediate property of an induced norm (as in the H_∞ or L_2-induced norm).*

It is, however, still difficult to compute $\|\mathcal{F}\|$ since $\mathcal{F}$ consists of an infinite number of columns. To alleviate this difficulty, we take an $N \in \mathbb{N}$, decompose $\mathcal{F}$ into

$$\mathcal{F} = \mathcal{F}_N^- + \mathcal{F}_N^+ \tag{17}$$

$$\mathcal{F}_N^- := \begin{bmatrix} \mathcal{D} & \cdots & \mathcal{CA}^N\mathcal{B} & 0 & 0 & \cdots \end{bmatrix} \tag{18}$$

$$\mathcal{F}_N^+ := \begin{bmatrix} 0 & \cdots & 0 & \mathcal{CA}^{N+1}\mathcal{B} & \mathcal{CA}^{N+2}\mathcal{B} & \cdots \end{bmatrix} \tag{19}$$

and compute the $L_\infty[0,h)$-induced norm $\|\mathcal{F}_N^-\|$ as accurately as possible while the computation of $\|\mathcal{F}_N^+\|$ is treated in a comparatively simple way (because this norm is expected to be small when N is large enough); we aim at computing upper and lower bounds of $\|\mathcal{F}\|$ through approximation of $\mathcal{F}_N^-$ and computing an upper bound of $\|\mathcal{F}_N^+\|$. The choice of N (as well as other parameters to be introduced) will be discussed in Subsection 4.4.

3.2 Fast-Lifted Representation of $\mathcal{F}_N^-$

As a key idea in the computation of the L_∞-induced norm of sampled-data systems within any prescribed error bound, we next apply fast-lifting [12]. For $M \in \mathbb{N}$

and $h' := h/M$, fast-lifting is defined as the mapping from $f \in \mathcal{K}_\nu$ to $\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(M)})]^T \in (\mathcal{K}'_\nu)^M$, and is denoted by $\check{f} = \mathbf{L}_M f$, where

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{20}$$

and $\mathcal{K}'_\nu$ is a shorthand notation for $(L_\infty[0,h'))^\nu$. It is easy to see that $\mathbf{L}_M$ is norm-preserving (i.e., $\|\mathbf{L}_M f\| = \|f\|$), which plays a crucial role in the following arguments. Unlike the conventional fast-sample/fast-hold (FSFH) approximation [1], which takes M equally spaced sampling points on the interval $[0,h)$, fast-lifting is used only to subdivide the sampling interval $[0,h)$ into M smaller pieces and hence no information is lost by its application. Approximations are applied later in the following section on the top of the fast-lifting treatment, by which signals on $[0,h/M)$ are constrained to constant functions or linear functions. *Piecewise* constant or linear approximations of signals on $[0,h)$ can be achieved easily in such a way.

It easily follows from the norm-preserving property of $\mathbf{L}_M$ that

$$\|\mathcal{F}_N^-\| = \left\| \left[\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} \ \cdots \ \mathbf{L}_M \mathcal{C} \mathcal{A}^N \mathcal{B} \mathbf{L}_M^{-1} \right] \right\| \tag{21}$$

To facilitate the treatment of the right-hand side, we introduce $\mathbf{D}'_{11}, \mathbf{B}'_1$ and $\mathbf{M}'_1$ defined as $\mathbf{D}_{11}$, $\mathbf{B}_1$ and $\mathbf{M}_1$, respectively, with the horizon $[0,h)$ replaced by $[0,h')$ $(= [0,h/M))$, and also introduce the matrices

$$A'_d := \exp(Ah'), \quad A'_{2d} := \exp(A_2 h'),$$
$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad J := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_u)\times n} \tag{22}$$

Then, (as in the standard arguments employing fast-lifting, e.g., [12]), it is easy to see that $\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1}$ and $\mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1}$ $(j = 0, \cdots, N)$ in (21) are described respectively by

$$\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1} \Delta_{M0} \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \tag{23}$$
$$\mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1} A'_{2dM} C_\Sigma \mathcal{A}^j J_\Sigma A'_{dM} \overline{\mathbf{B}'_1} \tag{24}$$

where

$$A'_{dM} := \left[(A'_d)^{M-1} \ \cdots I \right], \ A'_{2dM} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix} \tag{25}$$

$$\Delta_{M0} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{2d})^{M-2} J & \cdots & J & 0 \end{bmatrix} \tag{26}$$

and $\overline{(\cdot)}$ denotes diag$[(\cdot), \ \cdots, \ (\cdot)]$ consisting of M copies of $(\cdot)$. Hence, the operator matrix on the right hand side of (21) admits the representation

$$\mathcal{F}_{MN}^- = \left[\overline{\mathbf{M}'_1} \Delta_{M0} \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \quad \overline{\mathbf{M}'_1} \mathcal{A}_{M0} \overline{\mathbf{B}'_1} \ \cdots \ \overline{\mathbf{M}'_1} \mathcal{A}_{MN} \overline{\mathbf{B}'_1} \right] \tag{27}$$

where

$$\mathcal{A}_{Mj} := A'_{2dM} C_\Sigma \mathcal{A}^j J_\Sigma A'_{dM} \quad (j = 0, \cdots, N) \tag{28}$$

4 Main Results

This section gives computation methods for the L_∞-induced norm of sampled-data systems by using the ideas of constant approximation and linear approximation of the operators $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ involved in the fast-lifted representation $\mathcal{F}_{MN}^-$. Without referring to fast-lifting, this could be interpreted as piecewise constant approximation and piecewise linear approximation of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$ involved in $\mathcal{F}_N^-$.

4.1 Piecewise Constant Approximation of $\mathcal{F}_N^-$

In this subsection, we suppose that N is given and aim at computing upper and lower bounds of $\|\mathcal{F}_N^-\|$ through piecewise constant approximation of $\mathcal{F}_N^-$.

In piecewise constant approximation, a central role is played by the 'averaging' operator $\mathbf{J}'_0$ defined by

$$(\mathbf{J}'_0 w)(\theta') = \frac{1}{h'} \int_0^{h'} w(\tau') d\tau' \quad (0 \le \theta' < h') \tag{29}$$

We introduce the operator $\mathbf{B}'_{p0} := \mathbf{B}'_1 \mathbf{J}'_0$, i.e.,

$$\mathbf{B}'_{p0} w = \int_0^{h'} \exp(A(h' - \theta')) B_1 \cdot (\mathbf{J}'_0 w)(\theta') d\theta' \tag{30}$$

which corresponds to restricting the input of $\mathbf{B}'_1$ to constant functions. Obviously, $\mathbf{B}'_{p0} w = \mathbf{B}'_1 w$ whenever w is a constant function. On the other hand, we further introduce the operator $\mathbf{M}'_{p0}$ and $\mathbf{D}'_{p0}$ defined by

$$\left(\mathbf{M}'_{p0} \begin{bmatrix} x \\ u \end{bmatrix} \right)(\theta') = [C_1 \ D_{12}] \begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le \theta' < h') \tag{31}$$
$$(\mathbf{D}'_{p0} w)(\theta') = D_{11} w(\theta') \quad (0 \le \theta' < h') \tag{32}$$

The output of $\mathbf{M}'_{p0}$ is a constant function corresponding to the zero-order approximation of the Taylor expansion of the output of $\mathbf{M}'_1$. The operator $\mathbf{D}'_{p0}$ means the operator of multiplication by the matrix D_{11}.

We are in a position to introduce the constant approximation $\mathcal{P}^-_{MN0}$ for $\mathcal{F}^-_{MN}$, by which we mean to replace $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ in (27) with $\mathbf{B}'_{p0}$, $\mathbf{M}'_{p0}$ and $\mathbf{D}'_{p0}$, respectively:

$$\mathcal{P}^-_{MN0} = [\overline{\mathbf{M}'_{p0}}\Delta_{M0}\overline{\mathbf{B}'_{p0}} + \overline{\mathbf{D}'_{p0}} \quad \overline{\mathbf{M}'_{p0}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{p0}} \quad \cdots \quad \overline{\mathbf{M}'_{p0}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{p0}}] \tag{33}$$

This corresponds to piecewise constant approximation of $\mathcal{F}^-_N$. This subsection shows that $\|\mathcal{P}^-_{MN0}\|$ can be computed exactly and converges, as $M \to \infty$, to $\|\mathcal{F}^-_N\|$ at the rate of $1/M$. To establish a more precise assertion relevant to upper and lower bounds of $\|\mathcal{F}^-_N\|$, the following two lemmas are important.

Lemma 1 ([15], Theorem 3) *The inequality*

$$\|(\overline{\mathbf{M}'_1}\Delta_{M0}\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}}) - (\overline{\mathbf{M}'_{p0}}\Delta_{M0}\overline{\mathbf{B}'_{p0}} + \overline{\mathbf{D}'_{p0}})\| \le \frac{K_{M\mathcal{D}0}}{M} \tag{34}$$

holds, where

$$K_{M\mathcal{D}0} := h\|C_1\|\cdot\|B_1\|e^{\|A\|h/M} + \frac{h^2}{M}\|A\|\cdot\|B_1\|e^{\|A\|h/M} \cdot \sum_{k=0}^{M-2}\left\{\|C_1(A'_d)^{k+1}\| + \|C_1(A'_d)^k\|e^{\|A\|h/M}\right\} \tag{35}$$

Furthermore, $K_{M\mathcal{D}0}$ has a uniform upper bound with respect to M given by

$$K^U_{\mathcal{D}0} := h\|C_1\|\cdot\|B_1\|e^{\|A\|h} + h^2\|C_1\|\cdot\|B_1\|\cdot\|A\|e^{2\|A\|h}\left(1 + e^{\|A\|h}\right) \tag{36}$$

Lemma 2 *The inequality*

$$\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{p0}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{p0}}\| \le \frac{K_{Mj0}}{M} \tag{37}$$

holds for $j = 0, \cdots, N$, where

$$K_{Mj0} := e^{\|A\|h/M}\cdot\|\mathcal{A}_{Mj}\|\cdot\frac{h^2}{M} \cdot\{\|[C_1\ D_{12}]A_2\|\cdot\|B_1\|e^{\|A_2\|h/M} + \|[C_1\ D_{12}]\|\cdot\|A\|\cdot\|A'_dB_1\|\} \tag{38}$$

Furthermore, K_{Mj0} has a uniform upper bound with respect to M and j given by

$$K^U_{\mathcal{CAB}0} := h^2e^{\|A\|h}\cdot\|B_1\|\cdot K_* \cdot\left\{\|[C_1\ D_{12}]A_2\|\cdot e^{\|A_2\|h} + \|[C_1\ D_{12}]\|\cdot\|A\|\cdot e^{\|A\|h}\right\} \tag{39}$$

where

$$K_* := \max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\|\cdot e^{(\|A\|+\|A_2\|)h}\cdot\|C_\Sigma\| \tag{40}$$

Remark 3 $\max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\|$ *exists since $\mathcal{A}^i \to 0$ as $i \to \infty$ by the stability assumption of Σ_{SD}.*

A series of remarks are given here as to the advances in the present arguments beyond the pertinent study in [15] dealing only with the $L_\infty[0,h)$-induced norm of the compression operator $\mathbf{D}_{11}$ associated with continuous-time LTI systems. Another aspect specific to the present paper dealing with sampled-data systems, which are h-periodic, is deferred to Remark 8.

Remark 4 *Because of the structure of Δ_{M0}, the left-hand side of (34) is actually independent of B_2 and D_{12} involved in (13); the statement of Lemma 1 has been tailored to the arguments of sampled-data systems by rephrasing the original statement in terms of other two operators $\mathbf{C}'_1$ and $\mathbf{C}'_{p0}$ (which correspond to $\mathbf{M}'_1J$ and $\mathbf{M}'_{p0}J$ in the present notation, respectively) for continuous-time LTI systems in [15]. Similarly for Lemma 3 given later.*

Remark 5 *As opposed to the treatment of continuous-time LTI systems (or its compression operator $\mathbf{D}_{11}$) [15], an essentially different feature of sampled-data systems is that we need to deal with the operators $\mathbf{B}_1$ and $\mathbf{M}_1$ on the top of $\mathbf{D}_{11}$. In contrast to the H_2 and H_∞ problems of sampled-data systems, for which the finite-rank nature of $\mathbf{B}_1$ and $\mathbf{M}_1$ allows us to discretize them 'exactly,' these operators cannot be exactly discretized in the L_1 problem of sampled-data systems. This fact has particularly been the reason that motivated the involved approach developed in [2] through the pre-adjoint notion. The significance of the present paper lies in developing alternative and more elementary approximation approaches to these finite-rank operators $\mathbf{B}_1$ and $\mathbf{M}_1$. Lemma 2 and Lemma 4 given later play central roles in such directions. Another relevant remark follows on the treatment of $\mathbf{B}_1$ and $\mathbf{M}_1$ specific to sampled-data systems.*

Remark 6 *As seen from (27), $\mathbf{B}_1$ and $\mathbf{M}_1$ fortunately reduce to $\mathbf{B}'_1$ and $\mathbf{M}'_1$ under their fast-lifting treatment. The latter operators have also appeared, although partially (see Remark 4), in the fast-lifting treatment of the compression operator $\mathbf{D}_{11}$ for continuous-time LTI systems [15], and the arguments therein would thus suggest parallel treatment of $\mathbf{B}'_1$ and $\mathbf{M}'_1$ in the present paper. Although Lemma 2 is indeed pertinent to approximating $\mathbf{B}'_1$ and $\mathbf{M}'_1$ with $\mathbf{B}'_{p0}$ and $\mathbf{M}'_{p0}$, respectively, Lemma 2 is, nevertheless, not a straightforward extension of a relevant result in the continuous-time LTI case [15, Lemma 1], and its derivation cannot simply follow the same line as the continuous-time LTI case. This is because in the present study these two operators arise in connection with sampled-data systems, for which we need to deal with* *the hybrid nature of continuous- and discrete-time signals. This can be explained in more details as follows. In the continuous-time case, $\mathcal{A}_{Mj}$ reduces to a simple form represented by the exponentials of A, and it can be handled in a combined fashion with other exponentials of A involved in $\overline{\mathbf{B}'_1}$, $\overline{\mathbf{B}'_{p0}}$, $\overline{\mathbf{M}'_1}$ and $\overline{\mathbf{M}'_{p0}}$ when we evaluate $\|\overline{\mathbf{M}'_1}\mathcal{A}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{p0}}\mathcal{A}_{Mj}\overline{\mathbf{B}'_{p0}}\|$. Hence, an essential part of the arguments is about expanding exponentials with powers of A, and thus it is not necessary to evaluate $\|\overline{\mathbf{B}'_1} - \overline{\mathbf{B}'_{p0}}\|$ and $\|\overline{\mathbf{M}'_1} - \overline{\mathbf{M}'_{p0}}\|$. However, this is not the case for the sampled-data case and more involved arguments are necessary (see Appendix A). Because of this different treatment tailored to sampled-data systems, the assertion of Lemma 2 does not reduce to that of Lemma 1 in [15] even if its special case were considered when the discrete-time controller is absent. Similarly for Lemma 4.*

Lemmas 1 and 2 readily lead to the following result.

Proposition 1 *The inequality*

$$\|\mathcal{F}^-_{MN} - \mathcal{P}^-_{MN0}\| \leq \frac{K_{M0}}{M} \tag{41}$$

holds, where

$$K_{M0} := K_{M\mathcal{D}0} + \sum_{j=0}^{N} K_{Mj0} \tag{42}$$

In addition, K_{M0} has a uniform upper bound with respect to M given by

$$K_0^U := K_{\mathcal{D}0}^U + (N+1) \cdot K_{\mathcal{CAB}0}^U \tag{43}$$

To evaluate $\|\mathcal{F}^-_N\| = \|\mathcal{F}^-_{MN}\|$ through the above result and the triangle inequality, we next provide a method for (*exactly*) computing $\|\mathcal{P}^-_{MN0}\|$. To facilitate the arguments, let us first suppose that $D_{11} = 0$ (so that $\overline{\mathbf{D}'_{p0}} = 0$). Since $\|w\| \geq \|\mathbf{J}'_0 w\|$ whenever $w \in \mathcal{K}_{n_w}$ and since $\mathbf{J}'_0 w$ is a constant function, it follows readily from (33) that the input of $\mathcal{P}^-_{MN0}$ may always be assumed to be a constant function when we evaluate $\|\mathcal{P}^-_{MN0}\|$. By (31), the output of $\mathcal{P}^-_{MN0}$ is also a constant function determined by the matrix $[C_1\ D_{12}]$. Hence, $\|\mathcal{P}^-_{MN0}\|$ coincides with the ∞-norm of the matrix obtained by replacing the operators $\overline{\mathbf{B}'_{p0}}$ and $\overline{\mathbf{M}'_{p0}}$ with $\overline{B'_{0d}}$ and $\overline{[C_1\ D_{12}]}$, respectively, where B'_{0d} is the matrix representing an 'equivalent operation' in (30) for constant functions w:

$$B'_{0d} := \int_0^{h'} \exp(A(h'-\theta'))B_1 d\theta' \tag{44}$$

Combining the above arguments leads to the following prelude to the first main result in this paper.

Theorem 1 *The inequality*

$$\|P^-_{MN0}\| - \frac{K_{M0}}{M} \leq \|\mathcal{F}^-_N\| \leq \|P^-_{MN0}\| + \frac{K_{M0}}{M} \tag{45}$$

holds, where

$$\begin{aligned} P^-_{MN0} := [\overline{D_{11}} \quad & \overline{[C_1\ D_{12}]}\Delta_{M0}\overline{B'_{0d}} \quad \overline{[C_1\ D_{12}]}\mathcal{A}_{M0}\overline{B'_{0d}} \\ & \cdots \quad \overline{[C_1\ D_{12}]}\mathcal{A}_{MN}\overline{B'_{0d}}\ ] \end{aligned} \tag{46}$$

Remark 7 *The above arguments under the assumption $D_{11} = 0$ immediately lead to (46) without the extra entry $\overline{D_{11}}$, but it is not hard to see that dealing with $D_{11} \neq 0$ and thus the corresponding multiplication operator $\overline{\mathbf{D}'_{p0}}$ in (33) simply leads to introducing this extra entry by the property of $L_\infty[0, h')$; the treatment of D_{11} is essentially the same as that in [19]. With such treatment of D_{11} in mind, and by noting that the piecewise constant approximation is norm-contractive, we can show that the lower bound of $\|\mathcal{F}^-_N\|$ in (45) can in fact be replaced by $\|P^-_{MN0}\|$.*

Remark 8 *The matrix P^-_{MN0} in (46) contains a larger number of rows than a similar matrix used in the computation of $\|\mathbf{D}_{11}\|$ for continuous-time LTI systems [15]. This is because sampled-data systems are h-periodic in continuous-time; the LTI nature in [15] allows us to focus only on the last block row in the fast-lifted compression operator $\mathbf{L}_M\mathbf{D}_{11}\mathbf{L}_M^{-1}$, while the h-periodic nature in the present paper requires us to deal with all block rows in the corresponding fast-lifted representation $\mathcal{F}^-_{MN}$. A similar observation applies to the later results for piecewise linear approximation.*

We can summarize the arguments in this subsection as follows: Computing the approximate value $\|\mathcal{F}^-_N\|$ for the L_∞-induced norm can be achieved by piecewise constant approximation through the fast-lifted treatment, its upper and lower bounds can be computed exactly through matrix manipulations, and the gap between these bounds tends to 0 at the rate of $1/M$ (since K_{M0} has a uniform upper bound K_0^U given in (43)).

Remark 9 *We would like to note that although the use of P^-_{MN0} (and thus the central part of the computation method in this subsection) has something in common with [2],[9],[19] (and would essentially recover the computations in these studies if we were to consider only the limit of $\|P^-_{MN0}\|$ for $N \to \infty$), the overall method with piecewise constant approximation here is completely different from that in these existing studies. This is because the present paper provides readily computable upper and lower bounds of the L_∞-induced norm (aside from the extension to piecewise linear approximation discussed in the following subsection), while the existing studies only show the convergence rate without providing any readily computable upper and lower bounds.*

4.2 Piecewise Linear Approximation of $\mathcal{F}_N^-$

Next, this subsection considers computing upper and lower bounds of $\|\mathcal{F}_N^-\|$ through piecewise linear approximation of $\mathcal{F}_N^-$.

A key idea in this direction is to use the 'linearizing' operator $\mathbf{J}_1'$ defined by

$$(\mathbf{J}_1' w)(\theta') = \int_0^{h'} f_0(\tau')w(\tau')d\tau' + \theta' \int_0^{h'} f_1(\tau')w(\tau')d\tau' \tag{47}$$

with the scalar functions $f_0(\tau')$ and $f_1(\tau')$ given by

$$f_0(\tau') = -\frac{6}{(h')^2}\tau' + \frac{4}{h'},\ f_1(\tau') = \frac{12}{(h')^3}\tau' - \frac{6}{(h')^2} \tag{48}$$

This specific operator was introduced in [15] in the relevant computation problem of $\|\mathbf{D}_{11}\|$, and satisfies $\mathbf{J}_1' w = w$ for any linear function w (among other technically important properties). We further introduce the operator $\mathbf{B}_{p1}' := \mathbf{B}_1' \mathbf{J}_1'$. This is equivalent to restricting the input of $\mathbf{B}_1'$ to linear functions. We further introduce the operators $\mathbf{M}_{p1}'$ and $\mathbf{D}_{p1}'$ defined by

$$\begin{aligned}\left(\mathbf{M}_{p1}' \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') &= [C_1\ D_{12}](I + A_2\theta') \begin{bmatrix} x \\ u \end{bmatrix} \\ &= ([C_1\ D_{12}] + C_1[A\ B_2]\theta') \begin{bmatrix} x \\ u \end{bmatrix}\ (0 \le \theta' < h') \end{aligned} \tag{49}$$

$$(\mathbf{D}_{p1}' w)(\theta') = C_1 B_1 \int_0^{\theta'} w(\tau')d\tau' + D_{11} w(\theta')\ (0 \le \theta' < h') \tag{50}$$

$\mathbf{M}_{p1}'$ gives the first-order approximation of the Taylor expansion of the output of $\mathbf{M}_1'$ and thus its output is also a linear function. $\mathbf{D}_{p1}'$ was also introduced in [15] to approximate $\mathbf{D}_{11}'$, but note that, unlike $\mathbf{B}_{p1}'$, introducing $\mathbf{D}_{p1}'$ is not equivalent to restricting the input of $\mathbf{D}_{11}'$ to linear functions even when $D_{11} = 0$. It may be quite interesting to note that the compact operator $\mathbf{D}_{11}' - D_{11}$ is approximated by the *infinite-rank but rather amenable integral operator* $\mathbf{D}_{p1}' - D_{11} = C_1 B_1 \int_0^{\theta} \cdot\, d\tau$.

We are in a position to introduce the linear approximation $\mathcal{P}_{MN1}^-$ for $\mathcal{F}_{MN}^-$, by which we mean to replace $\mathbf{B}_1'$, $\mathbf{M}_1'$ and $\mathbf{D}_{11}'$ in (27) with $\mathbf{B}_{p1}'$, $\mathbf{M}_{p1}'$ and $\mathbf{D}_{p1}'$, respectively:

$$\mathcal{P}_{MN1}^- = [\overline{\mathbf{M}_{p1}'}\Delta_{M0}\overline{\mathbf{B}_{p1}'} + \overline{\mathbf{D}_{p1}'}\quad \overline{\mathbf{M}_{p1}'}\mathcal{A}_{M0}\overline{\mathbf{B}_{p1}'}\ \cdots\ \overline{\mathbf{M}_{p1}'}\mathcal{A}_{MN}\overline{\mathbf{B}_{p1}'}] \tag{51}$$

This in turn defines piecewise linear approximation of $\mathcal{F}_N^-$. This subsection shows that $\|\mathcal{P}_{MN1}^-\|$ can be computed exactly and converges to $\|\mathcal{F}_N^-\|$ at the rate of $1/M^2$. The following two lemmas are important in establishing a more precise assertion.

Lemma 3 ([15], Theorem 8) *The inequality*

$$\|(\overline{\mathbf{M}_1'}\Delta_{M0}\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{11}'}) - (\overline{\mathbf{M}_{p1}'}\Delta_{M0}\overline{\mathbf{B}_{p1}'} + \overline{\mathbf{D}_{p1}'})\| \le \frac{K_{M\mathcal{D}1}}{M^2} \tag{52}$$

holds, where

$$\begin{aligned} K_{M\mathcal{D}1} &:= \frac{1}{2}\|A\|^2 \cdot \|B_1\| e^{\|A\|h/M} \frac{h^3}{M} \\ &\cdot \sum_{k=0}^{M-2} \left\{ \|C_1(A_d')^k\| e^{\|A\|h/M} + \sup_{0 \le \theta' < h'} \|C_1(I + A\theta')(A_d')^{k+1}\| \right\} \\ &\quad + \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h/M} \end{aligned} \tag{53}$$

Furthermore, $K_{M\mathcal{D}1}$ has a uniform upper bound with respect to M given by

$$\begin{aligned} K_{\mathcal{D}1}^U &:= \frac{1}{2}\|C_1\| \cdot \|A\|^2 \cdot \|B_1\| e^{2\|A\|h} h^3 \left(e^{\|A\|h} + 1 + \|A\|h\right) \\ &\quad + \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h} \end{aligned} \tag{54}$$

Lemma 4 *The inequality*

$$\|\overline{\mathbf{M}_1'}\mathcal{A}_{Mj}\overline{\mathbf{B}_1'} - \overline{\mathbf{M}_{p1}'}\mathcal{A}_{Mj}\overline{\mathbf{B}_{p1}'}\| \le \frac{K_{Mj1}}{M^2} \tag{55}$$

holds for $j = 0, \cdots, N$, where

$$\begin{aligned} K_{Mj1} &= \frac{1}{2} e^{\|A\|h/M} \|\mathcal{A}_{Mj}\| \frac{h^3}{M} \\ &\cdot \Big\{ \max_{\theta' \in \{0, h'\}} \|[C_1\ D_{12}]\,(I + A_2\theta')\|\, \|A\|^2 \cdot \|A_d' B_1\| \\ &\quad + \left\|[C_1\ D_{12}]A_2^2\right\| e^{\|A_2\|\frac{h}{M}} \|B_1\| \Big\} \end{aligned} \tag{56}$$

Furthermore, K_{Mj1} has a uniform upper bound with respect to M and j defined as

$$\begin{aligned} K_{\mathcal{CAB}1}^U &:= \frac{1}{2} h^3 e^{\|A\|h} \|B_1\| K_* \\ &\cdot \Big\{ (\|[C_1\ D_{12}]\| + \|[C_1\ D_{12}]A_2\|\, h)\, \|A\|^2 e^{\|A\|h} \\ &\quad + \left\|[C_1\ D_{12}]A_2^2\right\| e^{\|A_2\|h} \Big\} \end{aligned} \tag{57}$$

where K_* is given by (40).

The proof of Lemma 4 is also given in Appendix A. From Lemmas 3 and 4, we readily obtain the following result.

Proposition 2 *The inequality*

$$\|\mathcal{F}_{MN}^- - \mathcal{P}_{MN1}^-\| \le \frac{K_{M1}}{M^2} \tag{58}$$

holds, where

$$K_{M1} := K_{M\mathcal{D}1} + \sum_{j=0}^{N} K_{Mj1} \tag{59}$$

In addition, K_{M1} has a uniform upper bound with respect to M given by

$$K_1^U := K_{\mathcal{D}1}^U + (N+1)\cdot K_{\mathcal{CAB}1}^U \tag{60}$$

With an application of the triangle inequality to (58) in mind, we now turn to giving a method for (*exactly*) computing the $L_\infty[0,h')$-induced norm

$$\|\mathcal{P}_{MN1}^-\| = \sup_{\|w\|\le 1} \|(\mathcal{P}_{MN1}^- w)(\cdot)\| \tag{61}$$

We note on the right hand side of (61) that

$$(\mathcal{P}_{MN1}^- w)(\theta') = \sum_{j=1}^{N+1} \left(\overline{\mathbf{M}'_{p1}}\mathcal{A}_{M,j-1}\overline{\mathbf{B}'_{p1}} w_j\right)(\theta') + \left((\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} + \overline{\mathbf{D}'_{p1}})w_0\right)(\theta') \tag{62}$$

where $w =: [w_0^T, \cdots, w_{N+1}^T]^T$.

We first consider the matrix function

$$\left((\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} + \overline{\mathbf{D}'_{p1}})w_0\right)(\theta') \tag{63}$$

in (62). Let us further introduce the partitioned notation $w_j =: [(w_j^{(1)})^T, \cdots, (w_j^{(M)})^T]^T$ $(j = 0, \cdots, N+1)$ by noting that w_j is in fact a fast-lifting representation of a signal on $[0,h)$. Then, for every $i \in \{1, \cdots, M\}$, $w_0^{(i)}$ appears only on the ith block row in $\overline{\mathbf{D}'_{p1}} w_0$ while it appears only on the kth block rows in $\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} w_0$ with $k > i$; this is because of the strict block lower triangular structure of Δ_{M0}. We further note in (61) that

$$\|(\mathcal{P}_{MN1}^- w)(\cdot)\| = \max_{i=1,\cdots,M} \|(\mathcal{P}_{MN1}^- w)_i(\cdot)\| \tag{64}$$

where $(\cdot)_i$ denotes the ith block row of $(\cdot)$. This implies that the block rows mentioned above can be handled one by one, and thus when $(\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} + \overline{\mathbf{D}'_{p1}})w_0$ is expanded into $\overline{\mathbf{D}'_{p1}} w_0$ and $\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} w_0$, the input w_0 in the first term may be handled independently of that in the second term (i.e., they may be regarded to be different functions), as long as we further take $\sup_{\|w\|\le 1}$ as in (61). This is equivalent to saying that $\mathcal{P}_{MN1}^-$ may be redefined as

$$\mathcal{P}_{MN1}^- = [\ \overline{\mathbf{D}'_{p1}} \quad \overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} \quad \overline{\mathbf{M}'_{p1}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{p1}} \quad \cdots \quad \overline{\mathbf{M}'_{p1}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{p1}}\ ] \tag{65}$$

without changing $\|\mathcal{P}_{MN1}^-\|$. Noting the definition of $\mathbf{D}'_{p1}$ in (50), let us further introduce the integral operator $\mathbf{D}'_{p10} := \mathbf{D}'_{p1} - D_{11}$. Then, it follows again from the property of $L_\infty[0,h')$ that $\mathcal{P}_{MN1}^-$ may be redefined further, without changing its norm, as

$$\mathcal{P}_{MN1}^- = [\ \overline{D_{11}} \quad \overline{\mathbf{D}'_{p10}} \quad \overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} \quad \overline{\mathbf{M}'_{p1}}\mathcal{A}_{M0}\overline{\mathbf{B}'_{p1}} \quad \cdots \quad \overline{\mathbf{M}'_{p1}}\mathcal{A}_{MN}\overline{\mathbf{B}'_{p1}}\ ] \tag{66}$$

Throughout the rest of this paper, we mean (66) by $\mathcal{P}_{MN1}^-$.

Summarizing the above arguments, we may replace the function $\left((\overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}} + \overline{\mathbf{D}'_{p1}})w_0\right)(\theta')$ in (62) by $\overline{D_{11}}w_{00} + \overline{\mathbf{D}'_{p10}}w_{01} + \overline{\mathbf{M}'_{p1}}\Delta_{M0}\overline{\mathbf{B}'_{p1}}w_{02}$, where $\|w_{0i}\| \le 1$ $(i = 0,1,2)$, as long as we consider evaluating $\|\mathcal{P}_{MN1}^-\|$. Note that the third term is a linear function by the definition of $\mathbf{M}'_{p1}$, as well as all the terms in (62) except the last. For simplicity, let us suppose $D_{11} = 0$ for a while, even though we will eventually deal with the case of $D_{11} \ne 0$. Then, since all the terms of (62) are linear functions except $\overline{\mathbf{D}'_{p10}}w_{01}$ and since $\mathbf{D}'_{p10}$ is simply an integral operator, it follows readily that w_{01} may be restricted to a constant function in its treatment for evaluating $\|\mathcal{P}_{MN1}^-\|$. An immediate consequence of this restriction is that $(\mathcal{P}_{MN1}^- w)(\theta')$ becomes a linear vector function, so that (61) reduces to

$$\begin{aligned}\|\mathcal{P}_{MN1}^-\| &= \sup_{\|w\|\le 1} \max_{\theta'=0,h'} \|(\mathcal{P}_{MN1}^- w)(\theta')\| \\ &= \sup_{\|w\|\le 1} \left\| \begin{bmatrix} (\mathcal{P}_{MN1}^- w)(0) \\ (\mathcal{P}_{MN1}^- w)(h') \end{bmatrix} \right\| \end{aligned} \tag{67}$$

where w is redefined as $[w_{00}^T, w_{01}^T, w_{02}^T, w_1^T, \cdots, w_{N+1}^T]^T$, and $(\mathcal{P}_{MN1}^- w)(h')$ is defined by continuity of a linear function.

By using the matrices $V^{[0]}$, $V^{[h']}$, $T_{ji}^{[0]}$ and $T_{ji}^{[h']}$ $(i = 1, \cdots, M; j = 1, \cdots, N+1)$ given in Appendix B, we can obtain the following prelude to the second main result. In particular, it gives an exact computation method for the $L_\infty[0,h')$-induced norm $\|\mathcal{P}_{MN1}^-\|$ given by (67). See Appendix B for the arguments leading to this result.

Theorem 2 *The inequality*

$$\|\mathcal{P}_{MN1}^{-}\| - \frac{K_{M1}}{M^2} \leq \|\mathcal{F}_N^{-}\| \leq \|\mathcal{P}_{MN1}^{-}\| + \frac{K_{M1}}{M^2} \tag{68}$$

holds. Furthermore, $\|\mathcal{P}_{MN1}^{-}\|$ *coincides with the* ∞*-norm of the finite-dimensional matrix*

$$P_{MN1}^{-} := \begin{bmatrix} U^{[0]} & T_1^{[0]} & \cdots & T_{N+1}^{[0]} \\ U^{[h']} & T_1^{[h']} & \cdots & T_{N+1}^{[h']} \end{bmatrix} \tag{69}$$

where

$$U^{[0]} := \begin{bmatrix} \overline{D_{11}} & 0 & V^{[0]} \end{bmatrix} \tag{70}$$

$$U^{[h']} := \begin{bmatrix} \overline{D_{11}} & \overline{C_1 B_1} h' & V^{[h']} \end{bmatrix} \tag{71}$$

$$T_j^{[0]} := \begin{bmatrix} T_{j1}^{[0]} & \cdots & T_{jM}^{[0]} \end{bmatrix} \quad (j = 1, \cdots, N+1) \tag{72}$$

$$T_j^{[h']} := \begin{bmatrix} T_{j1}^{[h']} & \cdots & T_{j1}^{[h']} \end{bmatrix} \quad (j = 1, \cdots, N+1) \tag{73}$$

with $V^{[0]} := [V_1^{[0]} \ \cdots \ V_M^{[0]}]$ *and* $V^{[h']} := [V_1^{[h']} \ \cdots \ V_M^{[h']}]$.

Remark 10 *In (69) (or more precisely in (70) and (71)), we have recovered the general case with* $D_{11} \neq 0$*, which can be validated as in Remark 7.*

To summarize, we have shown in this subsection that similar arguments to the preceding subsection can be developed by piecewise linear approximation through the fast-lifted treatment, in which the gap between the upper and lower bounds of $\|\mathcal{F}_N^{-}\|$ tends to 0 at the rate of $1/M^2$.

4.3 Upper Bound of $\|\mathcal{F}_N^{+}\|$ *and Computation of* $\|\mathcal{F}\|$

This subsection gives a computation method for an upper bound of $\|\mathcal{F}_N^{+}\|$, which together with the arguments in the preceding subsections lead to methods for computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{F}\|$ of the sampled-data system $\Sigma_{\rm SD}$. These bounds are ensured to converge to each other as the parameters M and N tend to ∞.

We first note that

$$\|\mathcal{F}_N^{+}\| \leq \|\mathbf{M}_1\| \cdot \|\mathbf{B}_1\| \cdot \left\| \begin{bmatrix} C_\Sigma \mathcal{A}^{N+1} & C_\Sigma \mathcal{A}^{N+2} & \cdots \end{bmatrix} \right\| \tag{74}$$

If we take an $L \in \mathbb{N}$ such that $\|\mathcal{A}^L\| < 1$, it readily follows that

$$\left\| [C_\Sigma \mathcal{A}^{N+1} \ C_\Sigma \mathcal{A}^{N+2} \ \cdots] \right\| \leq \frac{\|C_\Sigma \mathcal{A}_{NL}\|}{1 - \|\mathcal{A}^L\|} \tag{75}$$

where $\mathcal{A}_{NL} := \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots & \mathcal{A}^{N+L} \end{bmatrix}$. Summarizing (74) and (75), we obtain the following result.

Proposition 3 *If* $\|\mathcal{A}^L\| < 1$*, then*

$$\begin{aligned} \|\mathcal{F}_N^{+}\| &\leq \frac{\|C_\Sigma \mathcal{A}_{NL}\|}{1 - \|\mathcal{A}^L\|} \|[C_1 \ D_{12}]\| e^{\|A_2\| h} h e^{\|A\| h} \|B_1\| \\ &=: K_{NL} \end{aligned} \tag{76}$$

and K_{NL} *converges to* 0 *regardless of* L *as* $N \to \infty$.

Proof. It readily follows that

$$\|\mathbf{M}_1\| \leq \|[C_1 \ D_{12}]\| e^{\|A_2\| h} \tag{77}$$

$$\|\mathbf{B}_1\| \leq h e^{\|A\| h} \|B_1\| \tag{78}$$

and hence the first assertion follows immediately. The second assertion is immediate from the fact that $\|C_\Sigma \mathcal{A}_{NL}\| \to 0$ as $N \to \infty$. Q.E.D.

Combining Theorems 1 and 2, Proposition 3 together with (17), we are led to the following main results.

Theorem 3 *If* $\|\mathcal{A}^L\| < 1$*, then*

$$\|P_{MN0}^{-}\| - \frac{K_{M0}}{M} - K_{NL} \leq \|\mathcal{F}\| \leq \|P_{MN0}^{-}\| + \frac{K_{M0}}{M} + K_{NL} \tag{79}$$

$$\|P_{MN1}^{-}\| - \frac{K_{M1}}{M^2} - K_{NL} \leq \|\mathcal{F}\| \leq \|P_{MN1}^{-}\| + \frac{K_{M1}}{M^2} + K_{NL} \tag{80}$$

Furthermore, K_{M0} *and* K_{M1} *have uniform upper bounds* K_0^U *and* K_1^U *defined as (43) and (60), respectively, and* K_{M0}/M *and* K_{M1}/M^2 *converge to* 0 *as* $M \to \infty$*, while* K_{NL} *converges to* 0 *regardless of* L *as* $N \to \infty$.

4.4 Guideline for taking approximation parameters

It should be noted in (79) and (80) that the uniform upper bounds K_0^U of K_{M0} and K_1^U of K_{M1} given in (43) and (60), respectively, depend on N, and increase as N is increased to reduce K_{NL}. However, K_{NL} is bounded from above in the exponential order ρ^N in N regardless of L, for any $\rho < 1$ larger than the spectral radius of $\mathcal{A}$ and thus should reduce relatively fast with respect to N. Hence, it is expected that we can keep the uniform upper bounds K_0^U and K_1^U modest, and thus K_{M0}/M and K_{M1}/M^2 can also be made small with a modest M.

Regarding a guideline for taking the parameters N, M and L, we can summarize the above arguments as follows. It may be reasonable to take a relatively small L as long as $\|\mathcal{A}^L\| < 1$; this is to avoid undue increase of K_{NL}, or in particular $\|\mathcal{A}_{NL}\|$ (or the computation time for them). Once L is fixed, the next step would be to take an N such that K_{NL} is as small as we wish; this is always possible by taking N sufficiently large. For example, if

$$\mathcal{A} = P_{\mathcal{A}} \Lambda_{\mathcal{A}} P_{\mathcal{A}}^{-1} \tag{81}$$

with a diagonal $\Lambda_{\mathcal{A}}$, then it is easy to see that

$$K_{NL} \le \rho^{N+1} K_{\mathcal{A}} \tag{82}$$

where

$$K_{\mathcal{A}} := L \frac{\|C_\Sigma P_{\mathcal{A}}\| \|P_{\mathcal{A}}^{-1}\|}{1 - \|\mathcal{A}^L\|} \|[C_1\ D_{12}]\| e^{\|A_2\|h} h e^{\|A\|h} \|B_1\| \tag{83}$$

This implies that $K_{NL} \le \epsilon$ whenever $N \ge N_\epsilon := (\log \epsilon - \log K_{\mathcal{A}})/\log \rho - 1$. Once N is also fixed, the uniform upper bounds K_0^U and K_1^U in (43) and (60), respectively, are determined, and thus the last step would be to take an M such that K_0^U/M and K_1^U/M^2 are as small as we wish. It is obvious that following this kind of guideline leads to computation methods for the L_∞-induced norm of Σ_{SD} (given by $\|\mathcal{F}\|$) to any degree of accuracy.

5 Numerical Example

In this section, we study a numerical example and examine effectiveness of the computation methods developed in the preceding section.

Consider the stable sampled-data system

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & -1.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0 & 1.5 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D_{11} = 1,\ D_{12} = 0 \tag{84}$$
$$A_\Psi = \begin{bmatrix} -0.4888 & 1.6687 \\ 0.0737 & -0.2547 \end{bmatrix},\ B_\Psi = \begin{bmatrix} -3.1180 \\ 0.4701 \end{bmatrix}$$
$$C_\Psi = \begin{bmatrix} -1.6601 & 5.7348 \end{bmatrix}, D_\Psi = -7.5709 \tag{85}$$

with $h = 0.5$. We compute estimates of the L_∞-induced norm $\|\mathcal{F}\|$ by taking the fast-lifting parameter M ranging from 50 to 500 on the condition that $L = 10$ and then $N = 50$, which follow in this order by the guideline in Subsection 4.4, leading to $K_{NL} = 3.01 \times 10^{-7}$. The results of the estimate $\|P_{MN0}^-\|$ with (46), the error bound $K_{M0}/M + K_{NL}$ (with K_{M0} given by (42)) and the computation time corresponding to the piecewise constant approximation method are shown in Table 1. In addition, the results of $\|P_{MN1}^-\|$ with (69), $K_{M1}/M^2 + K_{NL}$ (with K_{M1} given by (59)) and computation time for piecewise linear approximation are shown in Table 2.

We can see from Tables 1 and 2 that the error bounds for the computation of $\|\mathcal{F}\|$ through its estimates $\|P_{MN0}^-\|$ and $\|P_{MN1}^-\|$ are decreasing by taking M larger. Hence, we can confirm validity of the piecewise constant and piecewise linear approximation methods for computing the L_∞-induced norm $\|\mathcal{F}\|$. In particular, we can also observe that $K_{M1}/M^2 + K_{NL}$ is much smaller than $K_{M0}/M + K_{NL}$ under the same parameter M. This demonstrates that the piecewise linear approximation method works much more effectively than the piecewise constant approximation method. In this respect, it should be observed that the latter method requires much larger computation time than the former method under the same parameter M. However, we can also see from these tables that the error $K_{M1}/M^2 + K_{NL}$ in piecewise linear approximation with $M = 50$ is much smaller than the error $K_{M0}/M + K_{NL}$ in piecewise constant approximation with $M = 500$, while the computation time for the former is smaller than that for the latter. These observations suggest that the piecewise linear approximation method drastically outperforms the piecewise constant approximation method, which essentially is the conventional FSFH approximation method.

Table 1

Computation results with piecewise constant approximation ($L = 10,\ N = 50$).

M	50	100	200	500
$\|P_{MN0}^-\|$	3.5632	3.5635	3.5635	3.5636
$\frac{K_{M0}}{M} + K_{NL}$	1.5979	0.7864	0.3901	0.1553
time (sec)	0.0626	0.1309	0.5626	7.3574

Table 2

Computation results with piecewise linear approximation ($L = 10,\ N = 50$).

M	50	100	200	500
$\|P_{MN1}^-\|$	3.5632	3.5635	3.5635	3.5636
$\frac{K_{M1}}{M^2} + K_{NL}$	0.0165	0.0041	0.0010	1.61×10^{-4}
time (sec)	2.6468	10.3856	41.6643	265.8337

6 Conclusion

In this paper, we developed two methods for computing the L_∞-induced norm of sampled-data systems by using ideas of piecewise constant and piecewise linear approximations, stimulated by the success in computing the $L_\infty[0, h)$-induced norm of a compression operator. We showed that upper and lower bounds of the L_∞-induced norm can be derived through such approximations and that the gap between the upper bound and lower bound is ensured to converge to 0 at the rate of $1/M$ and $1/M^2$ in piecewise constant and piecewise linear approximations, respectively, where M is the parameter for fast-lifting underlying these approximations. This demonstrates the effectiveness of our approaches developed in the present paper; this assertion is justified since the conventional FSFH approximation method gives just an asymptotic result that the convergence rate is $1/M$. We then examined effectiveness of the developed methods through a numerical study, and it was confirmed that the piecewise linear approximation method works far more effectively than the piecewise constant approximation method.

Finally, we give some remark on why this paper confines itself to piecewise constant and linear approximation approaches and does not deal with piecewise higher-order-polynomial approximation. Simply constructing the ith-order approximant $\mathbf{B}'_{pi}$ to $\mathbf{B}'_1$ (with desired properties from the ith-order approximation viewpoint) could be carried out even for $i \geq 2$ by following the same line of arguments as in [15]. The ith-order approximant $\mathbf{M}'_{pi}$ to $\mathbf{M}'_1$ can also be introduced readily through the Taylor series expansion. Nevertheless, extension of the present studies to $i \geq 2$ is nontrivial because it seems very hard to find a way to uniquely fix the input of $\mathbf{M}'_{pi}$ to such a value that is ensured to be 'the one we may assume in our induced-norm computation.' Hence, we cannot predetermine the timing $\theta' \in [0, h')$ such that the output of $\mathbf{M}'_{pi}$ at θ' does correspond to our induced-norm computation. This is in sharp contrast with the present paper dealing only with $i = 0$ and $i = 1$ (i.e., constant and linear functions), in which it is obvious that considering only $\theta' = 0$ and $\theta' \to h'$ suffices *whatever the input of* $\mathbf{M}'_{pi}$ *may be* (i.e., despite that even $i = 0$ or $i = 1$ does not allow us to uniquely fix its input, either). Note that this strong feature was the key in successfully circumventing the reference to θ' when (B.5) was reduced to (B.10) and (B.15) (and similarly for the θ' in (B.8)) and leading to finite-dimensional discretization. Another obstacle may be how to construct and deal with suitable approximants $\mathbf{D}'_{pi}$ to $\mathbf{D}'_{11}$ for $i \geq 2$, which is also nontrivial. Resolving all these issues might lead to an extension of the results in this paper to $i \geq 2$, and such a direction might be qualified as a possible future study.

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A Proofs of Lemmas 2 and 4

This appendix is concerned with the proofs of Lemmas 2 and 4. They are based on the Taylor expansion of the matrix exponential of Ah' (or $A\theta'$), and the proof of Lemma 2 proceeds in essentially the same way as that of Lemma 4. Hence, only the proof of the latter lemma is given here.

By the Taylor expansion of $\exp(A_2\theta')$,

$$\left((\mathbf{M}_1' - \mathbf{M}_{p1}')w\right)(\theta') = [C_1\ D_{12}]\sum_{i=2}^{\infty}\frac{(A_2\theta')^i}{i!}w \quad (A.1)$$

Hence,

$$\|\mathbf{M}_1' - \mathbf{M}_{p1}'\| \le \frac{1}{2}(h')^2 \left\|[C_1\ D_{12}]A_2^2\right\| e^{\left\|A_2 h'\right\|} \quad (A.2)$$

On the other hand, since f_0 and f_1 are scalar functions, it follows from the Taylor expansion of $\exp(A\theta')$ that

$$\begin{aligned}&(\mathbf{B}_1' - \mathbf{B}_{p1}')w\\ &= \int_0^{h'}\{\exp(A(h'-\tau')) - A_{0d}'f_0(\tau') - A_{1d}'f_1(\tau')\}B_1 w(\tau')d\tau'\end{aligned} \quad (A.3)$$

where

$$A_{0d}' = \int_0^{h'}\exp(A(h'-\theta'))d\theta' = \int_0^{h'}\exp(-A\theta')d\theta' A_d' \quad (A.4)$$

$$A_{1d}' = \int_0^{h'}\exp(A(h'-\theta'))\theta' d\theta' = \int_0^{h'}\exp(-A\theta')\theta' d\theta' A_d' \quad (A.5)$$

As shown in [15], we have

$$\begin{aligned}&\exp(A(h'-\tau')) - A_{0d}'f_0(\tau') - A_{1d}'f_1(\tau')\\ &= \left\{\sum_{i=2}^{\infty}\frac{(-A)^i(\tau')^i}{i!} - \left(\frac{6i}{(i+2)!}(-A)^i(h')^{i-1}\right)\tau'\right\}A_d'\\ &\quad + \left\{\sum_{i=2}^{\infty}\frac{2(i-1)}{(i+2)!}(-A)^i(h')^i\right\}A_d'\\ &=: L_A(\tau')A_d'\end{aligned} \quad (A.6)$$

where

$$\int_0^{h'}\|L_A(\tau')\|d\tau' \le \frac{1}{2}(h')^3\|A\|^2 e^{\|A\|h'} \quad (A.7)$$

Hence, we readily see that

$$\|\mathbf{B}_1' - \mathbf{B}_{p1}'\| \le \frac{1}{2}(h')^3\|A\|^2 e^{\|A\|h'}\|A_d'B_1\| \quad (A.8)$$

The assertion (55) now follows by applying (A.2) and (A.8) to

$$\begin{aligned}&\|\overline{\mathbf{M}_1'}\mathcal{A}_{Mj}\overline{\mathbf{B}_1'} - \overline{\mathbf{M}_{p1}'}\mathcal{A}_{Mj}\overline{\mathbf{B}_{p1}'}\|\\ &\le \|(\overline{\mathbf{M}_1'} - \overline{\mathbf{M}_{p1}'})\mathcal{A}_{Mj}\overline{\mathbf{B}_1'}\| + \|\overline{\mathbf{M}_{p1}'}\mathcal{A}_{Mj}\left(\overline{\mathbf{B}_1'} - \overline{\mathbf{B}_{p1}'}\right)\|\end{aligned} \quad (A.9)$$

and then noting the following inequalities:

$$\|\mathbf{B}_1'\| \le h' e^{\|A\|h'}\|B_1\| \quad (A.10)$$

$$\|\mathbf{M}_{p1}'\| \le \max_{\theta'\in\{0,h'\}}\|[C_1\ D_{12}](I + A_2\theta')\| \quad (A.11)$$

The second assertion can be proved easily if we note that

$$\|A_d'\| \le e^{\|A\|h} \quad (A.12)$$

$$\|\mathcal{A}_{Mj}\| \le M e^{(\|A\|+\|A_2\|)h}\|C_\Sigma\| \cdot \max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\| \quad (A.13)$$

regardless of M and j and that

$$\|[C_1\ D_{12}]\,(I + A_2\theta')\| \le \|[C_1\ D_{12}]\| + \|[C_1\ D_{12}]A_2 h\| \quad (A.14)$$

B Computation method for $\|\mathcal{P}_{MN1}^-\|$

This appendix is devoted to the derivation of Theorem 2. We begin by giving a concise way for representing $(\mathcal{P}_{MN1}^- w)(0)$ and $(\mathcal{P}_{MN1}^- w)(h')$. A direct computation shows that

$$\begin{aligned}\mathbf{B}_{p1}'w_j^{(i)} &= \int_0^{h'}(B_{0d}'f_0(\tau') + B_{1d}'f_1(\tau'))w_j^{(i)}(\tau')d\tau'\\ &= \int_0^{h'}(G_0 + G_1\tau')w_j^{(i)}(\tau')d\tau'\end{aligned} \quad (B.1)$$

where

$$B_{1d}' := \int_0^{h'}\exp(A(h'-\theta'))\theta' B_1 d\theta' \quad (B.2)$$

$$G_0 := -\frac{6}{(h')^2}B_{1d}' + \frac{4}{h'}B_{0d}' \quad (B.3)$$

$$G_1 := \frac{12}{(h')^3}B_{1d}' - \frac{6}{(h')^2}B_{0d}' \quad (B.4)$$

Hence, noting (49), we readily see that the function $\left(\overline{\mathbf{M}_{p1}'}\mathcal{A}_{M,j-1}\overline{\mathbf{B}_{p1}'}w_j\right)(\theta')$ in (62) equals the linear function

$$\sum_{i=1}^{M}(H_{ji0} + H_{ji1}\theta')\int_0^{h'}(G_0 + G_1\tau')w_j^{(i)}(\tau')d\tau' \quad (B.5)$$

where the matrices H_{ji0} and H_{ji1} $(i = 1, \cdots, M; j = 1, \cdots, N+1)$ are defined as

$$H_{ji0} := \overline{[C_1\ D_{12}]} A'_{2dM} C_\Sigma \mathcal{A}^{j-1} J_\Sigma (A'_d)^{M-i} \quad \text{(B.6)}$$
$$H_{ji1} := \overline{C_1 [A\ B_2]} A'_{2dM} C_\Sigma \mathcal{A}^{j-1} J_\Sigma (A'_d)^{M-i} \quad \text{(B.7)}$$

Similarly, under the notation $w_{0k} = [(w_{0k}^{(1)})^T, \cdots, (w_{0k}^{(M)})^T]$ $(k = 1, 2)$, it follows that $\left(\overline{\mathbf{M}'_{p1}} \Delta_{M0} \overline{\mathbf{B}'_{p1}} w_{02}\right)(\theta')$ equals the linear function

$$\sum_{i=1}^{M} (S_{i0} + S_{i1}\theta') \int_0^{h'} (G_0 + G_1\tau') w_{02}^{(i)}(\tau') d\tau' \quad \text{(B.8)}$$

where

$$S_{i0} := \overline{[C_1\ D_{12}]} \Delta_{M0i}, \quad S_{i1} := \overline{C_1[A\ B_2]} \Delta_{M0i} \quad \text{(B.9)}$$

and Δ_{M0i} is the ith block column of Δ_{M0}.

It follows from a direct computation with (B.5) and (B.8) together with the definition of $\mathbf{D}'_{p10}$ that $(\mathcal{P}^-_{MN1} w)(0)$ with $w = [w_{00}^T, w_{01}^T, w_{02}^T, w_1^T, \cdots, w_{N+1}^T]^T$ is determined by the mappings

$$w_j^{(i)} \mapsto \int_0^{h'} (Y_{ji0}^{[0]} + Y_{ji1}^{[0]}\tau') w_j^{(i)}(\tau') d\tau' \quad (j = 1, \cdots, N+1) \quad \text{(B.10)}$$
$$w_{00}^{(i)} \mapsto 0, \quad w_{01}^{(i)} \mapsto 0 \quad \text{(B.11)}$$
$$w_{02}^{(i)} \mapsto \int_0^{h'} (Z_{i0}^{[0]} + Z_{i1}^{[0]}\tau') w_{02}^{(i)}(\tau') d\tau' \quad \text{(B.12)}$$

where

$$Y_{ji0}^{[0]} := H_{ji0} G_0, \quad Y_{ji1}^{[0]} := H_{ji0} G_1 \quad \text{(B.13)}$$
$$Z_{i0}^{[0]} := S_{i0} G_0, \quad Z_{i1}^{[0]} := S_{i0} G_1 \quad \text{(B.14)}$$

Similarly, since w_{01} is assumed to be a constant function (whose value equals $w_{01}(0)$), it follows that $(\mathcal{P}^-_{MN1} w)(h')$ is determined by the mappings

$$w_j^{(i)} \mapsto \int_0^{h'} (Y_{ji0}^{[h']} + Y_{ji1}^{[h']}\tau') w_j^{(i)}(\tau') d\tau' \quad (j = 1, \cdots, N+1) \quad \text{(B.15)}$$
$$w_{00}^{(i)} \mapsto 0, \quad w_{01}^{(i)} \mapsto C_1 B_1 h' w_{01}^{(i)}(0) \quad \text{(B.16)}$$
$$w_{02}^{(i)} \mapsto \int_0^{h'} (Z_{i0}^{[h']} + Z_{i1}^{[h']}\tau') w_{02}^{(i)}(\tau') d\tau' \quad \text{(B.17)}$$

where

$$Y_{ji0}^{[h']} := (H_{ji0} + H_{ji1}h') G_0, \ Y_{ji1}^{[h']} := (H_{ji0} + H_{ji0}h') G_1 \quad \text{(B.18)}$$
$$Z_{i0}^{[h']} := (S_{i0} + S_{i1}h') G_0, \ Z_{i1}^{[h']} := (S_{i0} + S_{i1}h') G_1 \quad \text{(B.19)}$$

The above mappings immediately lead us to a procedure for the computation of $\|\mathcal{P}^-_{MN1}\|$ given in (67). This can be summarized as follows if we note that computing the induced norm of the operator representing the action (B.10) would require us to compute the $L_1[0, h')$ norm of each entry of $Y_{ji0}^{[0]} + Y_{ji1}^{[0]}\tau'$; by the properties of the $L_\infty[0, h')$ norm, t suffices us to repeat essentially the same arguments:

Let $T_{ji}^{[0]}$ $(j = 1, \cdots, N+1;\ i = 1, \cdots, M)$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix function $Y_{ji0}^{[0]} + Y_{ji1}^{[0]}\tau'$ involved in (B.10), while let $T_{ji}^{[h']}$ $(j = 1, \cdots, N+1;\ i = 1, \cdots, M)$ be the matrix constructed in the same way from $Y_{ji0}^{[h']} + Y_{ji1}^{[h']}\tau'$ involved in (B.15). Similarly, let $V_i^{[0]}$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix function $Z_{i0}^{[0]} + Z_{i1}^{[0]}\tau'$ involved in (B.12), while let $V_i^{[h']}$ be the matrix made in the same way from $Z_{i0}^{[h']} + Z_{i1}^{[h']}\tau'$ involved in (B.17). Note that each $L_1[0, h')$ norm can easily be computed exactly, since we only deal with linear functions. Theorem 2 now follows immediately from Proposition 2 by applying the triangle inequality.