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Scaling relations for percolation in the 2D high temperature Ising Model

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1 Ising model

1.1 Definition

We define the Ising model on the two-dimensional square lattice $\mathbb{Z}^2$: The spin configuration space is denoted by $\Omega := \{-1,+1\}^{\mathbb{Z}^2}$. This model has two parameters; the temperature $T \in [0,\infty)$ and the external magnetic field $h \in \mathbb{R}$. For a finite region $V \subset \mathbb{Z}^2$ and a boundary condition $\omega \in \Omega$, the interaction energy for a spin configuration $\sigma \in \Omega_V := \{-1,+1\}^V$ is given by

$$H^{\omega}_{V,h}(\sigma) := -\frac{1}{2}\sum_{u,v\in V,\,|u-v|=1}\sigma(u)\sigma(v) - \sum_{v\in V}\left(h + \sum_{u\notin V,\,|u-v|=1}\omega(u)\right)\sigma(v),$$

where $|x| := |x^1|+|x^2|$ for $x=(x^1,x^2)\in\mathbb{Z}^2$. The function $H^{\omega}_{V,h}(\sigma)$ is called *Hamiltonian.* The *finite volume Gibbs distribution* is defined by

$$q^{\omega}_{V,T,h}(\sigma) = (Z^{\omega}_{V,T,h})^{-1}\exp\{-H^{\omega}_{V,h}(\sigma)/(\mathfrak{K}T)\},$$

where

$$Z^{\omega}_{V,T,h} := \sum_{\sigma'\in\Omega_V}\exp\{-H^{\omega}_{V,h}(\sigma')/(\mathfrak{K}T)\}$$

is a normalizing constant called the *partition function*, and $\mathfrak{K}$ denotes the Boltzmann constant.

Let $\mathcal{F}_V$ denote the σ-algebra generated by the spin variables in $V\subset\mathbb{Z}^2$, and $\mathcal{F} := \mathcal{F}_{\mathbb{Z}^2}$. Then we have

$$\begin{aligned}&q^{\omega}_{V,T,h}[\sigma(v)=+1\,|\,\mathcal{F}_{\{v\}^c}](\sigma)\\ &= \left[1+\exp\left\{\frac{-2}{\mathfrak{K}T}\left(h+\sum_{u\in V,\,|u-v|=1}\sigma(u)+\sum_{u\notin V,\,|u-v|=1}\omega(u)\right)\right\}\right]^{-1}.\end{aligned}$$

Note that the case $T=\infty$ corresponds to the independent site percolation problem: Letting $T\to\infty$ with $h/(\mathfrak{K}T)\to H$, we can see that $\sigma(v)=+1$ with probability $p = 1/(1+e^{-2H})$, independent of each other. This probability law is denoted by P_p.

1.2 Phase transition in Ising model

A probability measure μ on Ω is called the *Gibbs measure* if it satisfies the following *Dobrushin-Lanford-Ruelle equation*:

$$\mu(\,\cdot\,|\,\mathcal{F}_{V^c})(\omega) = q^{\omega}_{V,T,h}(\,\cdot\,) \quad \text{for } \mu\text{-almost all } \omega.$$

- Every limit point of the finite volume Gibbs distribution $q^{\omega}_{V,T,h}$ as $V \nearrow \mathbb{Z}^2$ is a Gibbs measure.
- By using stochastic monotonicity, the following limiting Gibbs measure with pure boundary conditions exist:

  $$\mu_+ := \lim_{V \nearrow \mathbb{Z}^2} q^{+}_{V,T,h}, \quad \text{and} \quad \mu_- := \lim_{V \nearrow \mathbb{Z}^2} q^{-}_{V,T,h}.$$

  Both μ_+ and μ_- are invariant under spatial translations. Moreover, we have $\mu_- \le \mu \le \mu_+$ for any Gibbs measure μ.
- The set of Gibbs measures is a convex set, and its extremal points correspond to 'pure phases'; μ_+ and μ_- are among them.
- There exists the *critical temperature* $T_c \in (0,\infty)$:
  - ▶ If $T < T_c$ and $h = 0$, then $\mu_+ \neq \mu_-$.
  - ▶ Otherwise not only $\mu_+ = \mu_-$ but also there is a unique Gibbs measure.

Aizenman (1980), and Higuchi (1981) showed that for the Ising model on $\mathbb{Z}^2$, there are only two extremal Gibbs measures μ_+ and μ_-, and for any Gibbs measure μ, there is an $\alpha \in [0,1]$ such that

$$\alpha\mu_+ + (1-\alpha)\mu_-.$$

For higher dimensional cases, there are non translation-invariant Gibbs measures (Dobrushin (1972)) , but every translation-invariant Gibbs measure is a convex combination of μ_+ and μ_- (Bodineau (2006)).

2 Percolation in the high-temperature Ising model

We consider the percolation problem in the high-temperature regime. (See Higuchi (1997) for a survey.) First we prepare basic terminologies for the percolation theory.

- A *path* [resp. $(*)$-path] is a sequence $x_1, x_2, \ldots, x_s$ in $\mathbb{Z}^2$ with $|x_i - x_{i-1}| = 1$ [resp. $|x_i - x_{i-1}|_\infty = 1$] for $1 < i \le s$, where $|x|_\infty := \max\{|x^1|, |x^2|\}$ for $x = (x^1, x^2) \in \mathbb{Z}^2$.
- A path on which all spin variable are $+$ is called a $(+)$-*path*. We define $(-*)$-path in a similar manner.
  - ▷ Let $\{x \overset{+}{\leftrightarrow} y\}$ denote the event that there is a $(+)$-path between x and y.

▷ More generally, for $V, V' \subset \mathbb{Z}^2$, let $\{V \overset{+}{\leftrightarrow} V'\}$ denote the event that some point in V is connected by a $(+)$-path to some point in V'.

- A sequence $x_1, x_2, \ldots, x_s$ in $\mathbb{Z}^2$ is called a *circuit* if

$$\{(i,j) : |x_i - x_j| = 1\} = \{(i,j) : |i-j| = 1 \text{ or } \{i,j\} = \{1,s\}\}.$$

We define a $(+)$-circuit and a $(-*)$-circuit as above.

- The $(+)$-*cluster* containing $x \in \mathbb{Z}^2$ is defined by $\mathbf{C}_x^+ := \{y \in \mathbb{Z}^2 : x \overset{+}{\leftrightarrow} y\}$. We often write $\{x \overset{+}{\leftrightarrow} \infty\}$ for $\{\#\mathbf{C}_x^+ = \infty\}$. We adopt a similar notation for $-$ spins.

- For each extremal Gibbs measure μ, it is known that $\mu\left(\bigcup_{x\in\mathbb{Z}^2}\{\#\mathbf{C}_x^+ = \infty\}\right)$ is either 0 or 1. If it is equal to 1, then we say that $(+)$-*percolation* occurs. We define $(-)$-percolation similarly.

When $T > T_c$, there exists a unique Gibbs measure for each $h \in \mathbb{R}$:

$$\text{For every } \omega \in \Omega, \quad \lim_{V \nearrow \mathbb{Z}^2} q^{\omega}_{V,T,h} = \mu_{T,h}.$$

The origin in $\mathbb{Z}^2$ is denoted by $\mathbf{O}$. We write $\mathbf{C}_0^+$ for the $(+)$-cluster containing $\mathbf{O}$, and define

$$h_c(T) := \inf\{h : \mu_{T,h}(\#\mathbf{C}_0^+ = \infty) > 0\}.$$

It is shown in Higuchi (1993a) that $h_c(T) > 0$ whenever $T > T_c$. (It is also known that $h_c(T) = 0$ for $T \le T_c$.) When $T > T_c$, the percolation transition at $h = h_c(T)$ is sharp (Higuchi (1993b)). Hereafter we fix a $T > T_c$, and abbreviate $\mu_{T,h}$ to μ_h, and $h_c(T)$ to h_c. The expectation under μ_h is denoted by E_h.

3 Scaling relations

We investigate the critical behavior of principal quantities in percolation. We adopt the following notation:

- $f(n) \approx n^{\zeta}$ means that $\lim_{n\to\infty} \dfrac{\log f(n)}{\log n} = \zeta$.

- $f(n) \asymp g(n)$ means that $C_1 g(n) \le f(n) \le C_2 g(n)$ for some positive constants C_1 and C_2.

3.1 Conjectured power laws and scaling relations

[Near the critical point]

- The percolation probability $\theta(h) := \mu_h(\#\mathbf{C}_0^+ = \infty)$.

$$\boxed{\theta(h) \approx (h - h_c)^\beta} \text{ as } h \searrow h_c.$$

- The expected size of the finite cluster $\chi(h) := E_h[\#\mathbf{C}_0^+ : \#\mathbf{C}_0^+ < \infty]$.

$$\boxed{\chi(h) \approx |h - h_c|^{-\gamma}} \text{ as } h \searrow h_c.$$

* Gap exponent Δ

$$\text{For any } k \geq 2, \boxed{\frac{E_h[(\#\mathbf{C}_0^+)^k : \#\mathbf{C}_0^+ < \infty]}{E_h[(\#\mathbf{C}_0^+)^{k-1} : \#\mathbf{C}_0^+ < \infty]} \approx |h - h_c|^{-\Delta}} \text{ as } h \to h_c.$$

- Correlation length

$$\xi(h) := \left[\frac{1}{\chi(h)} \sum_{v \in \mathbf{Z}^2} |v|^2 \mu_h(\mathbf{O} \overset{+}{\leftrightarrow} v,\, \#\mathbf{C}_0^+ < \infty) \right]^{1/2}.$$

$$\boxed{\xi(h) \approx |h - h_c|^{-\nu}} \text{ as } h \to h_c.$$

[At the critical point] Let $S(n) := [-n, n]^2$ and $\partial S(n) := \{x \in \mathbb{Z}^2 : |x|_\infty = n\}$.

- The 1-arm probability $\pi_h(n) := \mu_h(\mathbf{O} \overset{+}{\leftrightarrow} \partial S(n))$.

$$\boxed{\pi_{h_c}(n) \approx n^{-1/\delta_r}} \text{ as } n \to \infty.$$

- The connectivity function $\tau_h(n) := \mu_h(\mathbf{O} \overset{+}{\leftrightarrow} (n, 0))$.

$$\boxed{\tau_{h_c}(n) \approx n^{-\eta}} \text{ as } n \to \infty.$$

- The size distribution of the finite cluster

$$\boxed{\mu_{h_c}(\#\mathbf{C}_0^+ \geq n) \approx n^{-1/\delta}} \text{ as } n \to \infty.$$

[Scaling relations]

$$\beta(\delta + 1) = \gamma + 2\beta = \Delta + \beta, \quad \gamma = \nu(2 - \eta).$$

[Hyperscaling relations] (only for small d)

$$d\nu = \beta(\delta + 1) = \gamma + 2\beta = \Delta + \beta, \quad d\delta_r = \delta + 1.$$

3.2 Finite-size scaling correlation length

Let $A^+(n,m)$ be the event that there exists a (+)-path between the left side and right side of $[-n,n] \times [-m,m]$; the (+)-path is called the *horizontal (+)-crossing*. Similarly we define $A^{-*}(n,m)$. The *finite-size scaling correlation length* $L(h,\varepsilon_0)$ by

$$L(h,\varepsilon_0) := \begin{cases} \min\{n : \mu_h(A^+(n,n)) \geq 1-\varepsilon_0\} & (h > h_c), \\ \infty & (h = h_c), \\ \min\{n : \mu_h(A^+(n,n)) \leq \varepsilon_0\} & (h < h_c). \end{cases}$$

Here ε_0 is a small positive constant. It is known that $\xi(h) \asymp L(h,\varepsilon_0)$.

The following theorems play important roles in deriving scaling relations. For the independent percolation, these theorems are proved by Kesten (1987b).

Theorem 3.1. (i) For any $n < L(h,\varepsilon_0)$, $C_1 \leq \dfrac{\pi_h(n)}{\pi_{h_c}(n)} \leq C_2$.

(ii) As $h \searrow h_c$,

$$\theta(h) \asymp \pi_h(L(h,\varepsilon_0)) \asymp \pi_{h_c}(L(h,\varepsilon_0)).$$

From this theorem, we can obtain a scaling relation involving β, δ_r, and ν, if they exist.

Theorem 3.2. (i) For $t > 1$, as $h \to h_c$,

$$E_h\left[(\#\mathbf{C}_0^+)^t : \#\mathbf{C}_0^+ < \infty\right] \asymp L(h,\varepsilon_0)^{2t}\pi_{h_c}(L(h,\varepsilon_0))^{t+1}.$$

(ii) For $t = 1$, as $h \to h_c$,

$$E_h\left[\#\mathbf{C}_0^+ : \#\mathbf{C}_0^+ < \infty\right] \approx L(h,\varepsilon_0)^2\pi_{h_c}(L(h,\varepsilon_0))^2.$$

This theorem suggests that the volume of a "large critical cluster" in $S(n) \approx n^2\pi_{h_c}(n)$.

Theorem 3.3. As $x \searrow 0$, $L(h_c - x, \varepsilon_0) \asymp L(h_c + x, \varepsilon_0)$.

This implies the symmetry of critical exponents on the left and right of h_c.

3.3 Scaling relations

We can show the following relations between critical exponents (Higuchi, Takei, and Zhang (2010, 2011)); those are obtained by Kesten (1986, 1987a,b) for the independent percolation on periodic lattices.

Theorem 3.4. (i) If one of

$$\pi_{h_c}(n) \approx n^{-1/\delta_r} \qquad \text{or} \qquad \tau_{h_c}(n) \approx n^{-\eta} \tag{1}$$

holds, then both statements as well as

$$\mu_{h_c}(\#\mathbf{C}_0^+ \geq n) \approx n^{-1/\delta}$$

hold, and

$$\begin{aligned}\theta(h) &\asymp L(h,\varepsilon_0)^{-1/\delta_\mathrm{r}} = L(h,\varepsilon_0)^{-2/(\delta+1)},\\ \delta &= 2\delta_\mathrm{r} - 1,\\ \eta &= \frac{2}{\delta_\mathrm{r}} = \frac{4}{\delta+1}.\end{aligned}$$

If, in addition, for some $\nu > 0$,

$$\xi(h) \approx |h - h_c|^{-\nu} \tag{2}$$

holds, then

$$\beta = \frac{2\nu}{\delta+1}.$$

(ii)

- For $t \geq 2$, $\dfrac{E_h[(\#\mathbf{C}_0^+)^t : \#\mathbf{C}_0^+ < \infty]}{E_h[(\#\mathbf{C}_0^+)^{t-1} : \#\mathbf{C}_0^+ < \infty]} \approx \xi(h)^2 \pi_{h_c}(\xi(h))$,
- For $t > 0$, $\left[\dfrac{1}{\chi(h)} \displaystyle\sum_{v \in \mathbf{Z}^2} |v|^t \mu_h(\mathbf{O} \overset{+}{\leftrightarrow} v,\ \#\mathbf{C}_0^+ < \infty)\right]^{1/t} \asymp \xi(h)$.

In addition, if (1) and (2) hold, then

- For $k \geq 2$, $\dfrac{E_h[(\#\mathbf{C}_0^+)^k : \#\mathbf{C}_0^+ < \infty]}{E_h[(\#\mathbf{C}_0^+)^{k-1} : \#\mathbf{C}_0^+ < \infty]} \approx |h - h_c|^{-\Delta_k}$,
- For $k \geq 1$, $\left[\dfrac{1}{\chi(h)} \displaystyle\sum_{v \in \mathbf{Z}^2} |v|^k \mu_h(\mathbf{O} \overset{+}{\leftrightarrow} v,\ \#\mathbf{C}_0^+ < \infty)\right]^{1/k} \approx |h - h_c|^{-\nu_k}$,

and

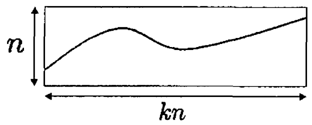
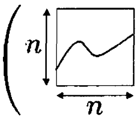
$$\gamma = 2\nu\frac{\delta-1}{\delta+1}, \qquad \Delta_k = 2\nu\frac{\delta}{\delta+1} \quad (k \geq 2), \qquad \nu_k = \nu \quad (k \geq 1).$$

4 Sketch of the proof

In this section, we present typical techniques to prove scaling relations.

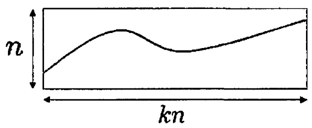
4.1 RSW-type estimates for crossing probabilities

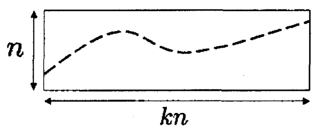
In the independent percolation, the RSW lemma (Russo (1978), Seymour and Welsh (1978)) gives the following estimate for crossing probabilities:

$$P_p\left(\right. $$  $$\left. \right) \geq f_k\left(P_p\left(\right.\right. $$  $$\left.\left. \right)\right),$$

where $f_k(\delta) \to 1$ as $\delta \to 1$. The following is an important consequence from the RSW lemma. It can be obtained for Ising percolation also.

Lemma 4.1 (RSW-type estimates). For each integer $k > 0$, there exists a constant δ_k such that for $n < L(h, \varepsilon_0)$,

$$\mu_h\big(A^+(kn,n)\big) = \mu_h\left(\right. $$  $$\left. \right) \geq \delta_k,$$

$$\mu_h\big(A^{-*}(kn,n)\big) = \mu_h\left(\right. $$  $$\left. \right) \geq \delta_k.$$

(Since μ_h is invariant under the rotation by right angle, same estimate can be obtained for vertical crossings.)

By the duality $\mu_h\big(A^+(n,n)\big) + \mu_h\big(A^{-*}(n,n)\big) = 1$, for any n,

$$0 < \delta_1 \leq \mu_{h_c}\big(A^+(n,n)\big) \leq 1 - \delta_1 < 1,$$

which suggests a kind of scale invariance.

Another standard tool in percolation is the Fortuin-Kasteleyn-Ginibre (FKG) inequality (For the independent case, Harris (1960) already noticed the inequality.) For two configuration $\sigma, \sigma' \in \Omega_V$, we say $\sigma \leq \sigma'$ if $\sigma(v) \leq \sigma'(v)$ for all $v \in V$.

- An event $A \in \mathcal{F}_V$ is called *increasing* if $1_A(\sigma) \leq 1_A(\sigma')$ whenever $\sigma \leq \sigma'$. For example, $A^+(kn,n)$ is increasing.
- An event $A \in \mathcal{F}_V$ is called *decreasing* if $1_A(\sigma) \geq 1_A(\sigma')$ whenever $\sigma \leq \sigma'$. For example, $A^{-*}(kn,n)$ is decreasing.

Lemma 4.2 (the FKG inequality). If A and B are both incresing [or both decreasing], then $\mu_h(A \cap B) \geq \mu_h(A)\mu_h(B)$.

As an application, we derive a power law estimate of the 1-arm probability $\pi_{h_c}(n)$ at the critical point (especially it does not decay exponentially).

Proposition 4.3. There are positive constants C_1, C_2, α such that for all n,

$$C_1 n^{-1} \leq \pi_{h_c}(n) \leq C_2 n^{-\alpha}.$$

Proof. First we give the upper bound. (The idea is related to that of Harris (1960).) For $j \geq 1$, we put $A_j := S(4^{j+1}) \setminus S(2 \cdot 4^j)$, and

$$X_j := \begin{cases} 1 & \text{If there exists a } (-*)\text{-circuit surrounding } \mathbf{O} \text{ in } A_j, \\ 0 & \text{otherwise.} \end{cases}$$

By the mixing property and the RSW-type estimate, we can find an integer j^* and a positive number δ such that for $j \geq j^*$,

$$\mu_{h_c}(X_j = 1 \mid X_1, \ldots, X_{j-1}) \geq \delta.$$

$$\pi_{h_c}(n) \leq \mu_{h_c}\left(\bigcap_{j=j^*}^{\lfloor \log_4 n - 1 \rfloor} \{X_j = 0\} \right) \leq (1-\delta)^{\lfloor \log_4 n - 1 \rfloor - j^* + 1}.$$

Now we turn to the lower bound. By the RSW-type estimate, we have

$$\mu_{h_c}\big(A^+(n,n)\big) \geq \delta_1 > 0.$$

On $A^+(n,n)$, we look at the lowest $(+)$-crossing L_n in $S(n)$, and put

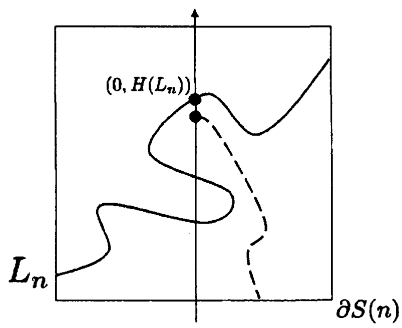
$$H(L_n) := \max\{y \in [-n,n] : (0,y) \in L_n\}.$$

Then we have

$$\begin{aligned} \mu_{h_c}\big(A^+(n,n)\big) &= \sum_{y \in [-n,n]} \mu_{h_c}\big(H(L_n) = y\big) \\ &\leq \sum_{y \in [-n,n]} \mu_{h_c}\Big((0,y) \overset{+}{\leftrightarrow} \partial\big((0,y) + S(n)\big)\Big) \\ &= (2n+1)\pi_{h_c}(n). \end{aligned}$$

□

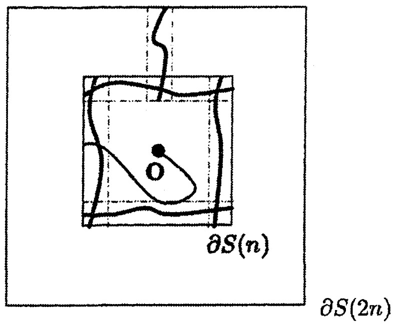
Remark 4.4. Two disjoint $(+)$-paths and one $(-*)$-path start from neighbors of $(0, H(L_n))$. In the independent percolation, we can obtain a better bound by the van den Berg-Kesten-Reimer inequality (see Kesten (1987b)). The trouble is that the inequality is not available for Ising percolation.



Proposition 4.5. $\pi_{h_c}(2n) \asymp \pi_{h_c}(n)$.

Proof. Obviously $\pi_{h_c}(2n) \le \pi_{h_c}(n)$. On the other hand, by the RSW-type estimate and the FKG inequality, we have

$$\pi_{h_c}(2n) \ge \mu_{h_c} \Bigg($$



$$\Bigg) \ge \delta_5 (\delta_8)^4 \pi_{h_c}(n).$$

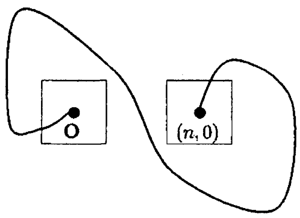
□

The following is the Ising version of Lemma in Kesten (1987a).

Proposition 4.6. $\tau_{h_c}(n) \asymp \pi_{h_c}(n)^2$.

Proof. For the upper bound, noting that

$$\tau_{h_c}(n) = \mu_{h_c} \Bigg($$



$$\Bigg)$$

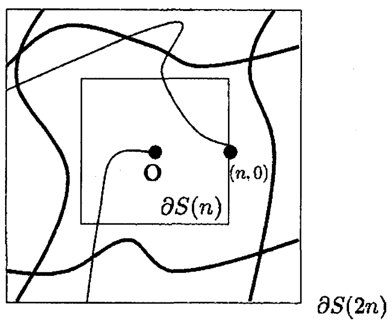
$$\le \mu_{h_c}\Big((0,0) \overset{+}{\leftrightarrow} \partial S(n/4),\ (n,0) \overset{+}{\leftrightarrow} \partial((n,0) + S(n/4)) \Big),$$

it follows from the translation-invariance and the mixing property that

$$\le \pi_{h_c}(n/4)^2 + C\,(n/4)^2 \cdot (n/2) \cdot e^{-\alpha n/2}.$$

For the lower bound, using the RSW-type estimate and the FKG inequality,

$$\tau_{h_c}(n) \ge \mu_{h_c} \Bigg($$



$$\Bigg) \ge (\delta_4)^4 \pi_{h_c}(2n) \pi_{h_c}(3n).$$

□

If we assume that the critical exponents η and δ_r exist, then a scaling relation $\eta = \dfrac{2}{\delta_r}$ follows from Proposition 4.6.

4.2 Ising version of Russo's formula

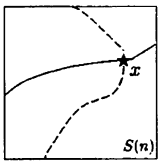
Theorem 3.1 relates the on-critical regime to the off-critical regime, and finite regions to the whole plane. To prove it, we estimate the derivative of $\pi_h(n)$. Let $H = h/(\mathfrak{K}T)$, and μ_H^N denote the finite volume Gibbs distribution on $S(N)$ with periodic boundary condition. For $n < N$ and $A \in \mathcal{F}_{S(n)}$, we have

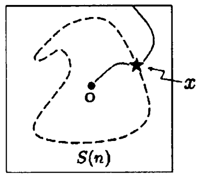
$$\begin{aligned}\frac{d}{dH}\mu_H^N(A) &= \sum_{x \in S(N)} \operatorname{Cov}_{\mu_H^N}(\sigma(x), 1_A(\sigma)) \\ &= \sum_{x \in S(N)} E_{\mu_H^N}\big[\{\sigma(x) - E_{\mu_H^N}[\sigma(x)]\} : A\big].\end{aligned}$$

A site x is *pivotal* for the event A in the configuration σ if $1_A(\sigma^x) \neq 1_A(\sigma)$, where σ^x is obtained from σ by flipping the spin at x. Let

$$\operatorname{Piv}_x A := \{\sigma \in \Omega_{S(n)} : x \text{ is pivotal for } A \text{ in } \sigma\}.$$

For example,

$\operatorname{Piv}_x A^+(n,n) =$ 

, $\operatorname{Piv}_x\{\mathbf{O} \overset{+}{\nleftrightarrow} \partial S(n)\} =$ 

.

Note that $\operatorname{Piv}_x A \in \mathcal{F}_{S(n)\setminus\{x\}}$.

We assume that A is an increasing event. Note that

$$\begin{aligned} A &= (A \cap \operatorname{Piv}_x A) \cup (A \cap (\operatorname{Piv}_x A)^c) \\ &= \big(\underbrace{\{\omega(x) = +1\}}_{\in \mathcal{F}_{\{x\}}} \cap \underbrace{\operatorname{Piv}_x A}_{\in \mathcal{F}_{S(n)\setminus\{x\}}}\big) \cup \underbrace{(A \cap (\operatorname{Piv}_x A)^c)}_{\in \mathcal{F}_{S(n)\setminus\{x\}},\ \text{increasing}}. \end{aligned}$$

In the independent percolation, we have

$$\frac{d}{dp}P_p(A) = \sum_{x \in S(n)} P_p(\operatorname{Piv}_x A) = E_{P_p}\big[\#(\text{pivotal sites for } A)\big],$$

which is called *Russo's formula* (Russo(1981)). In the Ising percolation, we can obtain

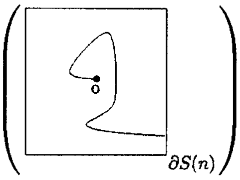
$$\frac{d}{dH}\mu_H^N(A) \geq c \sum_{x \in S(n)} \mu_H^N(\operatorname{Piv}_x A),$$

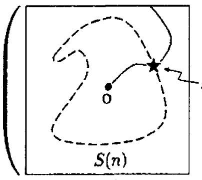
since

$$E_{\mu_H^N}\big[\{\sigma(x) - E_{\mu_H^N}[\sigma(x)]\} : A \cap (\operatorname{Piv}_x A)^c\big] \geq 0$$

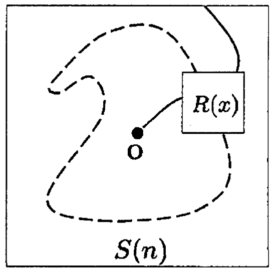
by the FKG inequality.

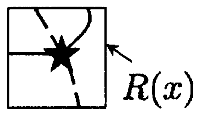
We show the strategy of the proof of Theorem 3.1(i) for the independent percolation (Kesten (1987b)).

$$\frac{d}{dp}\pi_p(n) := \frac{d}{dp}P_p\Bigg($$


$$\Bigg) = \sum_{x\in S(n)} P_p\Bigg($$


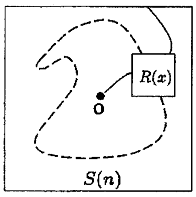
$$\Bigg)$$

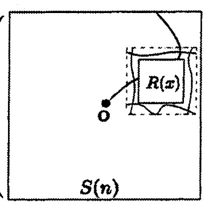
$$= \sum_{x\in S(n)} P_p\Bigg($$


$$\cap$$


$$\Bigg).$$

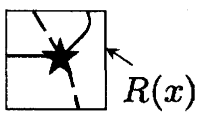
If $n < L(h, \varepsilon_0)$, then both the (+)-crossing probability and the $(-*)$-crossing probability are bounded away from 0 as in the critical case; in a similar manner as in Proposition 4.5, we have

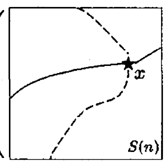
$$P_p\Bigg($$


$$\Bigg) \le CP_p\Bigg($$


$$\Bigg) \le C\pi_p(n).$$

The key idea in Kesten (1987b) is to extend (+)-paths and $(-*)$-paths simultaneously: Roughly,

$$P_p\Bigg($$


$$\Bigg) \le C'P_p\Bigg($$


$$\Bigg) \le CP_p(\mathrm{Piv}_x A^+(n,n)).$$

By independence,

$$\frac{d}{dp}\log \pi_p(n) \le C'' \sum_{x\in S(n)} P_p(\mathrm{Piv}_x A^+(n,n)) = C''\frac{d}{dp}P_p(A^+(n,n)).$$

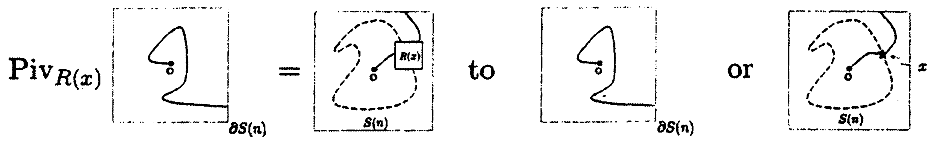
Integrating it from p_c to $p(\neq p_c)$, we have

$$\left|\log\frac{\pi_p(n)}{\pi_{p_c}(n)}\right| \le C''\left|P_p(A^+(n,n)) - P_{p_c}(A^+(n,n))\right|.$$

In the Ising case, when A is the 1-arm, 4-arm, or crossing events, we can prove

$$c\sum_{x\in S(n)} \mu_H(\mathrm{Piv}_x A) \le \frac{d}{dH}\mu_H(A) \le C_1\mu_H(A) + C_2\sum_{x\in S(n)} \mu_H(\mathrm{Piv}_x A).$$

This is sufficient for our purpose. The key idea of the proof for the 1-arm event is to reduce

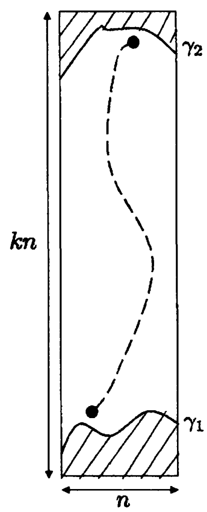


by extension arguments.

4.3 Connection lemma

Given a horizontal crossing γ of $S(n)$, we can divide $S(n)$ into two regions; the upper [resp. lower] one is denoted by $S^+(n,\gamma)$ [resp. $S^-(n,\gamma)$]. On $A^+(n,n)$, let L_n be the lowest $(+)$-crossing in $S(n)$. Note that $\{L_n = \gamma\} \in \mathcal{F}_{\gamma \cup S^-(n,\gamma)}$. In the independent case, L_n plays the same role as the stopping time. This property together with the RSW lemma gives a lower bound of the probability that there exists a $(-*)$-path from the top side of $S(n)$ to some point above L_n. It is important, for example, to estimate the number of pivotal points for $A^+(n,n)$. In the Ising case, we can 'approximately' use this property, summarized as in the following lemma.

Lemma 4.7 (Connection lemma). Let $V(n) = [0,n] \times [0,kn]$. By a horizontal crossing γ of $V(n)$, we can divide $V(n)$ into two regions; the upper [resp. lower] one is denoted by $V^+(n,\gamma)$ [resp. $V^-(n,\gamma)$]. Let γ_1 be a horizontal crossing of $[0,n] \times [0,n]$, and γ_2 be a horizontal crossing of $[0,n] \times [(k-1)n, kn]$. There exists an integer n_0 such that if $L(h,\varepsilon_0) \geq n \geq n_0$, then for any k and



$$E \in \mathcal{F}_{V(n)^c},\ F \in \mathcal{F}_{\gamma_1 \cup V^-(n,\gamma_1) \cup \gamma_2 \cup V^+(n,\gamma_2)},$$

we have

$$\mu_{T,h}\left(\begin{array}{c} (\gamma_1 + (0,1)) \overset{s}{\leftrightarrow} (\gamma_2 + (0,-1)) \\ \text{in } V^+(n,\gamma_1) \cap V^-(n,\gamma_2) \end{array} \middle| \, E \cap F \right) \geq \delta_{8k}/4,$$

where $s \in \{+, -*\}$.

4.4 Arm events

We have already introduced $\pi_h(n)$ (1-arm probability) and $\mathrm{Piv}_{\mathbf{0}} A^+(n,n)$ (4-arm event). More generally, *arm events* refer that there exist some number of crossings ("arms") of $S(N) \setminus S(n)$ $(N > n)$. For an integer $k \geq 1$ and a sequence $\sigma = (\sigma_1, \ldots, \sigma_k) \in \{+,-\}^k$, we define the event

$$A_{k,\sigma}(n,N) = \left\{ \partial S(n) \overset{k,\sigma}{\leftrightarrow} \partial S(N) \right\}$$

that there exist k disjoint crossings in $S(N)\setminus S(n)$, whose signs are those prescribed by σ in counterclockwise order. We also define k-arm events for half-planes: Let $S^+(n,N) := \{S(N)\setminus S(n)\}\cap(\mathbb{Z}_+\times\mathbb{Z})$ and

$$B_{k,\sigma}(n,N) = \left\{\partial S^+(n) \overset{k,\sigma}{\leftrightarrow} \partial S^+(N) \text{ in } S^+(n,N)\right\}.$$

A conjecture in Aizenman, Duplantier, and Aharony (1999) for the independent percolation is the following:

$$\text{For } k_+, k_- \geq 1, \qquad \mu_{h_c}(A_{k,\sigma}(n,N)) \asymp \left(\frac{n}{N}\right)^{(k^2-1)/12},$$

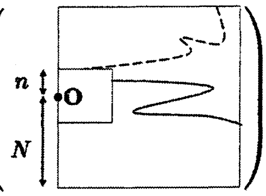
$$\text{For } k \geq 1, \qquad \mu_{h_c}(B_{k,\sigma}(n,N)) \asymp \left(\frac{n}{N}\right)^{(k(k+1))/6},$$

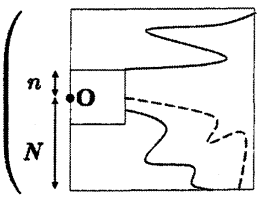
where

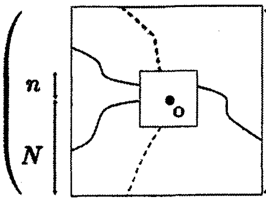
$$k_+ := \#\{1\leq i\leq k : \sigma_i = +\}, \text{ and } k_- := \#\{1\leq i\leq k : \sigma_i = -\}.$$

For the independent percolation on the planar triangular lattice, this conjecture is proved to be true in the sense of $\approx$ (Smirnov and Werner (2001), and Lawler, Schramm, and Werner (2002)). For two-dimensional periodic lattices, using the RSW-type estimates, the conjecture is verified for $k = 2,3$ in the half-plane (essentially done by Zhang (1995)) and $k = 5$ in the whole plane (Kesten, Sidoravicius, and Zhang (1998)). (See also Nolin (2008) and Werner (2009).)

Theorem 4.8. In the Ising percolation case, we can prove the following:

- 2-arm in half-planes: $\mu_{h_c}(B_{2,(+,-)}(n,N)) = \mu_{h_c}\Bigg($ 

$\Bigg) \asymp \frac{n}{N}.$
- 3-arm in half-planes: $\mu_{h_c}(B_{3,(+,-,+)}(n,N)) = \mu_{h_c}\Bigg($ 

$\Bigg) \asymp \left(\frac{n}{N}\right)^2.$
- 5-arm: $\mu_{h_c}(A_{5,(+,-,+,-,+)}(n,N)) = \mu_{h_c}\Bigg($ 

$\Bigg) \asymp \left(\frac{n}{N}\right)^2.$

We can use the above 2- and 3-arm estimates to control the boundary effect.

We remark that our techniques are also applicable to the Ising percolation on the triangular lattice, and they might be useful for studying the scaling limit problem, posed by Bálint, Camia, and Meester (2010).

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