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# THE NELSON MODEL ON STATIC LORENTZIAN MANIFOLDS

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## 1 The Nelson model on static Lorentzian manifolds

### 1.1 The standard Nelson model

We are concerned with the Nelson model defined on static Lorentzian manifolds. Static Lorentzian manifold is defined by a Lorentzian manifold with a metric depending on position but independent of time. The Nelson model is a simple but non-trivial model describing the strong interaction in quantum field theory. It is however assumed that fermions are governed by Schrödinger operator. Then it is the so called non-relativistic quantum field theory. From mathematical point of view the model is defined as a self-adjoint operator acting on some tensor product of Hilbert spaces, and we are interested in studying the spectrum of the self-adjoint operator rigorously. In particular the existence and the absence of ground state, property of continuous spectrum and spectral scattering theory are the main topics. For the Nelson model some physical folklore has been established rigorously. E.g., the absence of ground state of the Nelson model under infrared singular condition and the existence of ground state under the infrared regular condition are established. In this note we extend the Nelson model to the model defined on static Lorentzian manifold and study its spectrum.

The Hilbert space of the state vectors is defined by

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}, \tag{1.1}$$

where $\mathscr{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$ denotes the boson Fock space over $L^2(\mathbb{R}^3)$. Then the standard Nelson model is defined by a self-adjoint operator of the form:

$$H = \left(-\frac{1}{2}\Delta_X + V(X)\right) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X). \tag{1.2}$$

Here $d\Gamma(\omega)\Phi^{(n)}(x_1, ..., x_n) = \left(\sum_{j=1}^{n} \omega(-i\nabla_{x_j})\right) \Phi^{(n)}(x_1, ..., x_n)$ is the free field Hamiltonian defined by the second quantization of the dispersion relation $\omega = \omega(-i\nabla_x) = \sqrt{-\Delta_x + m^2}$ with boson mass $m \geq 0$. The scalar field is defined by

$$\phi(f) = \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f)), \tag{1.3}$$

where $a(f)$ and $a^\dagger(f)$ denote the annihilation operator and the creation operator smeared by cutoff function $f \in L^2(\mathbb{R}^3)$, respectively. In particular we set

$$\phi_\rho(X) = \phi(\omega^{-1/2}\rho(\cdot - X)), \tag{1.4}$$

where $0 \leq \rho \in \mathscr{S}$ is an UV cutoff function and $\mathscr{S}$ the set of Schwartz test functions on $\mathbb{R}^3$. The Hamiltonian $H$ describes the energy of a particle linearly interacting with a scalar field $\phi_\rho$. A relationships between the stability of ground state and boson mass is also known. Let

$$\mathrm{I}_{\mathrm{IR}} = \int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\omega(k)^3} dk. \tag{1.5}$$

It is known that under some conditions on $V$ there exists a ground state of $H$ if and only if $\mathrm{I}_{\mathrm{IR}} < \infty$. If $\omega(k) = \sqrt{|k|^2 + m^2}$ and $\hat{\rho}(0) > 0$, then $\mathrm{I}_{\mathrm{IR}} < \infty$ if and only if $m > 0$.

## 1.2 Klein-Gordon equation on static Lorentzian manifolds

In quantum field theory the dispersion relation $\omega = \sqrt{-\Delta + m^2}$ can be derived from the Klein-Gordon equation:

$$\frac{\partial^2}{\partial t^2}\phi(x,t) = (\Delta_x - m^2)\phi(x,t). \tag{1.6}$$

Let $e^{-itH}\phi(f)e^{itH} = \int \phi(t,x)f(x)dx$ and $e^{-itH}Xe^{itH} = X_t$. The standard Nelson model satisfies that

$$(\partial_t^2 - \Delta_X + m^2)\phi(t,x) = \rho(x - X_t),$$
$$\partial_t^2 X_t = -\nabla V(X_t) - \int \phi(t,x)\nabla_X \rho(x - X_t)dx.$$

Now we consider the Klein-Gordon equation on Lorentzian manifolds. Let $\underline{x} = (t,x) = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3$. Suppose that $g = (g_{\mu\nu})$, $\mu, \nu = 0, 1, 2, 3$, is a metric tensor on $\mathbb{R}^4$ such that

(1) $g_{\mu\nu}(\underline{x}) = g_{\mu\nu}(x)$, i.e., it is independent of time $t$,

(2) $g_{0j}(\underline{x}) = g_{j0}(\underline{x}) = 0$, $j = 1, 2, 3$,

(3) $g_{ij}(\underline{x}) = -\gamma_{ij}(x)$, where $\gamma = (\gamma_{ij})$ denotes a 3-dimensional Riemannian metric.

Namely

$$g = \begin{pmatrix} g_{00} & 0 \\ 0 & -\gamma \end{pmatrix}. \tag{1.7}$$

Let $\mathscr{M} = (\mathbb{R}^4, g)$ be a Lorentzian manifold equipped with the metric tensor $g$ satisfying (1)-(3) above. Then the line element on $\mathscr{M}$ is given by

$$ds^2 = g_{00}(x) dt \otimes dt - \sum_{i,j=1}^{3} \gamma_{ij}(x) dx^i \otimes dx^j. \tag{1.8}$$

Let $g^{-1} = (g^{\mu\nu})$ denote the inverse of $g$. In particular $1/g_{00} = g^{00}$. We also denote the inverse of $\gamma$ by $\gamma^{-1} = (\gamma^{ij})$. The Klein-Gordon equation on the static Lorentzian manifold $\mathscr{M}$ is generally given by

$$\Box_g \phi + (m^2 + \eta \mathcal{R}) \phi = 0, \tag{1.9}$$

where $\eta$ is a constant, $\mathcal{R}$ the scalar curvature of $\mathscr{M}$, and $\Box_g$ the d'Alembertian operator given by

$$\Box_g = \sum_{\mu,\nu=0}^{3} \frac{1}{\sqrt{|\mathrm{det} g|}} \partial_\mu g^{\mu\nu} \sqrt{|\mathrm{det} g|} \partial_\nu. \tag{1.10}$$

Let us assume that $g_{00}(x) > 0$. Then (1.9) is rewritten as

$$\frac{\partial^2 \phi}{\partial t^2} = K\phi, \tag{1.11}$$

where

$$K = g_{00} \left( \frac{1}{\sqrt{|\mathrm{det} g|}} \sum_{i,j=1}^{3} \partial_j \sqrt{|\mathrm{det} g|} \gamma^{ji} \partial_i - m^2 - \eta \mathcal{R} \right). \tag{1.12}$$

The operator $K$ is symmetric on a weighted $L^2$ space $L^2(\mathbb{R}^3; \rho(x)dx)$, where

$$\rho = \frac{\sqrt{|\mathrm{det} g|}}{g_{00}} = g_{00}^{-1/2} \sqrt{|\mathrm{det} \gamma|}. \tag{1.13}$$

Now let us transform the operator $K$ on $L^2(\mathbb{R}^3; \rho(x)dx)$ to the one on $L^2(\mathbb{R}^3; dx)$. Define the unitary operator $U : L^2(\mathbb{R}^3; \rho(x)dx) \to L^2(\mathbb{R}^3; dx)$ by

$$Uf = \rho^{1/2} f. \tag{1.14}$$

Let $\rho_i = \partial_i\rho$ and $\partial_i\partial_j\rho = \rho_{ij}$ for notational simplicity. Furthermore we set $\alpha^{ij} = g_{00}\gamma^{ij}$ and $\partial_k\alpha^{ij} = \alpha^{ij}_k$. Since $U^{-1}\partial_j U = \partial_j + \frac{\rho_j}{2\rho}$, we see that as an operator identity

$$U^{-1}\left(\sum_{i,j=1}^{3}\partial_i g_{00}\gamma^{ij}\partial_j\right)U = g_{00}\sum_{i,j=1}^{3}\gamma^{ij}\partial_i\partial_j + V_1 + V_2, \tag{1.15}$$

where

$$\begin{aligned} V_1 &= \sum_{i,j=1}^{3}\left(\alpha^{ij}_i + \alpha^{ij}\frac{\rho_i}{\rho}\right)\partial_j, \\ V_2 &= \frac{1}{4}\sum_{i,j=1}^{3}\left(2\alpha^{ij}_i\frac{\rho_j}{\rho} + 2\alpha^{ij}\frac{\rho_{ij}}{\rho} - \alpha^{ij}\frac{\rho_i}{\rho}\frac{\rho_j}{\rho}\right). \end{aligned}$$

Directly we can see that

$$g_{00}\frac{1}{\sqrt{|\det g|}}\sum_{i,j=1}^{3}\partial_i\sqrt{|\det g|}\gamma^{ij}\partial_j = V_1 + g_{00}\sum_{i,j=1}^{3}\gamma^{ij}\partial_i\partial_j. \tag{1.16}$$

Comparing (1.15) with (1.16) we obtain that

$$U^{-1}\left(\sum_{i,j=1}^{3}\partial_i g_{00}\gamma^{ij}\partial_j - V_2\right)U = g_{00}\frac{1}{\sqrt{|\det g|}}\sum_{i,j=1}^{3}\partial_i\sqrt{|\det g|}\gamma^{ij}\partial_j. \tag{1.17}$$

Then we proved the lemma below.

**Lemma 1.1** *It follows that*

$$UKU^{-1} = \sum_{i,j=1}^{3}\partial_i g_{00}\gamma^{ij}\partial_j - v, \tag{1.18}$$

*where* $v = g_{00}(m^2 + \eta\mathcal{R}) + V_2$.

By Lemma 1.1, (1.11) is transformed to the equation:

$$\frac{\partial^2\phi}{\partial t^2} = \left(\sum_{i,j=1}^{3}\partial_i g_{00}\gamma^{ij}\partial_j - v\right)\phi \tag{1.19}$$

on $L^2(\mathbb{R}^3)$. Hence the dispersion relation on static Lorentzian manifold is given by

$$\omega = \left(-\sum_{i,j=1}^{3}\partial_i g_{00}\gamma^{ij}\partial_j + v\right)^{1/2}. \tag{1.20}$$

We here give an example of a Klein-Gordon equation defined on a static Lorentzian manifold $\mathscr{M}$ such that a short range potential $v(x) = \mathcal{O}(\langle x\rangle^{-\beta-2})$ appears. Let

$$g(\underline{x}) = g(x) = (g_{ij}(x)) = \begin{pmatrix} e^{-\theta(x)} & 0 & 0 & 0 \\ 0 & -e^{-\theta(x)} & 0 & 0 \\ 0 & 0 & -e^{-\theta(x)} & 0 \\ 0 & 0 & 0 & -e^{-\theta(x)} \end{pmatrix}. \tag{1.21}$$

We compute the scalar curvature $\mathscr{R}$ of the Lorentzian manifold $\mathscr{M} = (\mathbb{R}^4, g)$.

**Lemma 1.2** *It follows that* $\mathscr{R} = e^{\theta}(-6\Delta\theta + \frac{11}{4}|\nabla\theta|^2)$.

*Proof:* As usual we set $g^{-1} = (g^{ij})$. Set $-\theta(x) = \Theta$ and $\Theta_j = \frac{\partial\Theta}{\partial x^j}$. Directly we have

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) = \begin{cases} \Gamma^k_{kk} & = \frac{3}{2}\Theta_k, \\ \Gamma^j_{kk}(j \neq k) & = \begin{cases} -\frac{1}{2}\Theta_j, & k \neq 0, \\ \frac{1}{2}\Theta_j, & k = 0, \end{cases} \\ \Gamma^k_{jk} = \Gamma^k_{kj}(j \neq k) & = \Theta_j, \\ \text{ow} & = 0 \end{cases}.$$

The Riemann curvature tensor $\mathscr{R}^l_{kij}$ is defined by

$$\mathscr{R}^l_{kij} = \frac{\partial \Gamma^l_{kj}}{\partial x^i} - \frac{\partial \Gamma^l_{ki}}{\partial x^j} + \sum_a \left(\Gamma^a_{kj}\Gamma^l_{ai} - \Gamma^a_{ki}\Gamma^l_{aj}\right)$$

and the Ricci tensor by $\mathscr{R}_{ji} = \sum_l \mathscr{R}^l_{ilj}$. Thus the scalar curvature $\mathscr{R}$ is represented by Riemann curvature tensor by

$$\mathscr{R} = \sum_{ij} g^{ij}\mathscr{R}_{ji} = \sum_{ijl} g^{ij}\mathscr{R}^l_{jli} = e^{-\Theta}\sum_l\left(\mathscr{R}^l_{0l0} - \sum_{j=1}^{3}\mathscr{R}^l_{jlj}\right).$$

Note that $\Theta_0 = 0$, since the metric $g$ is static. We have

$$\mathscr{R}^l_{0l0} = \frac{\partial\Gamma^l_{00}}{\partial x^l} - \frac{\partial\Gamma^l_{0l}}{\partial x^0} + \sum_a \left(\Gamma^a_{00}\Gamma^l_{al} - \Gamma^a_{0l}\Gamma^l_{a0}\right) = \frac{1}{2}\Theta_{ll} + \sum_a\left(\frac{1}{2}\Theta_a^2\right) - \Theta_l^2, \quad l \neq 0,$$

$$\mathscr{R}^0_{000} = 0.$$

We also have for $l \neq j$,

$$\begin{aligned}\mathscr{R}^l_{jlj} &= \frac{\partial\Gamma^l_{jj}}{\partial x^l} - \frac{\partial\Gamma^l_{jl}}{\partial x^j} + \sum_a \{\Gamma^a_{jj}\Gamma^l_{al} - \Gamma^a_{jl}\Gamma^l_{aj}\} \\ &= -\frac{1}{2}\Theta_{ll} - \Theta_{jj} + \frac{3}{4}\Theta_j^2 - \frac{3}{4}\Theta_l^2 + \frac{1}{2}\Theta_l^2 - \Theta_j^2 \\ &= -\frac{1}{2}\Theta_{ll} - \Theta_{jj} - \frac{1}{4}\Theta_j^2 - \frac{1}{4}\Theta_l^2,\end{aligned}$$

and $\mathscr{R}^l_{lll} = 0$. Hence we see that

$$\begin{aligned}\mathscr{R} &= e^{-\Theta}\sum_l(\frac{1}{2}\Theta_{ll} + \frac{1}{2}\sum_a \Theta_a^2 - \Theta_l^2) - e^{-\Theta}\sum_l\sum_{j=1}^{3}(-\frac{1}{2}\Theta_{ll} - \Theta_{jj} - \frac{1}{4}\Theta_j^2 - \frac{1}{4}\Theta_l^2) \\ &= e^{-\Theta}(6\Delta\Theta + \frac{11}{4}|\nabla\Theta|^2) \\ &= e^{\theta}(-6\Delta\theta + \frac{11}{4}|\nabla\theta|^2)\end{aligned}$$

□

The Klein-Gordon equation on $\mathscr{M}$ is

$$\Box_g\phi + (m^2 + \eta\mathscr{R})\phi = 0, \tag{1.22}$$

where the d'Alembertian operator is defined by

$$\Box_g = e^{\theta(x)}\partial_t^2 - e^{2\theta(x)}\sum_j \partial_j e^{-\theta(x)}\partial_j. \tag{1.23}$$

Thus the Klein-Gordon equation (1.22) is reduced to the equation

$$\frac{\partial^2\phi}{\partial t^2} = K_0\phi, \tag{1.24}$$

where

$$K_0 = e^{\theta(x)}\sum_j \partial_j e^{-\theta(x)}\partial_j - e^{-\theta(x)}(m^2 + \eta\mathscr{R}). \tag{1.25}$$

The operator $K_0$ is symmetric on the weighted $L^2$ space $L^2(\mathbb{R}^3; e^{-\theta(x)}dx)$. Now we transform the operator $K_0$ to the one on $L^2(\mathbb{R}^3)$. This is done by the unitary map $U_0 : L^2(\mathbb{R}^3; e^{-\theta(x)}dx) \to L^2(\mathbb{R}^3)$, $f \mapsto e^{-(1/2)\theta}f$. Hence the Klein-Gordon equation (1.24) is transformed to the equation

$$\frac{\partial^2\phi}{\partial t^2} - \Delta\phi + v\phi = 0 \tag{1.26}$$

on $L^2(\mathbb{R}^3)$, and the dispersion relation is given by $\sqrt{-\Delta + v}$ and

$$v = e^{-\theta}(m^2 + \eta\mathcal{R}) - \frac{\Delta\theta}{2} + \frac{|\nabla\theta|^2}{4}. \tag{1.27}$$

Taking $\eta = 0$, $m = 0$, and $\theta(x) = 2a\langle x\rangle^{-\beta}$, we obtain

$$v(x) = -a\langle x\rangle^{-\beta-4}(\beta(\beta-1)|x|^2 - 3\beta) + a^2\beta^2\langle x\rangle^{-2\beta-4}|x|^2. \tag{1.28}$$

| | $m(x) \geq a\langle x\rangle^{-1}$ | $m(x) \leq a\langle x\rangle^{-\beta}$, $\beta > 1$ |
|---|---|---|
| ground state | exist | not exist |

Figure 1: Existence and absence of ground state

In the case of $0 \leq \beta \leq 1$ and $a > 0$, we see that $v \geq 0$ and $v = \mathcal{O}(\langle x\rangle^{-\beta-2})$. Furthermore $-\Delta + v$ has no non-positive eigenvalues. In the case of $\beta > 1$ and $a < 0$, we see that however $v \not\geq 0$. We can estimate the number of non-positive eigenvalues of $-\Delta + v$ by the Lieb-Thirring inequality. This yields that $-\Delta + v$ has no non-positive eigenvalues for sufficiently small $a$.

**Proposition 1.3** [GHPS09] *There exist functions $\theta$ and $v$ such that $U_0 K_0 U_0^{-1} = \Delta - v$, $v(x) = \mathcal{O}(\langle x\rangle^{-\beta-2})$ for $\beta \geq 0$, and $-\Delta + v$ has no non-positive eigenvalues.*

### 1.3 Nelson model on static Lorentzian manifold

We define the Nelson model on a static Lorentzian manifold. Let

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega) + \phi_\rho(X), \tag{1.29}$$

where

$$K = -\sum_{i,j=1}^{3} \partial_i A^{ij}(X)\partial_j + V(X) \tag{1.30}$$

is a divergence form,

$$\omega = \left(-\sum_{\mu,\nu=1}^{3} c(x)^{-1}\partial_\mu a_{\mu\nu}(x)\partial_\nu c(x)^{-1} + m^2(x)\right)^{1/2} \tag{1.31}$$

denotes the dispersion relation with variable mass $m(x)$ and the scalar field is given by

$$\phi(X) = \phi(\omega^{-1/2}\rho(\cdot - X)). \tag{1.32}$$

In the next section we review the absence and the existence of ground state of $H$.

## 2 Spectrum of the Nelson model

### 2.1 Existence of ground state

We introduce assumptions on dispersion relation $\omega$ and divergence form $K$:

**Assumption 2.1** *We suppose that*

$$\begin{aligned}&(1) C_0 \mathbb{1} \le \left[a^{ij}(x)\right] \le C_1 \mathbb{1},\\&(2) \partial^\alpha a^{ij}(x) \in O(\langle x\rangle^{-1}), \quad |\alpha| \le 1,\\&(3) C_0 \le c(x) \le C_1, \quad \partial^\alpha c(x) \in O(1), \quad |\alpha| \le 2,\\&(4) \partial^\alpha m(x) \in O(1), \quad |\alpha| \le 1.\end{aligned}$$

*We also suppose that*

$$\begin{aligned}&(5) C_0 \mathbb{1} \le \left[A^{ij}(X)\right] \le C_1 \mathbb{1},\\&(6) V(X) \ge C_0 \langle X\rangle^{2\delta} - C_1.\end{aligned}$$

**Theorem 2.2** [GHPS11] *Suppose Assumption 2.1, $m(x) \ge a\langle x\rangle^{-1}$ for some $a > 0$, and $\delta > 3/2$. Then $H$ has a ground state.*

The proof of Theorem 2.2 is based on the proposition below:

**Proposition 2.3** [BD04] *Suppose that*

$$\begin{aligned}&(1) \omega \ge 0 \ \textit{and} \ \mathrm{Ker}\omega = 0,\\&(2) \sup_X \|\omega^{-1/2}\rho(\cdot - X)\| < \infty,\\&(3) (K + \mathbb{1})^{-1/2} \ \textit{is compact},\\&(4) \omega^{-1}\rho(\cdot - X)(K + \mathbb{1})^{-1/2} \ \textit{is compact},\\&(5) \omega^{-3/2}\rho(\cdot - X)(K + \mathbb{1})^{-1/2} \ \textit{is compact}.\end{aligned}$$

*Then $K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X)$ has a ground state.*

The condition (5) in Proposition 2.3 corresponds to the infrared regular condition $\mathrm{I}_{\mathrm{IR}} < \infty$ in the standard Nelson model.

*Proof of Theorem 2.2*: Assumptions (1)-(4) in Proposition 2.2 can be checked directly. We check (5). The key estimate is to show that $\omega^{-3/2}\langle x\rangle^{-3/2-\epsilon}$ is bounded, and $\langle X\rangle^{3/2+\epsilon}(K + \mathbb{1})^{-1/2}$ is compact. Then we can see that

$$\omega^{-3/2}\rho(\cdot - X)(K+\mathbb{1})^{-1/2} = \omega^{-3/2}\langle x\rangle^{-3/2-\epsilon}\langle x\rangle^{3/2+\epsilon}\rho(x - X)\langle X\rangle^{-3/2-\epsilon}\langle X\rangle^{3/2+\epsilon}(K + \mathbb{1})^{-1/2}$$

is also compact. □

## 2.2 Absence of ground state

The standard way of showing the absence of ground state of the model in quantum field theory is an application of the so called pull through formula. In our case however the pull through formula can not be applied directly. Instead of it we apply functional integrations developed in [LMS02].

Let $\varphi_{\mathrm{p}}$ be the ground state of $K$. We know that the function $\varphi_{\mathrm{p}}$ is strictly positive and $\varphi_{\mathrm{p}} \in D(e^{-C|x|^{\delta+1}})$ with some constant $C_1$. We introduce the so-called ground state transform by $U : L^2(\varphi_{\mathrm{p}}^2 dx) \to L^2(dx)$, $f \mapsto \varphi_{\mathrm{p}} f$, and set

$$L = U(K - \inf \sigma(K))U^{-1}. \tag{2.1}$$

Thus $L$ is a positive self-adjoint operator acting on $L^2$-space over the probability space $(\mathbb{R}^3, \varphi_{\mathrm{p}}^2 dx)$. We also see that $\mathscr{F} \cong L^2(\mathscr{S}'_{\mathbb{R}}, d\nu)$ with a Gaussian measure $\nu$ on $\mathscr{S}'_{\mathbb{R}}$ such that

$$\int_{\mathscr{S}'_{\mathbb{R}}} e^{\alpha\phi(f)} d\nu(\phi) = e^{(\alpha^2/4)\|f\|^2}.$$

Then the total Hilbert space and the Nelson Hamiltonian are given by

$$L^2(\mathbb{R}^3) \otimes \mathscr{F} \cong L^2(\mathbb{R}^3 \times \mathscr{S}'_{\mathbb{R}}, \varphi_{\mathrm{p}}^2 dx \otimes d\nu) \tag{2.2}$$

and

$$H \cong L \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X). \tag{2.3}$$

**Theorem 2.4** [GHPS12-a] *Suppose $m(x) \le a\langle x\rangle^{-1-\epsilon}$ with some $\epsilon > 0$ and $\delta > 0$. Then $H$ has no ground state.*

*Proof:* We show the outline of the proof. We show that $e^{-TH}$ is positivity improving. Then if $H$ has a ground state $\varphi_{\mathrm{g}}$, then $\varphi_{\mathrm{g}} > 0$. Let $\mathbb{1} = \mathbb{1}_{L^2} \otimes \Omega$ and define $\varphi_{\mathrm{g}}^T = e^{-TH}\mathbb{1}/\|e^{-TH}\mathbb{1}\|$. Let

$$\gamma = \lim_{T\to\infty} (\mathbb{1}, \varphi_{\mathrm{g}}^T)^2 = \lim_{T\to\infty} \frac{(\mathbb{1}, e^{-TH}\mathbb{1})^2}{(\mathbb{1}, e^{-2TH}\mathbb{1})}. \tag{2.4}$$

It is a fundamental fact [LMS02] that $H$ has a ground state if and only if $\gamma > 0$. Let $\mathscr{X} = C(\mathbb{R}, \mathbb{R}^3)$. There exists a diffusion process $(X_t)_{t\in\mathbb{R}}$ on a probability space $(\mathscr{X}, B(\mathscr{X}), P^x)$ such that

$$(f, e^{-tL} g)_{L^2(\varphi_{\mathrm{g}}^2 dx)} = \mathbb{E}\left[\overline{f(X_0)} g(X_t)\right],$$

where $\mathbb{E}[\cdots] = \int \varphi_{\mathrm{p}}^2(x)dx \int \cdots dP^x$. We have

$$(\mathbb{1}, e^{-TH}\mathbb{1})_{\mathscr{H}} = \mathbb{E}\left[e^{\int_0^T dt \int_0^T ds W(X_t, X_s, |t-s|)}\right]$$

with the pair potential

$$W = W(X, Y, |t|) = \frac{1}{2}(\rho(\cdot - X), \omega^{-1} e^{-|t|\omega} \rho(\cdot - Y)).$$

The denominator of $\gamma$ is

$$(\mathbb{1}, e^{-2TH} \mathbb{1}) = \mathbb{E}\left[e^{\int_0^{2T} \int_0^{2T} W}\right] = \mathbb{E}\left[e^{\int_{-T}^{T} \int_{-T}^{T} W}\right]$$

by the reflection symmetry and the numerator is estimated as

$$(\mathbb{1}, e^{-TH} \mathbb{1})^2 \leq \mathbb{E}\left[e^{\int_{-T}^{T} \int_{-T}^{T} - 2\int_{-T}^{0} \int_0^T W}\right].$$

Together with them we have

$$\gamma \leq \lim_{T \to \infty} \frac{\mathbb{E}\left[e^{\int_{-T}^{T} \int_{-T}^{T} - 2\int_{-T}^{0} \int_0^T W}\right]}{\mathbb{E}\left[e^{\int_{-T}^{T} \int_{-T}^{T} W}\right]} = \lim_{T \to \infty} \mathbb{E}_{\mu_T}\left[e^{-2\int_{-T}^{0} \int_0^T W}\right].$$

Here the probability measure $\mu_T$ is defined by $\mathbb{E}_{\mu_T}[\cdots] = \frac{1}{Z_T}\mathbb{E}[\cdots e^{-2\int_{-T}^{0} \int_0^T W}]$. Let

$$\mathbb{E}_{\mu_T}\left[e^{-2\int_{-T}^{0} \int_0^T W}\right] = \mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T} \cdots\right] + \mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c} \cdots\right],$$

where $A_T = \{(x, w) \in \mathbb{R}^3 \times \mathscr{X} \,|\, \sup_{|s| \leq T} |X_s(w)| \leq T^\lambda, X_0(w) = x\}$. When $m(x) \leq a\langle x \rangle^{-1-\epsilon}$, the Gaussian bound:

$$C_1 e^{-C_2 t \omega_\infty^2}(x, y) \leq e^{-t\omega^2}(x, y) \leq C_3 e^{-C_4 t \omega_\infty^2}(x, y)$$

can be derived, where $\omega_\infty^2 = -\Delta$. Hence we can see that

$$C_1 W_\infty(x, y, C_2|t|) \leq W(x, t, |t|) \leq C_3 W_\infty(x, y, C_4|t|), \tag{2.5}$$

$$W_\infty(X, Y, |t|) = \frac{1}{4\pi^2} \int \frac{\rho(x)\rho(y)}{|x - y + X - Y|^2 + t^2} dx dy. \tag{2.6}$$

Thus we have

$$\mathbb{1}_{A_T} \int_{-T}^{0} \int_0^T W \geq \mathbb{1}_{A_T} \text{cons.} \int \int dx dy \rho(x)\rho(y) \log\left\{\frac{8T^{2\lambda} + 2|x - y|^2 + cT^2}{8T^{2\lambda} + 2|x - y|^2}\right\} \to \infty$$

as $T \to \infty$. Next we have

$$\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c} e^{-\int_{-T}^{0} \int_0^T W}\right] \leq C e^{TC} \mathbb{E}\left[A_T^c\right]. \tag{2.7}$$

It is established that $\mathbb{E}\left[A_T^c\right] \leq T^{-\lambda}(a + bT)^{1/2} e^{-T^{\lambda(\delta+1)}}$. Hence $\lambda(\delta + 1) > 1$ implies that $\mathbb{E}_{\mu_T}\left[\mathbb{1}_{A_T^c} \cdots\right] \to 0$ as $T \to \infty$. Then the proof is completed. □

## 2.3 Removal of UV cutoff

Finally we discuss the removal of UV cutoff of the Nelson model defined on a static Lorentzian manifold. Let

$$\hat{\rho}_\Lambda(k) = \begin{cases} (2\pi)^{-3/2} & |k| \le \Lambda \\ 0 & |k| > \Lambda \end{cases} \tag{2.8}$$

$$E_\Lambda = -\frac{1}{2}(2\pi)^{-3} \int \frac{|\mathbb{1}_{|k|<\Lambda}}{|k|(|k|^2/2+|k|)} dk. \tag{2.9}$$

We have $\lim_{\Lambda\to\infty} \hat{\rho}_\Lambda(k) = (2\pi)^{-3/2}$. Let external potential $V$ be vanished. Then $H$ is commutative with respect to the total momentum:

$$P = -i\nabla \otimes \mathbb{1} + \mathbb{1} \otimes \int k a^\dagger(k) a(k) dk.$$

Thus $H$ can be decomposed in the spectrum of $P$ and we have $H = \int_{\mathbb{R}^3}^{\oplus} H(p)dp$, where

$$H(p) = \frac{1}{2}\left(p - \int k a^\dagger(k) a(k) dk\right)^2 + \mathrm{d}\Gamma(\omega).$$

The effective mass $m_{\mathrm{eff}}$ is defined by $\frac{1}{m_{\mathrm{eff}}} = -\frac{1}{3}\Delta_p E(p)\lceil_{p=0}$, and thus

$$m_{\mathrm{eff}} = 1 + g^2 E_\Lambda + O(|g|^3).$$

**Proposition 2.5** [Nel64-a] *There exists a self-adjoint operator $H_\infty$ bounded from below such that* $s-\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)} = e^{-tH_\infty}$.

Another derivation of $E_\Lambda$ is seen in [GHL12]. In [GHL12] the existence of a self-adjoint operator without UV cutoff is given by means of functional integrations. See also [Nel64-b].

Let $\rho_\Lambda(\cdot) = \Lambda^3 \rho(\Lambda \cdot)$

$$E_\Lambda(X) = -\frac{1}{2}(2\pi)^{-3} \int (h_0(X,\xi)+1)^{-1/2} \frac{K(X,\xi)}{(K(X,\xi)+1)^2} |\hat{\rho}(\xi/\Lambda)^2| d\xi, \tag{2.10}$$

$$h_0(X,\xi) = \sum \xi_i a^{ij}(X)\xi_j, \tag{2.11}$$

$$K(X,\xi) = \sum \xi_i A^{ij}(X)\xi_j. \tag{2.12}$$

Note that $\rho_\Lambda(x-X) \to \delta(x-X)\int \rho(y)dy$ as $\Lambda \to \infty$. The term $(h_0(X,\xi)+1)^{-1/2}$ in (2.11) corresponds to $|k|^{-1}$ in (2.9) and $\frac{K(X,\xi)}{(K(X,\xi)+1)^2}$ in (2.12) to $(|k|^2/2+|k|)^{-1}$ in (2.9).

**Theorem 2.6** [GHPS12-b] *There exists a self-adjoint operator $H_{\mathrm{ren}}$ bounded from below such that* $s-\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda(X))} \to e^{-tH_{\mathrm{ren}}}$.

The standard Nelson model without UV cutoff also has a ground state [HHS05]. However it is unknown the uniqueness of the ground state.

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