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Ground States for 2D Spin Glasses

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Edwards-Anderson (EA) Spin Glass

For $G_N = (V_N, E_N)$, graph with $N$ vertices, let

$$H_N(\sigma) = - \sum_{(x,y) \in E_N} J_{xy} \sigma_x \sigma_y .$$

with $J := \{J_{xy}\}$ indep. mean zero Gaussian (or other).

- For SK model [SK], $G_N$ is the complete graph.
- For EA model [EA], $G_N \subset \mathbb{Z}^d$.
- Spin glasses exhibit many, very different, states of low energy that lead to complex behavior.
- How many ground states as $N \to \infty$?
- How different is EA than SK? For what $d$?
Gibbs States and Ground States

- The Gibbs measure is concentrated on \( \sigma \)'s with small \( H_N \).

\[
\mathcal{G}_{\beta,N}(\sigma) = \frac{\exp(-\beta H_N(\sigma))}{Z_N(\beta)}
\]

- \( \mathcal{G}_{\beta,N} \) gives information on \( \sigma \)'s close to the minimum.

Take \( G_N = (V_N, E_N) \), a box in \( \mathbb{Z}^d \) (of volume \( N \)).

- Take periodic b.c. (so \( H_N(\sigma) = H_N(-\sigma) \)).
- Write \( \alpha := \{\sigma, -\sigma\} \), a **Ground State Pair** (GSP).
- Study the measure on the unique GSP \( \alpha_N(J) : \delta_{\alpha_N(J)} \).
- Equivalent to studying Gibbs state with

\[
\beta \to \infty, \text{ then } N \to \infty.
\]
Metastate [AW, NS1]: a measure on GSP's

- $\alpha_{N,J}$ could change drastically between $N$ and $N'$.
- Consider the joint distribution of $(\alpha_{N,J}, J)$,

$$K_N := \delta_{\alpha_{N,J}} \nu_N(dJ).$$

- Take a (subsequence) limit of $K_N$ as $N \to \infty$.
- Express the limit $K$ as $K_J \nu(dJ)$.

- $K_J \nu(dJ)$ is translation invariant. (Periodic b.c.)
- $K_J$ is a measure on GSP's on $\mathbb{Z}^d$ for given $J$. 
Ground States in Infinite Volume

Definition (Ground State Property)
\( \alpha = \{ \sigma, -\sigma \} \) is a GSP on \( \mathbb{Z}^d \) for \( J \) if for any finite \( A \subset \mathbb{Z}^d \)

\[
- \sum_{(x,y) \in \partial A} J_{xy} \sigma_x \sigma_y < 0 .
\]
Ground States in Low Dimension

Conjecture (Uniqueness of Ground States)

For low $d$ ($d < d_c = 6?, = 8?, = \infty?$),

- the limit $K = K_J \nu(dJ)$ exists (no subseq. needed);
- $K_J$ is supported on a single GSP.

($d = 2$ numerics of Palassini-Young [PY], Middleton [M])

Strategy:

1. Let $\alpha$ and $\alpha'$ be replica GSP's.
2. Study the interface:

\[ \alpha\Delta\alpha' := \{(x, y) : \alpha_{xy} \neq \alpha'_{xy}\}, \]

where $\alpha_{xy} := \sigma_x \sigma_y$.
3. Show $\alpha\Delta\alpha'$ is empty (hence $\alpha = \alpha'$).
Interfaces between GSP's

- $\alpha\Delta\alpha' = \{(x, y) : \alpha_{xy} \neq \alpha'_{xy}\}$.
- Put a dual edge whenever $(x, y) \in \alpha\Delta\alpha'$. 
Full Plane Partial Result

If $\alpha$ and $\alpha'$ from metastate are distinct; then

- $\alpha \Delta \alpha'$ cannot have dangling ends (or 3-branching points).
- cannot contain loops.
- cannot have 4-branching points.

So $\alpha \Delta \alpha'$ is one or more doubly-infinite (self-avoiding) paths.

Theorem (Newman, Stein [NS2])

*In fact, non-empty $\alpha \Delta \alpha'$ can only be a single path.*
Uniqueness of GSP’s in the Half-Plane

Take $G_N = [-N, N] \times [0, 2N] \cap \mathbb{Z}^2$ with horizontal periodic and vertical free b.c. and let $N \to \infty$.

**Horizontal but not vertical translation invariance**

Theorem (Arguin, Damron, Newman, Stein [ADNS])

*In the half-plane,*

1. *the limit $K = K_J \nu(dJ)$ exists (no subseq.);*
2. *$K_J$ is supported on a single GSP.*

This is the first complete result for $d > 1$.

Strategy — show interface between replicas must be empty.
Interface of GSP’s in the Half-Plane

If $\alpha \Delta \alpha' \neq \emptyset$, then there are infinitely many tethered paths:
From Half to Full Plane

Let $\mu^* = \lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} T^{-l} \mu$ (along subseq.) where $T$ is the vertical shift and $\mu$ is distrib. of $(\alpha, \alpha', J)$.

- $\mu^*$ is a measure on the full plane,
- translation invariant in full plane by construction.

Because we see many tethered paths, if $\alpha \Delta \alpha' \neq \emptyset$, then $\alpha \Delta \alpha'$ is not a single path, contradicting full plane result of [NS2].

To do list for the future:
- Uniqueness of GSP’s in the full plane.
- Uniqueness (or not) of GSP’s for $d > 2$?
- Metastates applied to $\beta < \infty$. 
References


