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Functional Integral Representation of Semi-Relativistic Schrödinger Operator Coupled to Klein-Gordon Field

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1 Introduction

This article is a short review of probabilistic representation of quantum particle system interacting with a quantum field considered in [6]. To analyze quantum physics model mathematicaly, the state space $\mathcal{H}$ of the system is given by a Hilbert space. In many cases, the total Hamiltonian H of the system is a self-adjoint operator on $\mathcal{H}$. Here assume that H is self-adjoint and bounded from below. Functional integral representation of H is a probabilistic representation of the strongly continuous semi-group $\{e^{-tH}\}_{t\geq 0}$ generated by H. It is seen that the spectral analysis of H can be stochastically investigaed by using functional integral representations. For the details of functional integral representation and its applicaion of quantum physics model, refer to [5].

We investigae the system of a semi-relativistic particle interacting with a Klein-Gordon field. Here we assume that the particle ovey the relativistic Schrödinger operator and the ultraviolet cutoff condition is imposed on the Klein-Gordon field. The functional representation of relativistic Schrödinger operator is investigated in [2]. The particle in the electromagnetic potential with spin is considered in [3] and the spatial decay of the bound states is estimated in [4]. In the following section, the functional integral representation of the relativistic Schrodinger operator is deriven by Levy subordinator according to [3]. In constructive quantum field theory, Klein-Gordon field is constructed by the methods of stochastic process, and the functional integral representation is derived by using Eucilidean field. By using the functional integral representations of relativistic Schrödinger operator and Klein-Gordon field, the functional integral representation of the interacting system between semi-relativistic particle and the Klein-Gordon field is derived. From the obtained functional integral representation, we see that if the ground states exist, it is unique and its decay rate can be estimated.

2 Relativistic Schrödinger Operator and Klein-Gordon Field

We assume that the partilcle's Hamiltonian is given the relativistic Schrödinger operator with potential $H_{\rm p} = \sqrt{-\Delta+M^2} - M + V$ on the Hilbert space $L^2(\mathbf{R}^d_{\mathbf{x}})$. Here $M>0$ is the rest mass of the particle. Let us set $h_{\rm rel}(s) = \sqrt{s+M^2}-M$, $s>0$. Since $h_{\rm rel}$ is a Bernstein function, it is seen that there exists a Lévy subordinator $\{T_t\}_{t\geq 0}$ on a probability space $(\Omega_{\rm rel}, \mathfrak{B}_{\rm rel}, P_{\rm rel})$ satisfying $\mathbb{E}_{\rm rel}[e^{-sT_t}] = e^{-th_{\rm rel}(s)}$ where $\mathbb{E}_{\rm rel}[X] = \int_{\Omega_{\rm rel}} X(\eta)dP_{\rm rel}(\eta)$. Let $\{\mathbf{B}_t\}_{t\geq 0}$ be d-dimensional Brownian motion starting $\mathbf{x}$ on the probability space $(\Omega_{\rm BM}, \mathfrak{B}_{\rm BM}, P^{\mathbf{x}}_{\rm BM})$. We introduce the probability space $(\Omega_{\rm p}, \mathfrak{B}_{\rm p}, P^{\mathbf{x}}_{\rm p}) = (\Omega_{\rm rel}\times\Omega_{\rm BM}, \mathfrak{B}_{\rm rel}\otimes\mathfrak{B}_{\rm BM}, P_{\rm rel}\otimes P^{\mathbf{x}}_{\rm BM})$. Then the following functional integral representation for the semi-relativistic particles holds.(See [3] ; Theorem 3.8):

$$e^{-tH_{\rm p}}\psi(\mathbf{x}) = \mathbb{E}^{\mathbf{x}}_{\rm p}[\psi(\mathbf{X}_t)e^{-\int_0^t V(\mathbf{X}_s)ds}], \tag{1}$$

where $\mathbf{X}_t\,((\eta,\omega)) = \mathbf{B}_{T_t(\eta)}(\omega)$ and $\mathbb{E}^{\mathbf{x}}_{\rm p}[Z] = \int_{\Omega_{\rm p}} Z(\xi)dP^{\mathbf{x}}_{\rm p}(\xi)$.

Klein-Gordon field is constructed as follows. The Hilbert space $\mathcal{K}_{\rm KG}$ is defined by the completion of the pre-Hilbert space which consists of real valued tempered distribution $f\in\mathcal{S}'_{\rm real}(\mathbf{R}^d)$ satisfying $\|\omega^{-1/2}f\|_{L^2}<\infty$. Here the inner product is given by $(g,f)_{\mathcal{K}_{\rm KG}} = (\omega^{-1/2}g, \omega^{-1/2}f)_{L^2}$. Then from a general theorem on Gausian random process indexed by Hilbert spaces, there exist a probability space $(Q_{\rm KG}, \mathfrak{B}_{\rm KG}, P_{\rm KG})$ and a random process $\{\phi_f\}_{f\in\mathcal{K}_{\rm KG}}$ which satisfies $\mathbb{E}[e^{-it\phi_f}] = \exp(-\|f\|^2_{\mathcal{K}_{\rm KG}}t^2/4)$. Let $H_{\rm KG} = d\Gamma(\omega)$ be the second quatization of ω. Then Klein-Gordon field is defined by the triplet $(L^2(Q_{\mathcal{K}_{\rm KG}}), H_{\rm KG}, \{\phi_f\}_{f\in\mathcal{K}_{\rm KG}})$. Let $\mathcal{K}_{\rm E}$ be the completion of the pre-Hilbert space which consists of real-valued tempered distribution $u\in\mathcal{S}'_{\rm real}(\mathbf{R}^{1+d})$ satisfying $\|\omega^{-1}u\|_{L^2}<\infty$. Here the inner product is given by $(v,u)_{\mathcal{K}_{\rm KG}} = (\omega^{-1}v, \omega^{-1}u)_{L^2}$. Then similar to the construction of the Klein-Gordon fied, it is seen that there exist a probability space $(Q_{\rm E}, \mathfrak{B}_{\rm E}, P_{\rm E})$ and a Gaussian random process $\{\phi^{\rm E}_u\}_{u\in\mathcal{K}_{\rm E}}$ satisfying $\mathbb{E}[e^{-it\phi^{\rm E}_u}] = \exp(-\|u\|^2_{\mathcal{K}_{\rm E}}t^2/4)$. Then the next functional integral representation follows : (Refer to e.g. [1]; Theorem 7.19).

$$(\Phi, e^{-t(H_{\rm b}+P(\phi(f)))}\Psi)_{L^2(Q_{\rm KG})} = \mathbb{E}_{\rm E}[\overline{(J_0\Phi)}(J_t\Psi)e^{-\int_0^t P(\phi^{\rm E}_{\delta_s\otimes f})ds}]. \tag{2}$$

where J_t is an isometric operator from $L^2(Q_{\mathcal{K}_{\rm KG}})$ to $L^2(Q_{\mathcal{K}_{\rm E}})$ and $P = \sum_{j=1}^{2n} c_j\lambda^j$.

3 Main Theorem

The interaction system between the semi-relativistic particle and a scalar Bose fields is defined as follows. The state space for the system is given by $\mathcal{H} = L^2(\mathbf{R}^d_{\mathbf{x}}) \otimes L^2(\mathcal{Q}_{\mathrm{b}})$. The total Hamiltonian of the system is defined by form sum of the free Hamiltonian and interaction

$$H_\kappa \; = \; H_0 \dotplus \kappa H_{\mathrm{I}}, \qquad \kappa \in \mathbf{R}, \tag{3}$$

where $H_0 = H_{\mathrm{p}} \otimes I + I \otimes H_{\mathrm{KG}}$ and $H_{\mathrm{I}} \; = \; P(\phi_{\rho_{\Lambda,\mathbf{x}}})$ with $P(\lambda) = \sum_{j=1}^{2n} c_j \lambda^j$, $c_j \in \mathbf{R}$, $j = 1, \cdots, 2n-1$, $c_{2n} > 0$ and $\rho_{\Lambda,\mathbf{x}}$ satisfies that $\hat{\rho}_{\Lambda,\mathbf{x}}(\mathbf{k}) = \hat{\rho}_\Lambda(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ with the characteristic funcion ρ_Λ on $\mathbf{R}^d$.

By appling Feynman-Kac formula of relativistic Shrödinger operator and Klein-Gordon and Trotter-Kato product formula, the functional integaral representation of H_κ is derived.

Theorem 1
Assume that V is relativistic Kato-class. Then it follows that

$$(\Phi, e^{-tH_\kappa}\Psi)_{\mathcal{H}} = \int_{\mathbf{R}^d} \mathbb{E}^{\mathbf{x}}_{\mathrm{p}\times\mathrm{E}}[\overline{(J_0\Phi(\mathbf{X}_0))}(J_t\Psi(\mathbf{X}_t))e^{-\int_0^t V(\mathbf{X}_s)ds} e^{-\kappa P(\phi^{\mathrm{E}}(\int_0^t \delta_s \otimes \rho_{\mathbf{X}_s} ds))}] d\mathbf{x}.$$

It is seen that $e^{-tH_{\mathrm{p}}}$ and $e^{-tH_{\mathrm{KG}}}$ are positivity improving operators. In addition, the exponetial decay of the bound states with respect to spacial variable is proven in [4]. Then from the functional integral representation of H, the next corollary immediately follows.

Corollary 2
Assume that V is relativistic Kato class and H_κ has the ground state. Then, following (1) and (2) holds.
(1) The ground state is unique.
(2) If $V(\mathbf{x}) \to 0$ or ∞ as $|\mathbf{x}| \to \infty$, the bound state has exponential decay with respecto to spatial vaiable.

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References

[1] A. Arai, *Functional integral methods in quantum mathematical physics*, kyoritsu Publication, 2010.(In Japanese)

[2] R. Carmona, M. C. Masters and B. Simon, Relativistic Schrödinger operators: Asymptotic behavior of the eigenvalues, *J. Funct. Anal.* **91** (1990) 117-142.

[3] F. Hiroshima, T. Ichinose and J. Lörinczi, Path integral representation for Schrödinger operators with Bernstein functions of laplacian, arxiv (2009).

[4] F. Hiroshima, T. Ichinose and J. Lorinczi, Probabilistic representation and fall-off of bound states of relativistic Schrödinger operators with spin $1/2$, preprint, arxiv (2011).

[5] J. Lörinczi, F. Hiroshima and V. Betz, *Feynman-Kac-type theorem and Gibbs Measure on path space*, De Gruyter (2011).

[6] T. Takaesu, Functional integral representation of semi-Relativistic quantum field model with $P(\phi)$ interaction, arxiv 2011.