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Temperature Dependence of the Solution to the BCS Gap Equation

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1 Introduction and preliminaries

We use the unit $k_B = 1$, where k_B stands for the Boltzmann constant. Let $\omega_D > 0$ and $k \in \mathbb{R}^3$ stand for the Debye frequency and the wave vector of an electron, respectively. Let $m > 0$ and $\mu > 0$ stand for the electron mass and the chemical potential, respectively. We denote by $T(\geq 0)$ the temperature, and by x the kinetic energy of an electron minus the chemical potential, i.e., $x = \hbar^2|k|^2/(2m) - \mu$. Note that $0 < \hbar\omega_D << \mu$.

In the BCS model [2, 4] of superconductivity, the solution to the BCS gap equation (1.1) below is called the gap function. We regard the gap function as a function of both T and x, and denote it by u, i.e., $u : (T, x) \mapsto u(T, x)$ (≥ 0). The BCS gap equation is the following nonlinear integral equation:

$$(1.1) \quad u(T, x) = \int_{\varepsilon}^{\hbar\omega_D} \frac{U(x, \xi)\, u(T, \xi)}{\sqrt{\xi^2 + u(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T}\, d\xi, \qquad \varepsilon \leq x \leq \hbar\omega_D\,,$$

where $U(x, \xi) > 0$ is the potential multiplied by the density of states per unit energy at the Fermi surface and is a function of x and ξ. In (1.1) we introduce $\varepsilon > 0$, which is small enough and fixed ($0 < \varepsilon << \hbar\omega_D$). It is known that the BCS gap equation (1.1) is based on a superconducting state called the BCS state. In this connection, see [11, (6.1)] for a new gap equation based on a superconducting state having a lower energy than the BCS state.

The integral with respect to ξ in (1.1) is sometimes replaced by the integral over $\mathbb{R}^3$ with respect to the wave vector k. Odeh [9], and Billard and Fano [3] established the existence and uniqueness of the positive solution to the BCS gap equation in the case $T = 0$. In the case $T \geq 0$, Vansevenant [10] determined the transition temperature (the critical temperature) and showed that there is a unique positive solution to the BCS gap equation. Recently, Frank, Hainzl, Naboko and Seiringer [5] gave a rigorous analysis of the asymptotic behavior of the transition temperature at weak coupling. Hainzl, Hamza, Seiringer and Solovej [6] proved that the existence of a positive solution to the BCS gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature. Moreover, Hainzl and Seiringer [7] derived upper and lower bounds on the transition temperature and the energy gap for the BCS gap equation.

Since the existence and uniqueness of the solution were established for T fixed in previous papers, the temperature dependence of the solution is not covered. Studying the temperature dependence of the solution to the BCS gap equation is very important. This is because, by dealing with the thermodynamical potential, this study leads to the mathematical challenge of showing that the transition to a superconducting state is a second-order phase transition.

In this paper, in order to show how the solution varies with the temperature, we first give another proof of the existence and uniqueness of the solution to the BCS gap equation (1.1). More precisely, we show that the solution belongs to V_T (see (2.1) below). Note that the set V_T depends on T. We define a certain subset W (see (2.2) below) of a Banach space consisting of continuous functions of both T and x. We approximate the solution by an element of W. We second show, under this approximation, that the transition to a superconducting state is a second-order phase transition. In other words, we show that the condition that the solution belongs to W is a sufficient condition for the second-order phase transition in superconductivity. We finally show that the solution to the BCS gap equation (1.1) is continuous with respect to both T and x when T satisfies a certain condition.

Let

$$U(x,\,\xi)=U_1 \qquad \text{at all} \quad (x,\,\xi)\in[\varepsilon,\,\hbar\omega_D]^2, \tag{1.2}$$

where $U_1>0$ is a constant. Then the gap function depends on the temperature T only. So we denote the gap function by Δ_1 in this case, i.e., $\Delta_1:\,T\mapsto\Delta_1(T)$. Then (1.1) leads to the simple gap equation

$$1=U_1\int_\varepsilon^{\hbar\omega_D}\frac{1}{\sqrt{\xi^2+\Delta_1(T)^2}}\,\tanh\frac{\sqrt{\xi^2+\Delta_1(T)^2}}{2T}\,d\xi. \tag{1.3}$$

It is known that superconductivity occurs at temperatures below the transition temperature. The following is the definition of the transition temperature, which originates from the simple gap equation (1.3).

Definition 1.1 ([2]). The transition temperature originating from the simple gap equation (1.3) is the temperature $\tau_1>0$ satisfying

$$1=U_1\int_\varepsilon^{\hbar\omega_D}\frac{1}{\xi}\,\tanh\frac{\xi}{2\tau_1}\,d\xi.$$

In the BCS model, we assume that there is a unique solution $\Delta_1:\,T\mapsto\Delta_1(T)$ to the simple gap equation (1.3), and that Δ_1 is of class C^2 with respect to the temperature T (see e.g. [2] and [16, (11.45), p.392]). The author [12] has given a mathematical proof of the assumption on the basis of the implicit function theorem. Set

$$\Delta=\frac{\sqrt{\left(\hbar\omega_D-\varepsilon\,e^{1/U_1}\right)\left(\hbar\omega_D-\varepsilon\,e^{-1/U_1}\right)}}{\sinh\frac{1}{U_1}}. \tag{1.4}$$

Proposition 1.2 ([12, Proposition 2.2]). *Let Δ be as in (1.4). Then there is a unique nonnegative solution $\Delta_1 : [0.\ \tau_1] \to [0, \infty)$ to the simple gap equation (1.3) such that the solution Δ_1 is continuous and strictly decreasing on the closed interval $[0.\ \tau_1]$:*

$$\Delta_1(0) = \Delta > \Delta_1(T_1) > \Delta_1(T_2) > \Delta_1(\tau_1) = 0, \qquad 0 < T_1 < T_2 < \tau_1.$$

Moreover, the solution Δ_1 is of class C^2 on the interval $[0.\ \tau_1)$ and satisfies

$$\Delta_1'(0) = \Delta_1''(0) = 0 \quad \text{and} \quad \lim_{T \uparrow \tau_1} \Delta_1'(T) = -\infty.$$

Remark 1.3. We set $\Delta_1(T) = 0$ for $T > \tau_1$.

Remark 1.4. In Proposition 1.2, $\Delta_1(T)$ is nothing but $\sqrt{f(T)}$ in [12, Proposition 2.2].

Let $0 < U_1 < U_2$, where $U_2 > 0$ is a constant. We assume the following condition on $U(\cdot, \cdot)$:

$$\text{(1.5)} \qquad U_1 \le U(x, \xi) \le U_2 \quad \text{at all} \quad (x, \xi) \in [\varepsilon, \hbar\omega_D]^2, \qquad U(\cdot, \cdot) \in C^2([\varepsilon, \hbar\omega_D]^2).$$

When $U(x, \xi) = U_2$ at all $(x, \xi) \in [\varepsilon, \hbar\omega_D]^2$, an argument similar to that in Proposition 1.2 gives that there is a unique nonnegative solution $\Delta_2 : [0.\ \tau_2] \to [0, \infty)$ to the simple gap equation

$$\text{(1.6)} \qquad 1 = U_2 \int_\varepsilon^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_2(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_2(T)^2}}{2T}\, d\xi, \qquad 0 \le T \le \tau_2.$$

Here, $\tau_2 > 0$ is defined by

$$\text{(1.7)} \qquad 1 = U_2 \int_\varepsilon^{\hbar\omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_2}\, d\xi.$$

We again set $\Delta_2(T) = 0$ for $T > \tau_2$. A straightforward calculation gives the following.

Lemma 1.5 ([13, Lemma 1.5]). (a) *The inequality $\tau_1 < \tau_2$ holds.*
(b) *If $0 \le T < \tau_2$, then $\Delta_1(T) < \Delta_2(T)$. If $T \ge \tau_2$, then $\Delta_1(T) = \Delta_2(T) = 0$.*

On the basis of Proposition 1.2, the author [12, Theorem 2.3] proved that the transition to a superconducting state is a second-order phase transition under the restriction (1.2).

As is mentioned above, we now introduce the thermodynamical potential Ω to study the phase transition in superconductivity. For more details on the thermodynamical potential, see e.g. [2, sec. III] or Niwa [8, sec. 7.7.3]. Let $N(x) \ge 0$ stand for the density of states per unit energy at the energy $x \quad (-\mu \le x < \infty)$ and set $N_0 = N(0) > 0$. Here, N_0 stands for the density of states per unit energy at the Fermi surface ($x = 0$). Note that the function $x \mapsto N(x)$ is continuous on $[-\mu, \infty)$. For the gap function u, set

$$\Omega_S(T) = \Omega_N(T) + \Psi(T),$$

where

$$\text{(1.8)} \qquad \Omega_N(T) = -2N_0 \int_\varepsilon^{\hbar\omega_D} x\, dx - 4N_0 T \int_\varepsilon^{\hbar\omega_D} \ln\left(1 + e^{-x/T}\right) dx + \Phi(T), \qquad T > 0,$$

$$(1.9)\qquad \Phi(T) = 2\int_{-\mu}^{-\hbar\omega_D} x\,N(x)\,dx - 2T\int_{-\mu}^{-\hbar\omega_D} N(x)\ln\left(1+e^{x/T}\right)dx$$
$$-2T\int_{\hbar\omega_D}^{\infty} N(x)\ln\left(1+e^{-x/T}\right)dx, \qquad T>0,$$

$$(1.10)\qquad \Psi(T) = -2N_0\int_{\varepsilon}^{\hbar\omega_D}\left\{\sqrt{x^2+u(T,\,x)^2}-x\right\}dx$$
$$+N_0\int_{\varepsilon}^{\hbar\omega_D}\frac{u(T,\,x)^2}{\sqrt{x^2+u(T,\,x)^2}}\tanh\frac{\sqrt{x^2+u(T,\,x)^2}}{2T}\,dx$$
$$-4N_0T\int_{\varepsilon}^{\hbar\omega_D}\ln\frac{1+e^{-\sqrt{x^2+u(T,\,x)^2}/T}}{1+e^{-x/T}}\,dx, \quad 0<T\le T_c\,.$$

Here, T_c is the transition temperature defined by Definition 2.4 below and originates from the BCS gap equation (1.1).

Remark 1.6. The integral $\displaystyle\int_{\hbar\omega_D}^{\infty} N(x)\ln\left(1+e^{-x/T}\right)dx$ on the right side of (1.9) is well defined for $T>0$, since $N(x)=O(\sqrt{x})$ as $x\to\infty$.

Definition 1.7. Let $\Omega_S(T)$ and $\Omega_N(T)$ be as above. The thermodynamical potential Ω in the BCS model is defined by (see e.g. Niwa [8, sec. 7.7.3])

$$\Omega(T) = \begin{cases} \Omega_S(T) & (0<T\le T_c), \\ \Omega_N(T) & (T>T_c). \end{cases}$$

Remark 1.8. Generally speaking, the thermodynamical potential Ω is a function of the temperature T, the chemical potential μ and the volume of our physical system. Fixing both μ and the volume of our physical system, we deal with the dependence of Ω on the temperature T only.

Remark 1.9. Hainzl, Hamza, Seiringer and Solovej [6] studied the BCS gap equation with a more general potential examining the thermodynamic pressure.

Remark 1.10. It is shown in [12, Lemmas 6.1 and 6.2] that both of the functions Ω_N (see (1.8)) and Φ (see (1.9)), regarded as functions of T, are of class C^2 on $(0,\,\infty)$.

Definition 1.11. We say that the transition to a superconducting state at the transition temperature T_c is a second-order phase transition if the following conditions are fulfilled.

(a) The thermodynamical potential Ω, regarded as a function of T, is of class C^1 on $(0,\,\infty)$.

(b) The thermodynamical potential Ω, regarded as a function of T, is of class C^2 on $(0,\,\infty)\setminus\{T_c\}$, and the second-order partial derivative $\left(\partial^2\Omega/\partial T^2\right)$ is discontinuous at $T=T_c$.

Remark 1.12. As is known in condensed matter physics, condition (a) implies that the entropy $S=-\left(\partial\Omega/\partial T\right)$ is continuous on $(0,\,\infty)$ and that, as a result, no latent heat is observed at $T=T_c$. On the other hand, (b) implies that the specific heat at constant volume, $C_V=-T\left(\partial^2\Omega/\partial T^2\right)$, is discontinuous at $T=T_c$ and that the gap ΔC_V in the specific heat at constant volume is observed at $T=T_c$. For more details on the entropy well as the specific heat at constant volume, see e.g. [2, sec. III] or Niwa [8, sec. 7.7.3].

2 Main results

Let $0 \le T \le \tau_2$ and fix T, where τ_2 is that in (1.7). We first consider the Banach space $C([\varepsilon,\, \hbar\omega_D])$ consisting of continuous functions of x only, and deal with the following subset V_T:

$$V_T = \{u(T,\,\cdot) \in C([\varepsilon,\, \hbar\omega_D]) : \ \Delta_1(T) \le u(T,\,x) \le \Delta_2(T) \ \text{ at } \ x \in [\varepsilon,\, \hbar\omega_D]\}\,. \tag{2.1}$$

Remark 2.1. The set V_T depends on T. So we denote each element of V_T by $u(T,\,\cdot)$.

As is mentioned in the introduction, the existence and uniqueness of the solution to the BCS gap equation were established in previous papers [3, 5, 6, 7, 9, 10]. However the temperature T was fixed, and hence the temperature dependence of the solution is not covered. So we give another proof of the existence and uniqueness of the solution to the BCS gap equation (1.1) so as to show how the solution varies with the temperature. More precisely, we show that the solution belongs to V_T. Note that Proposition 1.2 and Lemma 1.5 point out how Δ_1 and Δ_2 depend on the temperature and how Δ_1 and Δ_2 vary with the temperature.

Theorem 2.2 ([13, Theorem 2.2]). *Assume condition* (1.5) *on* $U(\cdot,\,\cdot)$. *Let* $T \in [0,\,\tau_2]$ *be fixed. Then there is a unique nonnegative solution* $u_0(T,\,\cdot) \in V_T$ *to the BCS gap equation* (1.1)*:*

$$u_0(T,\,x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x,\,\xi)\,u_0(T,\,\xi)}{\sqrt{\xi^2 + u_0(T,\,\xi)^2}}\,\tanh\frac{\sqrt{\xi^2 + u_0(T,\,\xi)^2}}{2T}\,d\xi, \quad x \in [\varepsilon,\, \hbar\omega_D].$$

Consequently, the solution is continuous with respect to x and varies with the temperature as follows:

$$\Delta_1(T) \le u_0(T,\,x) \le \Delta_2(T) \quad \text{at} \quad (T,\,x) \in [0,\,\tau_2] \times [\varepsilon,\, \hbar\omega_D].$$

Remark 2.3. In fact, Theorem 2.2 holds true under

$$U_1 \le U(x,\,\xi) \le U_2 \quad \text{at all} \quad (x,\,\xi) \in [\varepsilon,\, \hbar\omega_D]^2, \qquad U(\cdot,\,\cdot) \in C([\varepsilon,\, \hbar\omega_D]^2).$$

But we assume condition (1.5) on $U(\cdot,\,\cdot)$ instead. This is because we deal with the subset W (see (2.2) below) so as to prove Theorem 2.9.

The existence of the transition temperature T_c is pointed out in previous papers [5, 6, 7, 10]. In our case, it is defined as follows.

Definition 2.4. Let $u_0(T,\,\cdot) \in V_T$ be as in Theorem 2.2. The transition temperature T_c originating from the BCS gap equation (1.1) is defined by

$$T_c = \inf\{T > 0 : \ u_0(T,\,x) = 0 \quad \text{at all } x \in [\varepsilon,\, \hbar\omega_D]\}\,.$$

Remark 2.5. Combining Definition 2.4 with Theorem 2.2 implies that $\tau_1 \le T_c \le \tau_2$. For $T > T_c$, we set $u_0(T,\,x) = 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$.

We next consider the Banach space $C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D])$ consisting of continuous functions of both T and x. Let us consider the following condition, which gives the behavior of functions as $T \to T_c$. We assume condition (1.5) on $U(\cdot,\, \cdot)$. Let T_c be as in Definition 2.4 and let $\varepsilon_1 > 0$ be arbitrary.

Condition (C). For $u \in C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D]) \cap C^2\left((0,\, T_c) \times [\varepsilon,\, \hbar\omega_D]\right)$, there are $v,\, w \in C([\varepsilon,\, \hbar\omega_D])$ satisfying the following.

(C1) $v(x) > 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$.

(C2) For $\varepsilon_1 > 0$, there is a $\delta > 0$ such that $|T_c - T| < \delta$ implies

$$\left| v(x) - \frac{u(T,\, x)^2}{T_c - T} \right| < T_c\, \varepsilon_1 \quad \text{and} \quad \left| v(x) + 2\, u(T,\, x) \frac{\partial u}{\partial T}(T,\, x) \right| < T_c\, \varepsilon_1\,,$$

where δ does not depend on $x \in [\varepsilon,\, \hbar\omega_D]$.

(C3) Set $f(T,\, x) = u(T,\, x)^2$. Then, for $\varepsilon_1 > 0$, there is a $\delta > 0$ such that $|T_c - T| < \delta$ implies

$$\left| \frac{w(x)}{2} + \frac{f(T,\, x) + (T_c - T)\, \dfrac{\partial f}{\partial T}(T,\, x)}{(T_c - T)^2} \right| < \varepsilon_1 \quad \text{and} \quad \left| w(x) - \frac{\partial^2 f}{\partial T^2}(T,\, x) \right| < \varepsilon_1\,,$$

where δ does not depend on $x \in [\varepsilon,\, \hbar\omega_D]$.

Remark 2.6. If $u \in C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D]) \cap C^2\left((0,\, T_c) \times [\varepsilon,\, \hbar\omega_D]\right)$ satisfies condition (C), then $u(T_c\,,\, x) = 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$.

We deal with the following subset W of the Banach space $C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D])$.

(2.2)

$$\begin{aligned} W \quad = \quad & \{u \in C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D]) \cap C^2((0,\, T_c) \times [\varepsilon,\, \hbar\omega_D]) : \Delta_1(T) \le u(T,\, x) \le \Delta_2(T) \\ & \quad \text{at } (T,\, x) \in [0,\, T_c] \times [\varepsilon,\, \hbar\omega_D], \quad u \text{ satisfies condition (C)}\}\,. \end{aligned}$$

Remark 2.7. Let $u \in W$. Then, for $T \ge T_c$, we set $u(T,\, x) = 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$.

Define a mapping A by

$$Au(T,\, x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x,\, \xi)\, u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T,\, \xi)^2}}{2T}\, d\xi, \qquad u \in W. \tag{2.3}$$

Proposition 2.8 ([13, Proposition 2.10]). *Assume condition* (1.5) *on* $U(\cdot,\, \cdot)$. *Let* W *be as above. Let* $u_0(T,\, \cdot) \in V_T$ *be as in Theorem 2.2.*

(a) *The mapping* $A : W \longrightarrow W$ *is continuous with respect to the norm of the Banach space* $C([0,\, T_c] \times [\varepsilon,\, \hbar\omega_D])$.

(b) *Let* $u \in W$. *Let* $0 \le T \le \tau_2$ *and fix* T. *Then all of* $u(T,\, \cdot)$, $Au(T,\, \cdot)$ *and* $u_0(T,\, \cdot)$ *belong to* V_T. *Consequently, at all* $(T,\, x) \in [0,\, \tau_2] \times [\varepsilon,\, \hbar\omega_D]$,

$$\Delta_1(T) \le u(T,\, x),\ Au(T,\, x),\ u_0(T,\, x) \le \Delta_2(T).$$

We choose U_1 and U_2 (see (1.5)) such that the following inequality holds:

$$\sup_{0 \le T \le \tau_2} |\Delta_2(T) - \Delta_1(T)| < \varepsilon_2, \tag{2.4}$$

where $\varepsilon_2 > 0$ is small enough. Then it follows from Proposition 2.8 (b) that for $u \in W$,

(2.5)
$$|u(T,\, x) - u_0(T,\, x)| < \varepsilon_2, \quad |Au(T,\, x) - u_0(T,\, x)| < \varepsilon_2, \quad |Au(T,\, x) - u(T,\, x)| < \varepsilon_2.$$

at all $(T,\, x) \in [0,\, \tau_2] \times [\varepsilon,\, \hbar\omega_D]$.

Approximation (A). The gap function on the right side of (1.10) is the solution u_0 to the BCS gap equation (1.1). But no one gives the proof of the statement that there is a unique solution in W to the BCS gap equation (1.1). In view of (2.5), we then approximate u_0 by a $u \in W$, and replace the gap function on the right side of (1.10) by this $u \in W$.

Let $g:\ [0,\, \infty) \to \mathbb{R}$ be given by

$$g(\eta) = \begin{cases} \dfrac{1}{\eta^2}\left(\dfrac{1}{\cosh^2\eta} - \dfrac{\tanh\eta}{\eta}\right) & (\eta > 0), \\ -\dfrac{2}{3} & (\eta = 0). \end{cases} \tag{2.6}$$

Note that $g(\eta) < 0$.

Theorem 2.9 ([13, Theorem 2.11]). *Assume condition* (1.5) *on* $U(\cdot,\, \cdot)$. *Let* U_1 *and* U_2 *be chosen such that* (2.4) *holds. We approximate* u_0 *of Theorem 2.2 by a* $u \in W$ *as stated in approximation (A) above. Let* $v \in C([\varepsilon,\, \hbar\omega_D])$ *be as in condition (C). Then the following hold.*

(a) *The transition to a superconducting state at the transition temperature* T_c *is a second-order phase transition. Consequently, the condition that the solution to the BCS gap equation* (1.1) *belongs to* W *is a sufficient condition for the second-order phase transition in superconductivity.*

(b) *The gap* ΔC_V *in the specific heat at constant volume at the transition temperature* T_c *is given by the form*

$$\Delta C_V = -\frac{N_0}{8\,T_c}\int_{\varepsilon/(2\,T_c)}^{\hbar\omega_D/(2\,T_c)} v(2T_c\,\eta)^2\, g(\eta)\, d\eta \quad (> 0).$$

We now assume the following weaker condition instead of (1.5):

$$U_1 \le U(x,\, \xi) \le U_2 \quad \text{at all} \quad (x,\, \xi) \in [\varepsilon,\, \hbar\omega_D]^2, \qquad U(\cdot,\, \cdot) \in C([\varepsilon,\, \hbar\omega_D]^2). \tag{2.7}$$

Let $U_0 > 0$ be a constant satisfying $U_0 < U_1 < U_2$. An argument similar to that in Proposition 1.2 gives that there is a unique nonnegative solution $\Delta_0 : [0.\ \tau_0] \to [0,\, \infty)$ to the simple gap equation

$$1 = U_0 \int_\varepsilon^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_0(T)^2}} \tanh\frac{\sqrt{\xi^2 + \Delta_0(T)^2}}{2T}\, d\xi, \qquad 0 \le T \le \tau_0.$$

Here, $\tau_0 > 0$ is defined by

$$1 = U_0 \int_\varepsilon^{\hbar\omega_D} \frac{1}{\xi} \tanh\frac{\xi}{2\tau_0}\, d\xi.$$

We set $\Delta_0(T) = 0$ for $T > \tau_0$. A straightforward calculation gives the following.

Lemma 2.10. (a) $\tau_0 < \tau_1 < \tau_2$.
(b) *If* $0 \le T < \tau_0$, *then* $0 < \Delta_0(T) < \Delta_1(T) < \Delta_2(T)$.
(c) *If* $\tau_0 \le T < \tau_1$, *then* $0 = \Delta_0(T) < \Delta_1(T) < \Delta_2(T)$.
(d) *If* $\tau_1 \le T < \tau_2$, *then* $0 = \Delta_0(T) = \Delta_1(T) < \Delta_2(T)$.
(e) *If* $\tau_2 \le T$, *then* $0 = \Delta_0(T) = \Delta_1(T) = \Delta_2(T)$.

Remark 2.11. Let the functions Δ_k $(k = 0,\, 1,\, 2)$ be as above. For each Δ_k, there is the inverse $\Delta_k^{-1} : [0,\, \Delta_k(0)] \to [0,\, \tau_k]$. Here,

$$\Delta_k(0) = \frac{\sqrt{\left(\hbar\omega_D - \varepsilon\, e^{1/U_k}\right)\left(\hbar\omega_D - \varepsilon\, e^{-1/U_k}\right)}}{\sinh \frac{1}{U_k}}$$

and $\Delta_0(0) < \Delta_1(0) < \Delta_2(0)$. See [12] for more details.

Let $(0 <)\, T_1 < \tau_0$ satisfy

$$\frac{\Delta_0(0)}{4\,\Delta_2^{-1}\left(\Delta_0(T_1)\right)} \tanh \frac{\Delta_0(0)}{4\,\Delta_2^{-1}\left(\Delta_0(T_1)\right)} > \frac{1}{2}\left(1 + \frac{4\hbar^2\omega_D^2}{\Delta_0(0)^2}\right). \tag{2.8}$$

Remark 2.12. Numerically, the temperature T_1 is very small.

Let T_1 be as in (2.8). We deal with the following subset V of the Banach space $C([0,\, T_1] \times [\varepsilon,\, \hbar\omega_D])$:

$$\begin{aligned} V \;=\; & \{u \in C([0,\, T_1] \times [\varepsilon,\, \hbar\omega_D]) : \Delta_1(T) \le u(T,\, x) \le \Delta_2(T) \\ & \quad at\;\; (T,\, x) \in [0,\, T_1] \times [\varepsilon,\, \hbar\omega_D]\}\,. \end{aligned} \tag{2.9}$$

Theorem 2.13 ([14, Theorem 2.3]). *Assume* (2.7). *Let* u_0 *be as in Theorem 2.2 and* V *as in* (2.9). *Then* $u_0 \in V$. *Consequently, the gap function* u_0 *is continuous on* $[0,\, T_1] \times [\varepsilon,\, \hbar\omega_D]$.

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