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Poly-Cauchy numbers

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1 Introduction.

In 1997 M. Kaneko ([7]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ for an integer k and a non-negative integer n by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!},$$

where

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}$$

denotes the k-th polylogarithm function ($k\geq 1$) It becomes a rational function if $k\leq 0$. When $k=1$, $B_n^{(1)}=B_n$ are the Bernoulli numbers (with $B_1=1/2$), defined by the generating function

$$\frac{xe^x}{e^x-1}=\sum_{n=0}^{\infty}B_n^{(1)}\frac{x^n}{n!}.$$

$B_n^{(k)}$ is explicitly expressed as

$$B_n^{(k)}=(-1)^n\sum_{m=0}^{n}\frac{(-1)^m m!}{(m+1)^k}\left\{{n\atop m}\right\},$$

where

$$\left\{{n\atop m}\right\}=\frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l}l^m$$

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are the Stirling numbers of the second kind ([7, Theorem 1]).

For a positive integer k and a non-negative integer n, *poly-Cauchy numbers* (of the first kind) $c_n^{(k)}$ are given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \tag{1}$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

are called the k-th *polylogarithm factorial* function. When $k=1$, $c_n^{(1)} = c_n$ are the Cauchy numbers ([4]) defined by the generating function defined by the generating function

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

$c_n^{(k)}$ is explicitly expressed as

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m}{(m+1)^k} \begin{bmatrix} n \\ m \end{bmatrix},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ are the (unsigned) Stirling numbers of the first kind appeared as the coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m.$$

We record the values of $c_n^{(k)}$ for $n = 0, 1, \dots, 5$.

$$\begin{aligned}
c_0^{(k)} &= 1, \\
c_1^{(k)} &= \frac{1}{2^k}, \\
c_2^{(k)} &= -\frac{1}{2^k} + \frac{1}{3^k}, \\
c_3^{(k)} &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k}, \\
c_4^{(k)} &= -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k},
\end{aligned}$$

$$c_5^{(k)} = \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k}\,.$$

Poly-Cauchy numbers (of the first kind) may be defined by using integrals.

Theorem 1. *For $n \ge 0$ and $k \ge 1$, let $C_n^{(k)}$ be*

$$C_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - 1)$$

$$\dots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k\,.$$

Then $C_n^{(k)} = c_n^{(k)}$.

As the Stirling numbers of the second kind is related to $e^t - 1$ and the Stirling numbers of the second is to $1/\ln(1-t)$ via Riordan arrays (see e.g. [9, 11, 12]), it may be natural to consider if some properties which hold on Bernoulli numbers (polynomials) would also hold on Cauchy numbers (polynomials).

2 Polylogarithm factorial function

Note that for $k \ge 2$

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z)\,,$$

so

$$\mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t} dt\,;$$

on the other hand,

$$\frac{d}{dz}(z\mathrm{Lif}_k(z)) = \mathrm{Lif}_{k-1}(z)\,,$$

so

$$\mathrm{Lif}_k(z) = \frac{1}{z}\int_0^z \mathrm{Lif}_{k-1}(t) dt\,. \tag{2}$$

A Riordan array is a pair $(d(t), h(t))$ where d and h are analytic functions and $d(0) \ne 0$ ([11, 12]). This pair then defines an infinite lower triangular array $\{d_{n,k}\}$, where

$$\sum_{n=0}^{\infty} d_{n,m} t^n = d(t)(t \cdot h(t))^m\,.$$

From this definition, $d(t)(t \cdot h(t))^m$ is the generating function of column k in the array. It is known that Pascal triangle $\{P_{n,m}\}_{n,k\geq 0}$ is represented by a Riordan array:

$$\frac{1}{1-t}\left(\frac{t}{1-t}\right)^m = \sum_{n=0}^{\infty}\binom{n}{m}t^n = \sum_{n=0}^{\infty}P_{n,m}t^n$$

(Unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$, which arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n}\left[{n \atop m}\right]x^m,$$

is represented by

$$1\cdot\left(\ln\frac{1}{1-t}\right)^m = \sum_{n=0}^{\infty}\frac{m!}{n!}\left[{n \atop m}\right]t^n$$

Stirling numbers of the second kind $\left\{{n \atop m}\right\}$, which are determined by

$$\left\{{n \atop m}\right\} = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n,$$

is represented by

$$1\cdot(e^t-1)^m = \sum_{n=0}^{\infty}\frac{m!}{n!}\left\{{n \atop m}\right\}t^n$$

Notice that $\mathrm{Li}_1(z) = -\ln(1-z)$ and $\mathrm{Lif}_1(z) = (e^z-1)/z$.

3 Poly-Bernoulli numbers and poly-Cauchy numbers

The generating function of the poly-Bernoulli numbers can be written in terms of iterated integrals:

$$e^x\cdot\underbrace{\frac{1}{e^x-1}\int_0^x\frac{1}{e^x-1}\int_0^x\cdots\frac{1}{e^x-1}\int_0^x}_{k-1}\frac{x}{e^x-1}\underbrace{dxdx\dots dx}_{k-1} = \sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!}.$$

The generating function of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ $(k \ge 2)$ are also written in terms of iterated integrals:

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)\ln(1+x)} \underbrace{dxdx\dots dx}_{k-1}$$

$$= \sum_{n=0}^\infty c_n^{(k)} \frac{x^n}{n!}.$$

It is known that the identity

$$\sum_{m=0}^n (-1)^m \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} B_m^{(k)} = \frac{n!}{(n+1)^k}$$

holds. On the other hand,

$$\sum_{m=0}^n \left\{ n \atop m \right\} c_m^{(k)} = \frac{1}{(n+1)^k}.$$

It is known that the duality theorem holds for poly-Bernoulli numbers. Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad \text{for } n, k \ge 0.$$

It is due to the symmetric formula:

$$\sum_{n=0}^\infty \sum_{k=0}^\infty B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

$$\sum_{n=0}^\infty \sum_{k=0}^\infty c_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = e^y (1+x)^{e^y}.$$

4 Poly-Cauchy polynomials

We introduce the *poly-Cauchy polynomials* (of the first kind) $c_n^{(k)}(z)$ for a positive integer k and a non-negative integer n, given by the generating function

$$\frac{\mathrm{Lif}_k(\ln(1+x))}{(1+x)^z}=\sum_{n=0}^\infty c_n^{(k)}(z)\frac{x^n}{n!}\,, \tag{3}$$

where

$$\mathrm{Lif}_k(z)=\sum_{m=0}^\infty\frac{z^m}{m!(m+1)^k}\,.$$

When $z=0$, $c_n^{(k)}(0)=c_n^{(k)}$ are the poly-Cauchy numbers. We may also define the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ by

$$c_n^{(k)}(z)=n!\underbrace{\int_0^1\dots\int_0^1}_{k}\binom{x_1x_2\dots x_k-z}{n}dx_1dx_2\dots dx_k\,.$$

The first several polynomials are

$$\begin{aligned}
c_0^{(k)}(z)&=1,\\
c_1^{(k)}(z)&=\frac{1}{2^k}-z,\\
c_2^{(k)}(z)&=-\frac{1}{2^k}+\frac{1}{3^k}+\left(1-\frac{2}{2^k}\right)z+z^2,\\
c_3^{(k)}(z)&=\frac{2}{2^k}-\frac{3}{3^k}+\frac{1}{4^k}+\left(-2+\frac{6}{2^k}-\frac{3}{3^k}\right)z\\
&\quad+\left(-3+\frac{3}{2^k}\right)z^2-z^3,\\
c_4^{(k)}(z)&=-\frac{6}{2^k}+\frac{11}{3^k}-\frac{6}{4^k}+\frac{1}{5^k}+\left(6-\frac{22}{2^k}+\frac{18}{3^k}-\frac{4}{4^k}\right)z\\
&\quad+\left(11-\frac{18}{2^k}+\frac{6}{3^k}\right)z^2+\left(6-\frac{4}{2^k}\right)z^3+z^4.
\end{aligned}$$

Poly-Bernoulli polynomials $B_n^{(k)}(z)$ were defined as

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}e^{xz}=\sum_{n=0}^\infty B_n^{(k)}(z)\frac{x^n}{n!}\,.$$

([1]). Note that $B_n^{(k)}(z)$ are defined in [5] by replacing e^{xz} by e^{-xz}. In [10], $B_n^{(k)}(z)$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{e^x-1}e^{zx}=\sum_{n=0}^{\infty}B_n^{(k)}(z)\frac{x^n}{n!}$$

Concerning the poly-Bernoulli polynomials, for an integer k and a positive integer n we have

$$\frac{d}{dz}B_n^{(k)}(z)=nB_{n-1}^{(k)}(z)$$

([1, Theorem 1.4]). The poly-Cauchy polynomials $c_n^{(k)}$, however, are not Appell sequences. By differentiating $c_n^{(k)}$, we have

$$\frac{d}{dz}c_n^{(k)}(z)=(-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)l!}c_l^{(k)}(z)\quad(n\ge 1)\,.$$

We have a recurrence formula for the poly-Cauchy polynomials $c_n^{(k)}(z)$ in terms of the poly-Cauchy numbers $c_n^{(k)}$ and the Cauchy polynomials $c_n(z)$.

Theorem 2. *For a positive integer k and a non-negative integer n we have*

$$c_n^{(k)}=(-1)^n n!\sum_{m=0}^{n}\frac{(-1)^m c_m^{(k-1)}}{m!}\sum_{l=0}^{n-m}\frac{(-1)^l c_l(z)}{(n-l+1)l!}\,.$$

Poly-Cauchy polynomials of the first kind can be also expressed explicitly in terms of the Stirling number of the first kind:

$$c_n^{(k)}(z)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{(-z)^i}{(m-i+1)^k}\,. \tag{4}$$

5 Poly-Cauchy numbers and polynomials of the second kind

The *poly-Cauchy polynomials of the second kind* $\hat{c}_n^{(k)}(z)$ are defined by

$$(1+x)^z\mathrm{Lif}_k(-\ln(1+x))=\sum_{n=0}^{\infty}\hat{c}_n^{(k)}(z)\frac{x^n}{n!}\,.$$

The first several polynomials are

$$\hat{c}_0^{(k)}(z) = 1,$$
$$\hat{c}_1^{(k)}(z) = -\frac{1}{2^k} + z,$$
$$\hat{c}_2^{(k)}(z) = \frac{1}{2^k} + \frac{1}{3^k} - \left(1 + \frac{2}{2^k}\right) z + z^2,$$
$$\hat{c}_3^{(k)}(z) = -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k} + \left(2 + \frac{6}{2^k} + \frac{3}{3^k}\right) z - \left(3 + \frac{3}{2^k}\right) z^2 + z^3,$$
$$\hat{c}_4^{(k)}(z) = \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} - \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right) z$$
$$+ \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 - \left(6 + \frac{4}{2^k}\right) z^3 + z^4.$$

If $z = 0$, then $\hat{c}_n^{(k)}(0) = \hat{c}_n^{(k)}$ are the *poly-Cauchy numbers of the second kind.* If $k = 1$, then $\hat{c}_n^{(1)}(z) = \hat{c}_n(z)$ are the Cauchy polynomials given in [3]. The generating function of $c_n(z)$ is given by

$$(1+x)^z \mathrm{Lif}_1(-\ln(1+x)) = \frac{x(1+x)^z}{(1+x)\ln(1+x)}$$
$$= \sum_{n=0} \hat{c}_n(z)\frac{x^n}{n!}.$$

Note that x is replaced by $-x$ in the generating function in [3]. Under these definitions we call $c_n^{(k)}$ and $c_n^{(k)}(z)$ poly-Cauchy numbers of the first kind and poly-Cauchy polynomials of the first kind, respectively. In similar methods, we have the corresponding results to those in the previous sections.

Proposition 1.

$$\sum_{n=0}^\infty \sum_{k=0}^\infty \hat{c}_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^y}{(1+x)^{e^y}}.$$

Theorem 3. *For a positive integer k and a non-negative integer n we have*

$$\hat{c}_n^{(k)}(z) = (-1)^n n! \sum_{m=0}^n \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m} \frac{(-1)^l \hat{c}_l(z)}{(n-l+1)l!}$$
$$+ (-1)^n n! \sum_{m=0}^{n-1} \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m-1} \frac{(-1)^{l+1} \hat{c}_l(z)}{(n-l)l!}.$$

Theorem 4.

$$\frac{d}{dz}\hat{c}_n^{(k)}(z) = (-1)^{n-1} n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)l!}\hat{c}_l^{(k)}(z) \quad (n \ge 1)\,.$$

6 Some generalizations of poly-Cauchy numbers and polynomials

The generating function of ordinary generalized poly-Bernoulli numbers $B_{n,\chi}^{(k)}$ ([10]) is given by

$$\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-fx})}{e^{fx}-1}e^{ax} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)}\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f}\,.$$

The generating function of generalized poly-Bernoulli polynomials $B_{n,\chi}^{(k)}(z)$ ([2]) is given by

$$\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-fx})}{e^{fx}-1}e^{(z+a)x} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)}(z)\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f}\,.$$

The generating function of poly-Bernoulli polynomials with a, b parameters $B_n^{(k)}(z;a,b)$ ([6]) is given by

$$\frac{\mathrm{Li}_k(1-(ab)^{-x})}{b^x - a^{-x}}e^{zx} = \sum_{n=0}^{\infty} B_n^{(k)}(z;a,b)\frac{x^n}{n!}\,.$$

The generating function of poly-Bernoulli polynomials with a, b, c parameters $B_n^{(k)}(z;a,b,c)$ ([6]) is given by

$$\frac{\mathrm{Li}_k(1-(ab)^{-x})}{b^x - a^{-x}}c^{zx} = \sum_{n=0}^{\infty} B_n^{(k)}(z;a,b,c)\frac{x^n}{n!}\,.$$

Mari Yokohama (Hirosaki University) proposes the following generalizations of Cauchy numbers. Let n and k be integers with $n \ge 0$ and $k \ge 1$. Let q be a real number with $q \ne 0$. Define the *poly-Cauchy numbers with q* *parameter* (of the first kind) $c_{n,q}^{(k)}$ by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - q) \dots (x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k .$$

Then for a real number $q \neq 0$

$$c_{n,q}^{(k)} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \geq 0,\ k \geq 1) .$$

The generating function of $c_{n,q}^{(k)}$ is given by

$$\mathrm{Lif}_k \left(\frac{\ln(1+qx)}{q} \right) = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!} \quad (q \neq 0) .$$

The generating function can be also written in the form of iterated integrals as that of the poly-Cauchy numbers. For $k \geq 2$ we have

$$\underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx)\ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)} \int_0^x}_{k-1} \frac{q((1+qx)^{1/q} - 1)}{(1+qx)\ln(1+qx)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!} .$$

For $k = 1$ we have

$$\frac{q((1+qx)^{1/q} - 1)}{\ln(1+qx)} = \sum_{n=0}^{\infty} c_{n,q} \frac{x^n}{n!} .$$

Similarly, define the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ by

$$\hat{c}_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - q) \dots (-x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k .$$

Then

$$\hat{c}_{n,q}^{(k)}=(-1)^n\sum_{m=0}^n\begin{bmatrix}n\\m\end{bmatrix}\frac{q^{n-m}}{(m+1)^k}\,.$$

The generating function of the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ is given by

$$\operatorname{Lif}_k\left(-\frac{\ln(1+qx)}{q}\right)=\sum_{n=0}^\infty\hat{c}_{n,q}^{(k)}\frac{x^n}{n!}\,.$$

For $k\ge 2$ we have

$$\underbrace{\frac{q}{\ln(1+qx)}\int_0^x\frac{q}{(1+qx)\ln(1+qx)}\int_0^x\cdots\frac{q}{(1+qx)\ln(1+qx)}\int_0^x}_{k-1}\frac{q(1-(1+qx)^{-1/q})}{(1+qx)\ln(1+qx)}\underbrace{dxdx\dots dx}_{k-1}$$

$$=\sum_{n=0}^\infty\hat{c}_{n,q}^{(k)}\frac{x^n}{n!}\,.$$

For $k=1$ we have

$$\frac{q(1-(1+qx)^{-1/q})}{\ln(1+qx)}=\sum_{n=0}^\infty\hat{c}_{n,q}\frac{x^n}{n!}\,.$$

Poly-Cauchy polynomials with q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ are also similarly defined.

Even more generalizations are possible. For example, define $c_{n,q}^{(k)}(l_1,l_2,\dots,l_k)$, where $l_1,l_2,\dots,l_k$ are nonzero real numbers, by

$$c_{n,q}^{(k)}(l_1,l_2,\dots,l_k)=\int_0^{l_1}\int_0^{l_2}\cdots\int_0^{l_k}(x_1x_2\dots x_k)(x_1x_2\dots x_k-q)\\ \dots(x_1x_2\dots x_k-(n-1)q)dx_1dx_2\dots dx_k\,.$$

Then, for a real number $q\ne 0$

$$c_{n,q}^{(k)}(l_1,l_2,\dots,l_k)=\sum_{m=0}^n\begin{bmatrix}n\\m\end{bmatrix}\frac{(-q)^{n-m}(l_1l_2\dots l_k)^{m+1}}{(m+1)^k}\quad(n\ge 0,\ k\ge 1)\,.$$

References

[1] A. Bayad and Y. Hamahata, *Polylogarithms and poly-Bernoulli polynomials*, Kyushu J. Math. **65** (2011), 15–24.

[2] A. Bayad and Y. Hamahata, *Arakawa-Kaneko L-functions and generalized poly-Bernoulli polynomials*, J. Number Theory **131** (2011), 1020–1036.

[3] G.-S. Cheon, S.-G. Hwang and S.-G. Lee, *Several polynomials associated with the harmonic numbers*, Discrete Appl. Math. **155** (2007), 2573–2584.

[4] L. Comtet, *Advanced Combinatorics*, Reidel, Doredecht, 1974.

[5] M.-A. Coppo and B. Candelpergher, *The Arakawa-Kaneko zeta functions*, Ramanujan J. **22** (2010), 153–162.

[6] H. Jolany, *Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a, b, c parameters*, (preprint).

[7] M. Kaneko, *Poly-Bernoulli numbers*, J. Th. Nombres Bordeaux **9** (1997), 199–206.

[8] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, Reading, 1994.

[9] D. Merlini, R. Sprugnoli and M. C. Verri, *The Cauchy numbers*, Discrete Math. **306** (2006) 1906–1920.

[10] Y. Ohno and N. Wakabayashi, *On basic properties of poly-Bernoulli polynomials*, (preprint).

[11] R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. **132** (1994) 267–290.

[12] R. Sprugnoli, *Riordan arrays and the Abel-Gould identity*, Discrete Math. **142** (1995) 213–233.