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Hoffman's basis conjecture and Two-one formula [1]

Yasuo Ohno (Department of Mathematics, Kinki University)

Francis Brown showed that Hoffman's conjectural basis generates the vector space spanned by all multiple zeta values. The remaining problem on Hoffman's basis conjecture is to show independence of the basis elements, and it seems to be completely out of reach nowadays. For example, irrationality of $\frac{\zeta(3,3)}{\zeta(2,2,2)}$ is the first step of this problem, it is equivalent to that of $\frac{\zeta(3)^2}{\zeta(6)}$, and is not clear until now.

Two-one formula is a conjectural formula for multiple zeta star-values. For some restricted cases, their proofs have been obtained, but not yet shown in general.

In this note, we shall review Hoffman's basis conjecture and Two-one formula.

1 Dimension and Direct sum conjectures

We call the n-tuple $\mathbf{k}=(k_1,k_2,\ldots,k_n)$ of integers $k_1\geq 2,\ k_j\in\mathbb{N}\ \ (j=1,2,\ldots,n)$ admissible index for multiple zeta values. The multiple zeta values (MZVs) and multiple zeta-star values (MZSVs) are defined, for admissible index $\mathbf{k}$, by

$$\zeta(\mathbf{k})=\zeta(k_1,k_2,\ldots,k_n)=\sum_{m_1>m_2>\cdots>m_n>0}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}}$$

and

$$\zeta^\star(\mathbf{k})=\zeta^\star(k_1,k_2,\ldots,k_n)=\sum_{m_1\geq m_2\geq\cdots\geq m_n>0}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$

respectively. The integers $\mathrm{wt}(\mathbf{k})=k_1+k_2+\cdots+k_n$, $\mathrm{dep}(\mathbf{k})=n$ and $\mathrm{ht}(\mathbf{k})=\#\{j\mid k_j\geq 2\}$ are called the weight, the depth and the height of $\mathbf{k}$, respectively. A multiple zeta-star value is a linear combination of multiple

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zeta values of the same weight, and vise versa. Thus MZSVs span the same $\mathbb{Q}$-vector space as MZVs.

We define a series of vector spaces $\{\mathcal{Z}_k\}$ by $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = \{0\}$ and $\mathcal{Z}_k = \sum_{\mathbf{k}\in I_0(k)} \mathbb{Q}\zeta(\mathbf{k})$ for $k \geq 2$, and $\mathcal{Z} = \sum_{k\geq 0} \mathcal{Z}_k$ is the vector space spanned by all multiple zeta values. Here, we meant by $I_0(k)$ the set of all admissible indices with weight k. It is known that $\mathcal{Z}_k \cdot \mathcal{Z}_{k'} \subset \mathcal{Z}_{k+k'}$ are satisfied.

We define an integer sequence $\{d_k\}$ by $d_0 = 1, d_1 = 0, d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$.

Two famous conjectures on multiple zeta values are as follows.

Conjecture 1 (Dimension Conjecture). *For any non-negative integer k, we conjecture*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

Conjecture 2 (Direct Sum Conjecture). *For the vector space spanned by MZVs, we conjecture*

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k.$$

The table of each values, based on numerical experimentation, is as follows.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\#I_0(k)$			1	2	4	8	16	32	64	128	256	512	1024
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12

The following result is known for Dimension conjecture.

Theorem 1 (Terasoma [9], Deligne-Goncharov [3]). *For any non-negative integer k, we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k.$$

2 {2,3}-Basis

In 1997, Hoffman [4] presented a conjectural basis for $\mathbb{Q}$-vector space of MZVs. The base is called $\{2,3\}$-basis because its basis elements are all multiple zeta values whose index is constructed by 2's and 3's. To be precise, Hoffman's basis conjecture can be stated as follows. First, we define sets of indices H and H_k by

$$\begin{aligned} H &= \{\mathbf{k} = (k_1, \dots, k_n) \mid n \in \mathbb{N}, k_i \in \{2,3\}, (i = 1, 2, \dots, n)\}, \\ H_k &= \{\mathbf{k} \in H \mid \mathrm{wt}(\mathbf{k}) = k\}. \end{aligned}$$

Conjecture 3 ({2,3}-Basis Conjecture [4]**).** *The set* $\{\zeta(\mathbf{k}) \mid \mathbf{k} \in H\} \cup \{1\}$ *is a basis of* $\mathcal{Z}$ *over* $\mathbb{Q}$.

Hoffman's {2,3}-basis conjecture is based on the following fact and numerical experimentation for some extent.
Fact. For any $k \in \mathbb{N}$,

$$\# H_k = d_k$$

is satisfied.

As we mentioned in the introduction of present note, Francis Brown [2] showed

$$\mathcal{Z} = \sum_{\mathbf{k} \in H} \mathbb{Q}\zeta(\mathbf{k}),$$

by using an argument on motivic zeta values and Don Zagier's result [11] on $\zeta(2, \dots, 2, 3, 2, \dots, 2)$.

If we shift, without much thought, the above conjectural basis to multiple zeta-star values, we find another conjecture as follows.

Conjecture 4 ({2,3}*-Basis Conjecture [5]**).** *The set* $\{\zeta^\star(\mathbf{k}) \mid \mathbf{k} \in H\} \cup \{1\}$ *is a basis of* $\mathcal{Z}$ *over* $\mathbb{Q}$.

We can get supporting evidence of {2,3}*-basis conjecture up to weight 15 by numerical experimentation. Also the above mentioned "Fact" supports this conjecture again.

For {2,3}*-basis conjecture, we obtain the following result. Here $\{2\}_k$ denotes the k-tuple $\underbrace{2, 2, \dots, 2}_{k}$.

Theorem 2 (Ihara-Kajikawa-O.-Okuda [5]**).** *For any integer* $k > 1$, *we have*

$$\zeta(k) \in \sum_{\mathbf{k} \in H_k} \mathbb{Q}\zeta^\star(\mathbf{k}).$$

Moreover, $\zeta(k)$ has the following expressions by $\{2,3\}^\star$-basis.

$$\zeta(2k) = \{2(1-2^{1-2k})\}^{-1}\zeta^\star(\{2\}_k),$$
$$\zeta(2k+1) = \{4k(1-2^{-2k})^{-1}\left(2\sum_{i=1}^{k-1}\zeta^\star(\{2\}_{i-1},3,\{2\}_{k-i}) + 3\zeta^\star(\{2\}_{k-1},3)\right).$$

Remark. The above formula for $\zeta(2k)$ is firstly obtained by Sergey Zlobin [12], and is a restricted case of the results in [1] and [6]. On the other hand, the formula for $\zeta(2k+1)$ is newly obtained in [5]. After releasing our draft of [5], Don Zagier [11] gave another proof of the formula for $\zeta(2k+1)$ of Theorem 2 by giving a linear relation between $\zeta^\star(2,\dots,2,3,2,\dots,2)$'s and $\zeta(2,\dots,2,3,2,\dots,2)$'s (that is to say, zeta and zeta-star values with many 2's and only one 3) and applying his new formula of relations between Riemann zeta values and $\zeta(2,\dots,2,3,2,\dots,2)$'s.

3 Two-one formula

In 2008, we presented in [8] a conjectural formula "Two-one formula" and its proof for some specific cases. The conjectural formula is not yet proved in general, but we believe that it might give some further information for understanding the precise structure of multiple zeta algebra $\mathcal{Z}$.

As we mentioned in the previous section, the formula

$$\zeta^\star(\{2\}_n) = 2(1-2^{1-2n})\zeta(2n)$$

is due to Sergey Zlobin [12], and the other formula

$$\zeta^\star(\{2\}_n,1) = 2\zeta(2n+1)$$

is also appeared in his paper. The second identity is also obtained as a special case of Cyclic sum formula for MZSVs given in [7]. The following identity was discovered experimentally when we searched for a generalization of the above formula (which is the particular case $l = 1$ of our finding).

Conjecture 5 (Two-one formula [8]). *For $k = 0,1,2,\dots$, μ_{2k+1} denotes $(\{2\}_k,1)$. Then for any admissible index $(s_1,s_2,\dots,s_l)$ with odd entries $s_1,s_2,\dots,s_l$, we conjecture*

$$\zeta^\star(\mu_{s_1},\mu_{s_2},\dots,\mu_{s_l}) = \sum_{\mathbf{p}} 2^{l-\sigma(\mathbf{p})}\zeta(\mathbf{p}),$$

where $\mathbf{p}$ *runs through all indices of the form* $(s_1 \circ s_2 \circ \cdots \circ s_l)$ *with "*$\circ$*" being either the symbol "*,*" or the sign "*+*", and the exponent* $\sigma(\mathbf{p})$ *denotes the number of signs "*+*" in* $\mathbf{p}$.

The right hand side of the equality in Conjecture 5 can also be written as

$$\sum_{\mathbf{p}}(-1)^{\sigma(\mathbf{p})}2^{l-\sigma(\mathbf{p})}\zeta^{\star}(\mathbf{p}).$$

In spite of a simple, but somehow mysterious, shape of Two-one formula, we cannot yet prove it in the full generality. The following two particular cases as well as our experimental results support the validity of the formula.

Theorem 3 (O.-Zudilin [8]). *For any integers* $n \geq i \geq 1$, *we have*

$$\zeta^{\star}(\{2\}_i, 1, \{2\}_{n-i}, 1) = 4\zeta^{\star}(2i+1, 2n-2i+1) - 2\zeta(2n+2).$$

Theorem 4 (O.-Zudilin [8]). *For any integer* $l \geq 1$, *we have*

$$\begin{aligned}\zeta^{\star}(2, \{1\}_l) &= \sum_{\circ =, \text{ or } +} 2^{l-\#\{\circ=+\}}\zeta(3 \circ \underbrace{1 \circ \cdots \circ 1}_{l-1}) \\ &= \sum_{i=0}^{l-1} 2^{l-i} \sum_{e_1+\cdots e_{l-i}=i} \zeta(3+e_1, 1+e_2, 1+e_3, \ldots, 1+e_{l-i}),\end{aligned}$$

where all e_j *are non-negative integers.*

A recent development on studying the algebraic structure (of the right hand side, especially) of Two-one formula given by Shuji Yamamoto can be found in [10].

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