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The joint universality theorem for automorphic L-functions

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1 Introduction

In 1910s, H. Bohr initiated the investigation of value distribution of the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\prod_{p}\left(1-\frac{1}{p^{s}}\right)^{-1}\quad \text{for } \sigma>1,$$

where $s=\sigma+it$ denotes a complex variable and the symbol p denotes a prime number as usual. Bohr and R. Courant [4] showed that for any fixed $1/2<\sigma_{0}<1$ the set

$$\{\zeta(\sigma_{0}+it)\in \mathbb{C}\mid t\in \mathbb{R}\}$$

is dense in the set $\mathbb{C}$ of all complex numbers. In 1975, S. M. Voronin [15] extended this denseness result to the infinite dimensional space, that is, the functional space and obtained the remarkable universality theorem. To state it in modern form which was established by B. Bagchi [1], we define a probability measure on $\mathbb{R}$. Let μ be the Lebesgue measure on the set $\mathbb{R}$ of all real numbers. For $T>0$ define

$$\nu_{T}(\cdots)=\frac{1}{T}\mu\{\tau\in[0,T]\ :\cdots\},$$

where in place of dots we write some conditions satisfied by a real number τ.

Theorem 1 (Voronin, [15]). *Let K be a compact subset in the strip $\frac{1}{2}<\sigma<1$ with connected complement and $h(s)$ be a non-vanishing and continuous function on K which is analytic in the interior of K. Then for any small positive number ε we have*

$$\liminf_{T\to\infty}\nu_{T}\left(\max_{s\in K}|\zeta(s+i\tau)-h(s)|<\varepsilon\right)>0.$$

This theorem asserts roughly that any analytic function can be approximated uniformly by suitable vertical translation of $\zeta(s)$. In order to prove the theorem, we need several analytic properties of the Riemann zeta function. Above all, the Euler product expression plays an essential role. In fact, for major zeta functions with Euler product the universality theorems have been established. The details will be described in §3 later.

After Theorem 1, Voronin [16], S. M. Gonek [6] and Bagchi [2] independently obtained the following joint universality theorem for Dirichlet L-functions

Theorem 2 (Voronin[16], Gonek[6], Bagchi[2]). *Let $\chi_{1},\cdots,\chi_{r}$ be pairwise non-equivalent Dirichlet characters. For each $1\leq j\leq r$, let K_{j} be a compact subset in $\frac{1}{2}<\sigma<1$ with connected complement* *and $h_j(s)$ be a non-vanishing and continuous function on K_j which is analytic in the interior of K_j. Then for any small positive number ε we have*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau\in[0,T] \,\middle|\, \max_{1\le j\le r}\max_{s\in K_j}|L(s+i\tau,\chi_j)-h_j(s)|<\varepsilon\right\}>0.$$

The above inequality implies that for a collection of Dirichlet L-functions the corresponding universalily properties hold simultaneously. Therefore the joint universality theorem is interpretted as the statistical independence of value distribution of Dirichlet L-functions. In the proof of this theorem, the periodicity of Dirichlet characters

$$\chi_i(n_1)=\chi_i(n_2) \quad \text{if} \quad n_1\equiv n_2 \pmod{Q},$$

where Q is the least common multiple of modulus q_i's, and the orthogonality of the characters

$$\frac{1}{\varphi(Q)}\sum_{n=1}^{Q}\chi_i(n)\overline{\chi_j(n)}=\begin{cases}1 & (i=j),\\ 0 & (i\neq j),\end{cases}$$

play essential roles. Similar properties also hold for a set of $\mathbb{C}$-linearly independent characters of $Gal(K/\mathbb{Q})$, where $K/\mathbb{Q}$ is an arbitrary finite Galois extension. H. Bauer [3] paid attention to this fact proved a joint universality theorem for a set of Artin L-functions associated with these charachters. In 2004, A. Laurinčikas and K. Matsumoto [7] obtained a joint universality theorem for automorphic L-functions which are associated with a single holomorphic newform and twisted by non-equivalent Dirichlet characters.

In this paper we give a new method to prove joint universality theorems without the need for the periodicity of coefficients. In particular, we will prove a joint universality theorem for pairs consisting of the Riemann zeta-function and the following two types of automorphic L-functions.

For an even positive integer k, let $\mathcal{F}_k$ denote the set of holomorphic Hecke eigen cusp forms of weight k for the full modular group $SL_2(\mathbb{Z})$. Put $\mathcal{F}=\cup_k\mathcal{F}_k$. For $f\in\mathcal{F}_k$ and $n\in\mathbb{N}$, let $\hat{\lambda}_f(n)$ be the n-th Fourier coefficient of f and put $\lambda_f(n)=\hat{\lambda}_f(n)n^{-\frac{k-1}{2}}$. For each prime p the coefficient $\lambda_f(p)$ is a real number satisfying Deligne's estimate $|\lambda_f(p)|\le 2$. Therefore there exist complex numbers $\alpha_{f,1}(p)$, $\alpha_{f,2}(p)$ such that

$$\alpha_{f,1}(p)+\alpha_{f,2}(p)=\lambda_f(p), \quad \text{and} \quad |\alpha_{f,1}(p)|=|\alpha_{f,2}(p)|=1. \tag{1}$$

Then the automorphic L-function $L(s,f)$ is given by

$$L(s,f)=\prod_p\prod_{i=1,2}\left(1-\frac{\alpha_{f,i}(p)}{p^s}\right)^{-1}=\prod_p\left(1-\frac{\lambda_f(p)}{p^s}+\frac{1}{p^{2s}}\right)^{-1}$$

for $\sigma>1$. The universality theorem for $L(s,f)$ was obtained by Laurinčikas and Matsumoto [7]. As we stated above, Laurinčikas and Matsumoto [8] also established the joint universality theorem for a set of twisted automorphic L-functions

$$L(s,f,\chi_j)=\prod_p\prod_{i=1,2}\left(1-\frac{\alpha_{f,i}(p)\chi_j(p)}{p^s}\right)^{-1} \quad (1\le j\le r),$$

where f is a fixed holomorphic newform and $\chi_j (1\le j\le r)$ are non-equivalent Dirichlet characters.

For cusp forms $f, g \in \mathcal{F}$, the Rankin-Selberg L-function $L(s, f \otimes g)$ is defined by

$$L(s, f \otimes g) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{f,i}(p)\alpha_{g,j}(p)}{p^s}\right)^{-1} \quad \text{for} \quad \sigma > 1,$$

where numbers $\alpha_{f,i}(p)$, $\alpha_{g,j}(p)$ are given by (1). The universality property for $L(s, f \otimes g)$ holds in the narrow strip $\frac{3}{4} < \sigma < 1$, which was shown by Matsumoto [9] when $f = g$, and by Nagoshi [11] when $f \neq g$.

Now we state our main results. In the following, denote by D_1 the strip $\{s \in \mathbb{C} \mid 1/2 < \sigma < 1\}$ and by D_2 the strip $\{s \in \mathbb{C} \mid 3/4 < \sigma < 1\}$.

Theorem 3. *The joint universality theorem holds for the following pairs of zeta functions:*

(i) $\zeta(s)$ *and* $L(s, f)$*,*

(ii) $L(s, f)$ *and* $L(s, g)$ $\quad (f \neq g)$*,*

(iii) $\zeta(s)$ *and* $L(s, f \otimes g)$*,*

(iv) $L(s, f)$ *and* $L(s, f \otimes g)$*.*

The joint universality for the pairs (i) and (ii) hold in the strip D_1*. The joint universality for the pairs (iii) and (iv) hold in the strip* D_2*.*

2 Outline of the proof of Theorem 3

In this section, we sketch the proof of the joint universality theorem for $\zeta(s)$ and $L(s, f)$.

Let D_1 be the same strip as in §1. Let $H(D_1)$ be the space of analytic functions on D_1 equipped with the topology of uniform convergence on compacta. Put $H(D_1)^2 = H(D_1) \times H(D_1)$. For a topological space X, let $\mathcal{B}(X)$ be the class of Borel subsets of X. For $T > 0$ define a probability measure P_T on the probability space $(H(D_1)^2, \mathcal{B}(H(D_1)^2))$ by

$$P_T(A) = \nu_T\left((\zeta(s + i\tau), L(s + i\tau, f)) \in A\right),$$

for $A \in \mathcal{B}(H(D_1)^2)$. ¿From Theorem 12.1 in [14], which is the joint limit theorem for a set of zeta functions with polynomial Euler products, we have the following limit theorem.

Lemma 1. *There exists the probability measure* P *on the space* $(H(D_1)^2, \mathcal{B}(H(D_1)^2))$ *such that the measure* P_T *converges weakly to* P *as* $T \to \infty$.

The limit measure P is given as follows. Let γ be the unit circle $\{s \in \mathbb{C} \mid |s| = 1\}$ and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. With the product topology and pointwise multiplication Ω is a compact Abelian group. Let m_H be the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Let $\omega = \{\omega(p)\} \in \Omega$. Put $\omega(1) = 1$ and

$$\omega(n) = \prod_{p^\alpha \| n} \omega(p)^\alpha \in \gamma$$

for a positive integer n. For $\omega \in \Omega$ and $s \in D_1$, define

$$\zeta(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1},$$

and

$$L(s,f,\omega) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{f,i}(p)\omega(p)}{p^s}\right)^{-1}.$$

For almost $\omega \in \Omega$, these infinite products converge uniformly on compact subsets of D_1. Therefore the products are considered as $H(D_1)$-valued random elements. The limit measure P is the distribution of a pair of these random elements. Namely,

$$P(A) = m_H\left(\{\omega \in \Omega \mid (\zeta(s,\omega), L(s,f,\omega)) \in A\}\right),$$

for $A \in \mathcal{B}(H(D_1)^2)$.

For $\sigma > \frac{1}{2}$ and $\omega \in \Omega$ we define functions g_p and h_p by

$$\log\left(1 - \frac{\omega(p)}{p^s}\right)^{-1} = \frac{\omega(p)}{p^s} + g_p(s)$$

and

$$\log \prod_{i=1}^{2} \left(1 - \frac{\alpha_{f,i}(p)\omega(p)}{p^s}\right)^{-1} = \frac{\lambda_f(p)\omega(p)}{p^s} + h_p(s).$$

Then for all $s \in D_1$ and almost all $\omega \in \Omega$

$$(\log\zeta(s,\omega), \log L(s,f,\omega)) = \sum_p \left(\frac{\omega(p)}{p^s}, \frac{\lambda_f(p)\omega(p)}{p^s}\right) + \sum_p (g_p(s), h_p(s)),$$

where the sum is taken over all prime numbers. Remark that the series $\sum_p (g_p(s), h_p(s))$ converges uniformly for $\omega \in \Omega$ and on any compact subset of D_1. For each prime p we set

$$f_p(s) = \left(\frac{1}{p^s}, \frac{\lambda_f(p)}{p^s}\right) \in H(D_1)^2.$$

Lemma 2 (Joint denseness lemma). *The set of convergent series*

$$\left\{\sum_p \omega(p) f_p(s) \in H^2(D_1) \;\middle|\; \omega \in \Omega\right\}$$

is dense in $H(D_1)^2$.

This lemma implies that the set $\{(\zeta(s,\omega), L(s,f,\omega)) \in H(D_1)^2 \mid \omega \in \Omega\}$ is also dense in the space $H(D_1)^2$. ¿From Lemma 1 and Lemma 2, the joint universality follows immediately.

Proof of Lemma 2. Let U be a bounded simply connected region in D_1. Let $\mathcal{H}$ be the Hardy space on U, which is the set of analytic and second integrable functions on U. Let $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. The space $\mathcal{H}$ becomes a complex Hilbert space with the inner product

$$\langle g_1, g_2 \rangle = \iint_U g_1(s)\overline{g_2(s)}\, d\sigma dt.$$

We will prove that the set $\{\sum_p a_p f_p(s) \in \mathcal{H}^2 \mid |a_p| = 1\}$ is dense in $\mathcal{H}^2$ by using the following general denseness lemma, which was essentially obtained by Voronin [15].

Lemma 3. *Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Suppose that a sequence $\{u_n\} \subset H$ satisfies*

(i) $\sum_n \|u_n\|^2 < \infty$,

(ii) for any non-zero element $u \in H$

$$\sum_n |\langle u_n, u \rangle| = \infty.$$

Then for any $m > 0$ the set

$$\left\{ \sum_{n \geq m} a_n u_n \in H \;\middle|\; |a_n| = 1 \right\}$$

is dense in H.

We return to the proof of Lemma 2. Let $\sigma_0 = \min\{\Re s \mid s \in \overline{U}\} > \frac{1}{2}$. Then

$$\sum_p \|f_p(s)\|^2 = \sum_p \iint_U \frac{1 + |\lambda_f(p)|^2}{p^{2\sigma}} d\sigma dt \ll_U \sum_p \frac{1}{p^{2\sigma_0}} < \infty.$$

Therefore the sequence $\{f_p(s)\}$ satisfies condition (i) in Lemma 3. For $g(s) = (g_1(s), g_2(s)) \in \mathcal{H}^2$ we have

$$\begin{aligned} \langle f_p(s), g(s) \rangle &= \iint_U \frac{1}{p^s} \overline{g_1(s)} d\sigma dt + \iint_U \frac{\lambda_f(p)}{p^s} \overline{g_2(s)} d\sigma dt \\ &= \Delta_1(\log p) + \lambda_f(p) \Delta_2(\log p), \end{aligned}$$

where we set

$$\Delta_j(z) = \iint_U e^{-sz} \overline{g_j(s)} d\sigma dt$$

for $z \in \mathbb{C}$ and $j = 1, 2$. It is enough to prove the following lemma.

Lemma 4. *Let $g(s) = (g_1(s), g_2(s))$ be a non-zero element of $\mathcal{H}^2$. Then*

$$\sum_p |\Delta_1(\log p) + \lambda_f(p) \Delta_2(\log p)| = \infty. \tag{2}$$

To prove Lemma 4, we need the following lemmas, which play key roles in our new method.

Lemma 5. *Let z_1 and z_2 be complex numbers.*

1. *If $\Re z_1$ and $\Re z_2$ have the same sign, then*

$$|z_1 + z_2| \geq |\Re z_1|.$$

2. *If $\Im z_1$ and $\Im z_2$ have the same sign, then*

$$|z_1 + z_2| \geq |\Im z_1|.$$

Lemma 6. *Assume that g_1 and g_2 are non-zero element in $\mathcal{H}$. Then there exists a sequence of intervals $I_n = [x_n, x_n + y_n]$ such that*

(I) $x_n \to \infty$ $(n \to \infty)$ and $y_n \sim x_n^{-38}$.

(II) For each $n \in \mathbb{N}$,

$$|\Re\Delta_1(x)| \geq \frac{1}{4}e^{-\sigma_2 x_n}, \quad or, \quad |\Im\Delta_1(x)| \geq \frac{1}{4}e^{-\sigma_2 x_n},$$

holds for $x \in I_n$, where $\sigma_2 = \max\{\Re s \mid s \in U\} < 1$.

(III) For each $n \in \mathbb{N}$, the functions $\Re\Delta_2(x)$ and $\Im\Delta_2(x)$ have no zeros on the interval I_n.

Proof Assertions (I) and (II) were obtained by Voronin [15] essentially. Assertion (III) was established by the author recently. □

Now we prove Lemma 4. The divergence of series (2) was established by Voronin [15] when $g_2 = 0$ and by Laurinčikas and Matsumoto [7] when $g_1 = 0$, respectively. Therefore we may assume that g_1 and g_2 are non-zero elements. For each $n \in \mathbb{N}$, define a set $\mathbb{P}_n$ of prime numbers. Let $\{I_n\}$ be a sequence of intervals as in Lemma 6. If n is an integer for which

$$\Re\Delta_1(x) > \frac{1}{4}e^{-\sigma_2 x} > 0 \quad (x \in I_n), \tag{3}$$

holds, define

$$\mathbb{P}_n = \begin{cases} \{p \mid \log p \in I_n,\ \lambda_f(p) \geq 0\} & (\text{if } \Re\Delta_2 > 0 \text{ on } I_n), \\ \{p \mid \log p \in I_n,\ \lambda_f(p) < 0\} & (\text{if } \Re\Delta_2 < 0 \text{ on } I_n). \end{cases}$$

Then from Lemma 5 and Lemma 6, we have

$$\sum_{p\in\mathbb{P}_n} |\Delta_1(\log p) + \lambda_f(p)\Delta_2(\log p)| \geq \sum_{p\in\mathbb{P}_n} |\Re\Delta_1(\log p)| \gg e^{-\sigma_2 x_n} \cdot \sharp\mathbb{P}_n.$$

Applying Deligne's estimate $|\lambda_f(p)| \leq 2$ and the following estimates

$$\sum_{p\leq x} \lambda_f(p) = O(x\exp(-c\sqrt{\log x})),$$

and

$$\sum_{p\leq x} |\lambda_f(p)|^2 = \mathrm{li(x)} + \mathrm{O(x\exp(-c\sqrt{\log x}))},$$

we obtain

$$\sharp\mathbb{P}_n \gg \frac{e^{x_n}}{x_n^{39}}.$$

Hence we have

$$\sum_{p\in\mathbb{P}_n} |\Delta_1(\log p) + \lambda_f(p)\Delta_2(\log p)| \gg \frac{e^{(1-\sigma_2)x_n}}{x_n^{39}}.$$

Remark that even if n is a sufficiently large integer for which inequality (3) does not hold, we can define set $\mathbb{P}_n$ for which the above estimate holds. Since $\sigma_2 < 1$, this sub-series diverges as $n \to \infty$. This completes the proof of Lemma 4.

3 A Conjecture

In 1989, A. Selberg [13] introduced a rather wide class of Dirichlet series with some arithmetic properties. The Selberg class $\mathcal{S}$ consists of all Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

having the Euler product over prime numbers, analytic continuation to the whole complex plane, a functional equation of Riemann type and some analytic axioms. It is expected that all major zeta functions are contained in the class $\mathcal{S}$. Recently Nagoshi and Steuding [12] showed that if a zeta function $L(s) = \sum_n a(n)n^{-s} \in \mathcal{S}$ satisfies an estimate type of the prime number theorem

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a(p)|^2 = \kappa, \tag{4}$$

where $\pi(x) = \sharp\{p \leq x | \ p : \text{prime}\}$ and κ is some positive constant, then $L(s)$ has universality property in the strip $\sigma_L < \sigma < 1$, where the number σ_l is determined from the corresponding functional equation.

In Chapter 12 of book [14], Steuding deals with joint universality for a set of zeta functions

$$L_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s} \in \mathcal{S} \quad (1 \leq j \leq r).$$

First he generalized the proof of the joint universality for Dirichlet L-functions and obtained Theorem 12.8 in [14], which is the joint universality theorem for $\{L_j(s)\}$ in the case that for each $1 \leq j \leq r$,

$$a_j(n) = a(n)\chi_j(n) \quad \text{for all } n \geq 1$$

holds, where $a(n)$ are Dirichet coefficients of a certain zeta function with universality property, and $\chi_j(n)$ are pairwise non-equivalent Dirichlet characters. Furthermore, Steuding predicts a necessary and sufficient condition that a given set $\{L_j(s)\}$ becomes joint universal. To describe it, we recall the Selberg conjecture on the class $\mathcal{S}$. Since all zeta functions which belong to $\mathcal{S}$ have Euler product, the class $\mathcal{S}$ is closed under multiplication. A zeta function $L(s) \in \mathcal{S}$ is called primitive if when

$$L(s) = L_1(s)L_2(s) \quad L_1, L_2 \in \mathcal{S},$$

holds, then $L = L_1$ or $L = L_2$. Regarding primitive zeta functions, Selberg [13] gives the following conjecture:

(1) Let $L(s) = \sum_n a(n)n^{-s}$ be a zeta function in $\mathcal{S}$ such that $L \not\equiv 1$. Then there exists a positive integer n_L such that

$$\sum_{p\leq x} \frac{|a(p)|^2}{p} = n_L \log\log x + O(1).$$

(2) For any primitive functions $L_j(s) = \sum_n a_j(n)n^{-s}$ $(j = 1, 2)$,

$$\sum_{p\leq x} \frac{a_1(p)\overline{a_2(p)}}{p} = \begin{cases} \log\log x + O(1) & \text{if } L_1 = L_2, \\ O(1) & \text{otherwise.} \end{cases}$$

Remark that assertion (1) of the conjecture implies that condition (4) must hold for any zeta functions in $\mathcal{S}$. Therefore the conjecture yields that the universality theorems hold for arbitrary zeta functions in $\mathcal{S}$. Assertion (2) means that the set of Dirichlet coefficients $\{a_j(n)\}$ has an orthogonality similarly to that of Dirichlet characters. In other words, primitive zeta functions are expected to form an orthonormal system of $\mathcal{S}$. E. Bombieri and D. A. Hejahl [5] proved that if we assume a strong version of the Selberg conjecture and some analytic conditions for zeta functions $L_j(s)$, then the statistical independence of zero distribution of $L_j(s)$ holds. Steuding take the result one step further and gives the following conjecture.

Conjecture 1 (Steuding, [14]). Any primitive zeta functions $L_1(s)$ and $L_2(s)$ become jointly universal if and only if

$$\sum_{p\leq x}\frac{a_1(p)\overline{a_2}(p)}{p}=O(1).$$

This conjecture, roughly speaking, yields that the joint universality for a given pair of zeta functions follows from the orthogonality of the Dirichlet coefficients, even if the periodicity of the coefficients does not hold.

Applying Lemma 5 and Lemma 6, we have succeeded in proving the joint universality theorem for automorphic L-functions without using the periodicity of the coefficients. However, our new method is insufficient to solve Steuding's conjecture. As we know, all Dirichlet coefficients of the automorphic L-functions are real numbers. This fact is indispensable to apply Lemma 5. Our method can not be applied to zeta functions with non-periodic and non-real coefficients. For instance, we have not proved the joint universality theorem for a set of Hecke L-functions over algebraic number fields associated with Grössenccharacters.

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