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**On saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$**

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# 1 Introduction

In this note, we consider Axiom A regular polynomial skew products on $\mathbb{C}^2$. It is of the form : $f(z,w)=(p(z),q(z,w))$, where $p(z)$ and $q(z,w)$ are polynomials of some degree $d$. Let $\Omega$ be the set of non-wandering points for $f$. Then $f$ is said to be Axiom A if $\Omega$ is compact, hyperbolic and periodic points are dense in $\Omega$. For polynomial skew products, Jonsson [J2] has shown that $f$ is Axiom A if and only if

$(a)$ $p$ is hyperbolic,
$(b)$ $f$ is *vertically expanding* over $J_p$,
$(c)$ $f$ is *vertically expanding* over $A_p :=\{$attracting periodic points of $p\}$.

We are interested in the dynamics of $f$ on $J_p\times\mathbb{C}$. Put $q_z(w)=q(z,w)$ and consider the critical set $C_{J_p}=\{(z,w)\in J_p\times\mathbb{C}; q_z'(w)=0\}$ over the *base Julia set* $J_p$. Let $K$ denotes the set of points with bounded orbits, $K_z :=\{w\in\mathbb{C};(z,w)\in K\}$ and $J_z=\partial K_z$. Then the condition $(b)$ implies that the postcritical set for $C_{J_p}$ is disjoint from the *second Julia set* $J_2=\overline{\cup_{z\in J_p}\{z\}\times J_z}$. It follows from $(b)$ that the map $z\mapsto J_z$ is continuous on $J_p$, hence $J_2=\cup_{z\in J_p}\{z\}\times J_z$.

For any subset $X$ in $\mathbb{C}^2$, its accumulation set is defined by

$$A(X)=\cap_{N\geq 0}\overline{\cup_{n\geq N}f^n(X)}.$$

DeMarco & Hruska define the *pointwise* and *component-wise* accumulation sets of $C_{J_p}$ respectively by

$$A_{pt}(C_{J_p})=\overline{\cup_{x\in C_{J_p}}A(x)} \text{ and } A_{cc}(C_{J_p})=\overline{\cup_{C\in\mathcal{C}(C_{J_p})}A(C)},$$

where $\mathcal{C}(C_{J_p})$ denotes the collection of connected components of $C_{J_p}$.

Let $\Lambda$ be the saddle part of $\Omega$ in $J_p\times\mathbb{C}$. It decomposes into a disjoint union of *saddle basic sets* : $\Lambda=\sqcup_{i=1}^m\Lambda_i$. Put

$$\begin{aligned} W^s(\Lambda) &= \{y\in\mathbb{C}^2; f^n(y)\to\Lambda\}, \\ W^u(\Lambda) &= \{y\in\mathbb{C}^2;\exists \text{ prehistory } \hat{y}=(y_{-k})\to\Lambda\}. \end{aligned}$$

Put $\Lambda_0 := \emptyset$, $W^s(\Lambda_0) := (J_p \times \mathbb{C}) \setminus K$ and $C_i := C_{J_p} \cap W^s(\Lambda_i)$ for $0 \le i \le m$. Then $C_{J_p} = \sqcup_{i=0}^m C_i$.

**Theorem 1.** (DeMarco-Hruska [DH1]) $\Lambda = A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C})$.

**Theorem 2.** (Nakane [N]) $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Longleftrightarrow \forall C \in \mathcal{C}(C_{J_p})$, $0 \le \exists i \le m$, *s.t.* $C \subset C_i$.

**Theorem 3.** (Nakane [N]) *For each* $i \ge 0$,

$$A(C_i) = \Lambda_i \Longleftrightarrow C_i \textit{ is closed }.$$

*Consequently,*

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \Longleftrightarrow \forall i \ge 0, C_i \textit{ is closed.}$$

We state a stability result on the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A(C_{J_p}) = A_{pt}(C_{J_p})$. See also [DH2].

**Theorem 4.** (Nakane [N]) *Both equalities* $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ *and* $A(C_{J_p}) = A_{pt}(C_{J_p})$ *are preserved in hyperbolic components.*

The proof of Theorem 4 is an application of the holomorphic motions of hyperbolic sets developped by Jonsson [J1] and DeMarco & Hruska [DH1, DH2]. In Section 2, we prepare the notion of holomorphic motions and stability results of hyperbolic sets. As another application, in Section 3, we give examples which have solenoids as saddle basic sets.

# 2 Holomorphic motion

Let $\{f_a; a \in \mathbb{D}\}$ be a holomorphic family of polynomial endomorphisms on $\mathbb{C}^2$ and $L$ be a hyperbolic set of $f = f_0$. A holomorphic motion of $L$ parametrized in $\mathbb{D}_r = \{|a| < r\}$ is a continuous map $\varphi : \mathbb{D}_r \times L \to \mathbb{C}^2$ such that

(1) $\varphi(0, \cdot) = id_L$,
(2) $\varphi(\cdot, x)$ is holomorphic in $\mathbb{D}_r$ for each fixed $x$,
(3) $\varphi_a = \varphi(a, \cdot)$ is injective for each fixed $a$.

If the fiber Julia sets $J_z$ and fiber saddle sets $\Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\}$ move holomorphically, it is easy to show Theorem 4. It is true for fiber Julia sets, but not for fiber saddle sets. However, we have continuous dependence on parameters for fiber saddle sets.

**Theorem 5.** ([J1]) *Let $\{f_a; a \in \mathbb{D}\}$ be a holomorphic family of polynomial endomorphisms on $\mathbb{C}^2$, uniformly expanding on $L$. Then there exist $r > 0$ and a holomorphic motion $\Phi : \mathbb{D}_r \times L \to \mathbb{C}^2$ such that $f_a$ is uniformly expanding on $L_a = \Phi_a(L)$ and $f_a = \Phi_a \circ f_0 \circ \Phi_a^{-1}$ on $L_a$ for all $a \in \mathbb{D}_r$.*

If the maps are Axiom A polynomial skew products, $f$ is uniformly expanding on $L = J_2$. Thus we have

**Theorem 6.** ([DH1]) *If the maps $f_a(z,w) = (p_a(z), q_a(z,w))$ are Axiom A polynomial skew products in Theorem 5, $\Phi_a$ is also a skew product $\Phi_a(z,w) = (\varphi_a(z), \psi_a(z,w))$, where :*

*(1) $\varphi : \mathbb{D}_r \times J(p_0) \to \mathbb{C}$ is a holomorphic motion of $J_{p_0}$ such that $p_a = \varphi_a \circ p_0 \circ \varphi_a^{-1}$ on $J_{p_a} = \varphi_a(J_{p_0})$,*
*(2) for each $z \in J_{p_0}$, $\psi_a(z,w)$ defines a holomorphic motion $\psi : \mathbb{D}_r \times J_z(f_0) \to \mathbb{C}$ such that $\psi_a(J_z(f_0)) = J_{\varphi_a(z)}(f_a)$.*

If the hyperbolic set $L$ is not uniformly expanding, we cannot expect the existence of holomorphic motion. See Example 1 below. Instead, we have a holomorphic motion for the natural extension $\hat{L} = \{\hat{y} = (y_{-n})_{n \geq 0}; y_{-n} \in L, f(y_{-n}) = y_{-n+1}\}$. Let $\pi : \hat{L} \to L$ be the projection : $\pi(\hat{y}) = y_0$. The map $f$ induces a homeomorphism $\hat{f} : \hat{L} \to \hat{L}$ by $\hat{f}((y_{-n})) = (y_{-n+1})$.

**Theorem 7.** ([J1]) *Let $\{f_a; a \in \mathbb{D}\}$ be a holomorphic family of polynomial endomorphisms on $\mathbb{C}^2$ and let $L$ be a hyperbolic set of $f_0$. Then there exist $r > 0$ and a holomorphic motion $\hat{h}$ of $\hat{L}$ on $\mathbb{D}_r$, which induces a continuous map $h = \pi \circ \hat{h} : \mathbb{D}_r \times \hat{L} \to \mathbb{C}^2$. That is,*

*(1) for each $a \in \mathbb{D}_r$, $\hat{f}_a = \hat{h}_a \circ \hat{f}_0 \circ \hat{h}_a^{-1}$ on $\hat{L}_a = \hat{h}_a(\hat{L})$,*
*(2) for each $a \in \mathbb{D}_r$, $L_a = h_a(\hat{L})$ is a hyperbolic set of $f_a$,*
*(3) for each $a \in \mathbb{D}_r$, $h_a$ satisfies $f_a \circ h_a = h_a \circ \hat{f}_0$,*
*(4) for each $\hat{x} \in \hat{L}$, the map $h(\cdot, \hat{x}) : \mathbb{D}_r \to \mathbb{C}^2$ is holomorphic in $\mathbb{D}_r$.*

**Theorem 8.** ([DH2]) *If the maps $f_a$ are polynomial skew products in Theorem 7, then $\hat{h}_a$ is also a skew product : $\hat{\pi}_a \circ \hat{h}_a = \hat{\varphi}_a \circ \hat{\pi}$. Here, $\hat{\pi}_a : \hat{\Lambda}_a \to \hat{J}_{p_a}$ is the projection and $\hat{\varphi}_a : \hat{J}_p \to \hat{J}_{p_a}$ is the lift of $\varphi_a$.*

# 3 Examples

First, we will give an example which does not admit holomorphic motions.

**Example 1.** ([J1]) $f_a(z,w) = (z^2, w^2 + az)$.

A remarkable fact is that it is semiconjugate to the product map $g_a(z,w) = (z^2, w^2+a)$ by the map $\rho(z,w) = (z^2, zw) : \rho \circ g_a = f_a \circ \rho$. As a consequence, we have $\Lambda(f_a) = \rho(\Lambda(g_a)) = \{(z^2, \alpha_a z); |z| = 1\}$, where $\alpha_a$ is the attracting fixed point of the map $w^2+a$. $\Lambda(f_0)$ is a circle while for $a \neq 0$, $\Lambda(f_a)$ is a fiber bundle over the circle with two point set as a fiber. Hence $\Lambda(f_a)$ is not a holomorphic motion of $\Lambda(f_0)$.

In this example, $\hat{\Lambda}(f_0)$ is a solenoid as a natural extension of the circle by angle doubling map. By Theorem 7, $\hat{\Lambda}(f_a)$ is also a solenoid for small $|a|$. It is natural to ask whether a solenoid appears as a saddle set by a further perturbation of $f_a$. This actually occurs.

**Example 2.** ([FS2], Example 7.3, [J2], Example 9.4)
$f_a(z,w) = (z^2, aw^2 + \frac{1}{10}w + \frac{1}{2}z)$.

It is well known that $f_0$ has a solenoid $\Lambda_0$ as a hyperbolic set. See for example, Devaney [D], Section 2.5. By Theorem 7, there exists a continuous map $h_a$ such that $\Lambda_a = h_a(\Lambda_0)$ is a hyperbolic set of $f_a$. Since $f_0$ is injective in a neighborhood of $\Lambda_0$, so is $f_a$, hence $\hat{\Lambda}_a \cong \Lambda_a$ for small $a$. Again, by Theorem 7, it follows that $\hat{h}_a$ is a homeomorphism between $\Lambda_0 \cong \hat{\Lambda}_0$ and $\Lambda_a \cong \hat{\Lambda}_a$. Thus we conclude that $\Lambda_a$ is also a solenoid for small $a$. For $a \neq 0$, the map $\tau_a(z,w) = (z, aw)$ satisfies $\tau_a \circ f_a = k_a \circ \tau_a$, where $k_a(z,w) = (z^2, w^2 + \frac{1}{10}w + \frac{a}{2}z)$. That is, it gives a conjugacy between $f_a$ and $k_a$. Hence for small $|a| > 0$, $k_a$ has a solenoid $\tau_a(\Lambda_a)$ as a saddle set. As $a \to 0$, these solenoids tend to the circle $S^1 \times \{0\}$, a saddle set of $k_0$. Note that $J_2(f_a) = \tau_a^{-1}(J_2(k_a))$ tends to $[0:1:0]$ as $a \to 0$. As a consequence, $f_0$ has no second Julia set.

**Example 3.** (Mihailescu & Urbański [MU], Theorem 4.1)
$f_\epsilon(z,w) = (z^2, w^2 + a + \epsilon(bz^2 + czw + z + dw))$.

Let $\alpha_a$ be the attracting fixed point of $w^2+a$. Since $f_0$ has a saddle set $\Lambda_0 = S^1 \times \{\alpha_a\}$, Theorem 7 assures that $f_\epsilon$ has a saddle set $\Lambda_\epsilon$. They showed that there exist $a(b,c,d), \epsilon(a,b,c,d) > 0$ such that $f_\epsilon : \Lambda_\epsilon \to \Lambda_\epsilon$ is injective for $0 < |a| < a(b,c,d), 0 < |\epsilon| < \epsilon(a,b,c,d)$. Recall that $\hat{\Lambda}_0$ is a solenoid. Thus, by the same argument as in Example 2, $\Lambda_\epsilon \cong \hat{\Lambda}_\epsilon$ is also a solenoid for such $a$ and $\epsilon$.

We consider a perturbation of the maps in Example 1. By a similar argument as in Example 3, we get the following.

**Proposition 1.** *Take $a$ so that $0 < |\alpha_a| < 1/4$. Then the map $f_{a,b}(z,w) = (z^2, w^2 + az + bw)$ is injective on a saddle set $\Lambda_{a,b}$ for small $|b| > 0$.*

*proof.* Note that $\Lambda_{a,0} = \{(z, \pm\alpha_a\sqrt{z}); |z| = 1\}$ and that

$$f_{a,0}(z, \pm\alpha_a\sqrt{z}) = (z^2, \alpha_a z), \quad f_{a,0}(-z, \pm\alpha_a\sqrt{-z}) = (z^2, -\alpha_a z). \tag{1}$$

Here we fix the branch of the square root $\sqrt{z}$. Now put

$$\beta = \max_{(z,w)\in\Lambda_{a,b}} \min(|w - \alpha_a\sqrt{z}|, \ |w + \alpha_a\sqrt{z}|)$$

and $\beta = |w_0 - \alpha_a\sqrt{z_0}|$ for some $(z_0, w_0) \in \Lambda_{a,b}$. If we take a preimage $(z, w)$ of $(z_0, w_0)$, it follows

$$z_0 = z^2, \quad w_0 = w^2 + az + bw.$$

Set $z = \sqrt{z_0}$. Then

$$\begin{aligned}
w_0 - \alpha_a\sqrt{z_0} &= w^2 + az - \alpha_a\sqrt{z_0} + bw \\
&= w^2 + az - (\alpha_a^2 + a)z + bw \\
&= w^2 - \alpha_a^2 z + bw \\
&= (w - \alpha_a\sqrt{z})^2 + 2\alpha_a\sqrt{z}(w - \alpha_a\sqrt{z}) + bw.
\end{aligned}$$

We may assume $|w + \alpha_a\sqrt{z}| \geq |w - \alpha_a\sqrt{z}|$. Then

$$\begin{aligned}
\beta = |w_0 - \alpha_a\sqrt{z_0}| &\leq |w - \alpha_a\sqrt{z}|^2 + 2|\alpha_a\sqrt{z}(w - \alpha_a\sqrt{z})| + |bw| \\
&\leq \beta^2 + 2|\alpha_a|\beta + |bw|.
\end{aligned}$$

Moreover, since $|w| \leq |w - \alpha_a\sqrt{z}| + |\alpha_a\sqrt{z}| \leq \beta + |\alpha_a|$, $\beta$ satisfies $\beta^2 + (2|\alpha_a| - 1 + |b|)\beta + |b\alpha_a| \geq 0$. Since $\beta$ is small for small $|b|$, it follows

$$\begin{aligned}
\beta &\leq \frac{1 - 2|\alpha_a| - |b| - \sqrt{(1 - 2|\alpha_a| - |b|)^2 - 4|b\alpha_a|}}{2} \\
&\leq \frac{|b\alpha_a|}{1 - 2|\alpha_a| - |b|} + O(|b|^2) \\
&= (\frac{|\alpha_a|}{1 - 2|\alpha_a|} + O(|b|))|b|
\end{aligned}$$

for small $|b|$.

Now suppose $(z, w), (z', w') \in \Lambda_{a,b}$ satisfy $f_{a,b}(z', w') = f_{a,b}(z, w)$. Then $z' = z$ or $z' = z$. From (1), $|f_{a,0}(-z, \pm\alpha_a\sqrt{-z}) - f_{a,0}(z, \pm\alpha_a\sqrt{z})| = 2|\alpha_a| > 0$. Hence, $z' = z$ for small $|b|$. Then $w'^2 + az' + bw' = w^2 + az + bw$ implies $(w' - w)(w' + w + b) = 0$. Suppose $w' \neq w$. Then $|w - \alpha_a\sqrt{z}| \leq |w + \alpha_a\sqrt{z}|$

implies $|w' + \alpha_a\sqrt{z}| \leq |w' - \alpha_a\sqrt{z}|$ for small $|b|$. Therefore $|w - \alpha_a\sqrt{z}| \leq \beta$ and $|w' + \alpha_a\sqrt{z}| \leq \beta$. Hence

$$|w' + w| \leq |w' + \alpha_a\sqrt{z}| + |w - \alpha_a\sqrt{z}| \leq 2\beta \leq (\frac{2|\alpha_a|}{1 - 2|\alpha_a|} + O(|b|))|b|.$$

Thus, if $|\alpha_a| < 1/4$, then $|w'+w| < (1+O(|b|))|b|$. This contradicts $w'+w+b = 0$ for small $|b|$. Thus we conclude $w' = w$. This completes the proof. □

Recall that $\hat{\Lambda}_{a,0}$ is a solenoid for small $|a|$. It is true for any $a$ such that $q_a(w) = w^2 + a$ has an attracting fixed point :

**Lemma 1.** *If $q_a$ has an attracting fixed point $\alpha_a$, $\hat{\Lambda}_{a,0}$ is a solenoid.*

*proof.* The semiconjugacy $\rho$ defined in Example 1 induces an endomorphism $\hat{\rho} : \hat{\Lambda}(g_a) \to \hat{\Lambda}_{a,0}$. It is expressed by $\hat{\rho}(\hat{z}, \hat{\alpha}_a) = (\hat{p}(\hat{z}), \alpha_a\hat{z})$. Here $p(z) = z^2$ and $\hat{\alpha}_a = (\cdots, \alpha_a, \alpha_a)$ is a fixed prehistory. Evidently it is a homeomorphism. Thus $\hat{\Lambda}_{a,0} \cong \hat{\Lambda}(g_a)$ is a solenoid. □

By Theorem 7, $\hat{\Lambda}_{a,b}$ is also a solenoid for small $|b|$, hence, so is $\Lambda_{a,b}$ for small $|b| > 0$.

We also give a higher degree analogue of the examples above. Set $f_\epsilon(z, w) = (z^d, q(w) + \epsilon z)$, where $d \geq 2$ and $q$ is a monic hyperbolic polynomial of degree $d$. Then $f_0$ is Axiom A. If $q$ has an attracting fixed point $\alpha$, $\Lambda_0 = S^1 \times \{\alpha\}$ is a saddle basic set of $f_0$. We consider the perturbation $\Lambda_\epsilon$ of $\Lambda_0$.

**Proposition 2.** *Suppose $q$ has an attracting fixed point $\alpha$ of multiplier $\rho$. If $0 < |\rho| < \dfrac{\sin(\pi/d)}{1 + \sin(\pi/d)}$, then $f_\epsilon : \Lambda_\epsilon \to \Lambda_\epsilon$ is injective, hence $\Lambda_\epsilon$ is a solenoid, for small $|\epsilon| > 0$.*

*proof.* As before, we put $\beta = \max_{(z,w)\in\Lambda_\epsilon} |w - \alpha|$. Here the maximum $\beta = |w_0 - \alpha|$ is supposed to be attained at $(z_0, w_0)$. If $f_\epsilon(z, w) = (z_0, w_0)$, then $w_0 = q(w) + \epsilon z$. Therefore

$$w_0 - \alpha = q(w) - q(\alpha) + \epsilon z = q'(\alpha)(w - \alpha) + \Sigma_{j=2}^d a_j(w - \alpha)^j + \epsilon z$$

for some $a_j$. Thus

$$\beta = |w_0 - \alpha| \leq |\rho|\beta + \Sigma_{j=2}^d |a_j|\beta^j + |\epsilon|,$$

that is, $\beta$ satisfies

$$\Sigma_{j=2}^d |a_j|\beta^j + (|\rho| - 1)\beta + |\epsilon| \geq 0.$$

Since $|\rho| < 1$, the equation $\sum_{j=2}^{d} |a_j|t^j + (|\rho| - 1)t = 0$ has a simple root 0. Hence, for small $|\epsilon|$, the equation $\sum_{j=2}^{d} |a_j|t^j + (|\rho| - 1)t + |\epsilon| = 0$ has a simple root $t_\epsilon$ near the origin. It is real analytic in $|\epsilon|$. Then we have

$$\beta \leq t_\epsilon = \frac{|\epsilon|}{1 - |\rho|} + O(|\epsilon|^2).$$

Now suppose $f_\epsilon(z', w') = f_\epsilon(z, w)$. Then $z'^d = z^d$ and $q(w') + \epsilon z' = q(w) + \epsilon z$. If we put $\sigma = e^{2\pi i/d}$, $z' = \sigma^j z$ for some $j \geq 0$. First we show $z' = z$. Then, since $|q'(\alpha)| > 0$, $q$ is injective in a neighborhood of $\alpha$. Thus we have $w' = w$ for small $|\epsilon| > 0$ and the proposition follows.

Since

$$|q(w') - q(w)| \leq |q(w') - q(\alpha)| + |q(\alpha) - q(w)| \leq (|q'(\alpha)| + O(|\epsilon|))2\beta,$$

we have

$$|\epsilon(1 - \sigma^j)z| = |q(w') - q(w)| \leq 2\left(\frac{|\epsilon|}{1 - |\rho|} + O(|\epsilon|^2)\right)(|\rho| + O(|\epsilon|)).$$

Then it follows $|1 - \sigma^j| \leq \dfrac{2|\rho|}{1 - |\rho|} + O(|\epsilon|)$. In other words, $z' \neq \sigma^j z$ if $|1 - \sigma^j| > \dfrac{2|\rho|}{1 - |\rho|}$. Since $|1 - \sigma^j| \geq |1 - \sigma| = 2\sin(\pi/d)$ for $1 \leq j \leq d - 1$, we conclude that $z' = z$ if $\sin(\pi/d) > \dfrac{|\rho|}{1 - |\rho|}$, which is equivalent to the assumption. This completes the proof. □

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