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Centralizing Monoids with Minimal Function Witnesses on a Three-Element Set

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Abstract

A centralizing monoid M on a fixed set A is a set of unary functions on A which commute with some set S of functions on A. We call S a witness of M. It is known that every maximal centralizing monoid has a singleton witness consisting of a minimal function where a minimal function is, by definition, a generator of a minimal clone.

In this paper we consider the case where A is a three-element set. Using the result of B. Csákány, we obtain the list of all centralizing monoids on A which have minimal functions as their witnesses. In particular, we determine all maximal centralizing monoids on a three-element set.

Keywords: clone; centralizing monoid; minimal clone

1 Preliminaries

Let A be a finite set. For a positive integer n denote by $\mathcal{O}_A^{(n)}$ the set of all n-variable functions defined over A, i.e., maps from A^n into A. Let $\mathcal{O}_A$ be the set of all functions defined over A, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. A function $e_i^n \in \mathcal{O}_A^{(n)}$ for $1 \leq i \leq n$ is the i-th n-ary *projection* which is defined by $e_i^n(a_1, \ldots, a_i, \ldots, a_n) = a_i$ for every $(a_1, \ldots, a_n) \in A^n$. Denote by $\mathcal{J}_A$ the set of all projections defined on A.

For functions $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that f *commutes* with g, or f and g *commute*, if

$$f(g({}^t\boldsymbol{c}_1), \ldots, g({}^t\boldsymbol{c}_n)) \;=\; g(f(\boldsymbol{r}_1), \ldots, f(\boldsymbol{r}_m))$$

holds for every $m \times n$ matrix M over A with rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$ and columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$. Note that, for $m = n = 1$, this means that $f(g(x)) = g(f(x))$ for every $x \in A$, i.e., an ordinary commutation for unary functions. We write $f \perp g$ when f commutes with g. The binary relation $\perp$ on $\mathcal{O}_A$ is obviously symmetric.

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For a subset $F \subseteq \mathcal{O}_A$ the *centralizer* F^* of F is defined by

$$F^* = \{ g \in \mathcal{O}_A \mid g \perp f \text{ for all } f \in F \}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is a clone. When $F = \{f\}$ we often write f^* for F^*. Also, we write F^{**} for $(F^*)^*$. It is easy to see that the map $F \mapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

A subset M of $\mathcal{O}_A^{(1)}$ is a *monoid* if it is closed with respect to composition and contains the identity $id\,(= e_1^1)$. The set $\mathcal{O}_A^{(1)}$ is the largest monoid and the set $\{id\}$ is the smallest monoid.

2 Centralizing Monoid

A centralizing monoid may be defined in three different ways.

Lemma 2.1 *For $M \subseteq \mathcal{O}_k^{(1)}$ the following conditions are equivalent.*

(1) $M = M^{**} \cap \mathcal{O}_k^{(1)}$

(2) $\exists F \subseteq \mathcal{O}_k, \quad M = F^* \cap \mathcal{O}_k^{(1)}$

(3) $\exists \mathcal{A} = (A;\, F)$ *: algebra,* $\quad M = \mathrm{End}\,(\mathcal{A})$

Definition 2.1 *For $M \subseteq \mathcal{O}_k^{(1)}$, M is a* centralizing monoid *if M satisfies the above conditions given in Lemma 2.1.*

The above condition (2) asserts that a centralizing monoid is the unary part of some centralizer.

For an algebra $\mathcal{A} = (A;\, F)$ and a map $\varphi : A \longrightarrow A$, i.e., $\varphi \in \mathcal{O}_A^{(1)}$, φ is an *endomorphism* of $\mathcal{A}$ if

$$f(\varphi(x_1), \ldots, \varphi(x_n)) = \varphi(f(x_1, \ldots, x_n))$$

holds for every $f \in F$ and all $(x_1, \ldots, x_n) \in A^n$. In other words, φ is an endomorphism of $\mathcal{A} = (A;\, F)$ if and only if $\varphi \perp f$ for all $f \in F$, i.e., $\varphi \in F^*$. This means that a centralizing monoid is the set of endomorphisms of some algebra.

From Lemma 2.1 it is easy to see the following, which we call the *Witness Lemma.*

Lemma 2.2 *For a monoid $M \subseteq \mathcal{O}_A^{(1)}$ and $S \subseteq \mathcal{O}_A$, suppose the conditions* (i) *and* (ii) *hold:*

(i) *For any $f \in M$ and any $u \in S$, f and u commute, i.e., $f \perp u$.*

(ii) *For any $g \in \mathcal{O}_A^{(1)} \setminus M$ there exists $w \in S$ such that g does not commute with w, i.e., $g \not\perp w$.*

Then M is a centralizing monoid.

A subset S in the lemma will be called a *witness* for a centralizing monoid M. We denote by $M(S)$ the centralizing monoid M with S as its witness, i.e., $M(S) = S^* \cap \mathcal{O}_A^{(1)}$. When f is a singleton, i.e., $S = \{f\}$, we write $M(f)$ instead of $M(\{f\})$.

By definition, M^* is a witness for M. Hence, we have:

Lemma 2.3 *Every centralizing monoid M has a witness.*

This result can be strenghened due to the assumption that A is finite.

Proposition 2.4 *For every centralizing monoid M there exists a finite subset of $\mathcal{O}_A$ which is a witness of M, that is, every centralizing monoid M has a* finite *witness.*

3 Maximal Centralizing Monoid and Minimal Clone

A centralizing monoid M is *maximal* if $\mathcal{O}_A^{(1)}$ is the only centralizing monoid properly containing M.

Proposition 3.1 *For every maximal centralizing monoid M, there exists $u \in \mathcal{O}_A$ such that*

$$M = M(u),$$

that is, every maximal centralizing monoid has a singleton *witness.*

For the proof see [MR 11].

Definition 3.1 *A function $f \in \mathcal{O}_A$ is called a* minimal function *if*

(i) *f generates a minimal clone C, and*

(ii) *f has the minimum arity among functions generating C.*

Theorem 3.2 *For any maximal centralizing monoid M, there exists a minimal function $f \in \mathcal{O}_A$ such that*

$$M = M(f),$$

that is, every maximal centralizing monoid has a witness which is a minimal *function.*

The reader is again referred to [MR 11] for the proof.

4 Ternary Case : $E_3 = \{0, 1, 2\}$

In the following we determine all maximal centralizing monoids on a three-element set. We write $E_3 = \{0, 1, 2\}$. In Table 0, we present all unary functions on E_3, named after [La 84, La 06], which will be used in the sequel.

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5
0	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1
1	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2
2	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

	s_1	s_2	s_3	s_4	s_5	s_6
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	2	1	2	0	1	0

Table 0: Unary Functions in $\mathcal{O}_3^{(1)}$

4.1 Minimal Clones on E_3

The complete list of minimal clones on E_3 was given by B. Csákány (1983).

Proposition 4.1 ([Cs 83]) *On E_3 there are 84 minimal clones. The number of minimal clones generated by each of five types of minimal functions is as follows:*

Unary functions	:	13	(4)
Binary idempotent functions	:	48	(12)
Ternary majority functions	:	7	(3)
Ternary semiprojections	:	16	(5)

The numbers in the parentheses indicate the numbers of conjugate classes.

For each minimal function $f \in \mathcal{O}_3^{(1)}$, let $\{f\}$ be a witness and construct a centralizing monoid $M(f)$. Then some of such centralizing monoids are maximal while some are not.

4.2 Centralizing Monoids with Minimal Functions as their Witnesses

We have explicitly determined all centralizing monoids on E_3 which have minimal functions as their witnesses. The complete list of such centralizing monoids is presented in Tables 1–4 at the end of this paper.

In [Cs 83], B. Csákány numbered each minimal function in the following way.

- A unary function $u_r(x)$ is numbered by:

$$r = u(0) \times 3^2 + u(1) \times 3^1 + u(2) \times 3^0$$

- A binary idempotent function $b_s(x, y)$ is numbered by:

$$\begin{aligned} s = \; & b(0,1) \times 3^5 + b(0,2) \times 3^4 + b(1,0) \times 3^3 \\ & + b(1,2) \times 3^2 + b(2,0) \times 3^1 + b(2,1) \times 3^0 \end{aligned}$$

- A ternary majority function $m_t(x, y, z)$ is numbered by:

$$\begin{aligned} t = \; & m(0,1,2) \times 3^5 + m(0,2,1) \times 3^4 + m(1,0,2) \times 3^3 \\ & + m(1,2,0) \times 3^2 + m(2,0,1) \times 3^1 + m(2,1,0) \times 3^0 \end{aligned}$$

- A ternary function $p(x_1, x_2, x_3)$ is called a *semiprojection* if there exists $j\ (\in \{1,2,3\})$ such that $p(x_1, x_2, x_3) = x_j$ whenever $|\{x_1, x_2, x_3\}| < 3$. A semiprojection $p_t(x, y, z)$ has a similar numbering as a majority function:

$$\begin{aligned} t \ = \ & p(0,1,2) \times 3^5 + p(0,2,1) \times 3^4 + p(1,0,2) \times 3^3 \\ & + p(1,2,0) \times 3^2 + p(2,0,1) \times 3^1 + p(2,1,0) \times 3^0 \end{aligned}$$

In Tables 1–4, we use these numberings to indicate minimal functions, except unary minimal functions for which we use D. Lau's naming introduced in Table 0.

In the tables, minimal functions f for all minimal clones on E_3 appear in the leftmost column. In the row with a minimal function f in the leftmost place, all members of the centralizing monoid $M(f)$ are shown as indicated by the circle "o".

4.3 Minimal Functions corresponding to Maximal Centralizing Monoids

Due to Theorem 3.2, one can determine all maximal centralizing monoids on E_3 by inspecting all centralizing monoids shown in Tables 1–4.

Proposition 4.2 *On E_3, there are 10 maximal centralizing monoids. Among them,*

- *3 maximal centralizing monoids have unary constant functions as their witnesses, and*
- *7 maximal centralizing monoids have ternary majority functions which generate minimal clones as their witnesses.*

Recall that there are exactly 7 minimal clones generated by ternary majority functions. Hence every minimal clone generated by a ternary majority function corresponds to a maximal centralizing monoid.

The following is the set of minimal functions which give maximal centralizing monoids as their witnesses.

(I) **Constant functions**

$$c_i(x) = i \quad \text{for any } x \in E_3 \qquad (i = 0, 1, 2)$$

(II) **Majority functions** (showing the values only for mutually distinct x, y and z.)

Let $\sigma = \{0,1,2), (1,2,0), (2,0,1)\}$ and $\tau = \{(0,2,1), (1,0,2), (2,1,0)\}$.

$$\begin{aligned} m_0\,(x,y,z) &= 0 \quad \text{if } |\{x,y,z\}| = 3 \\ m_{364}(x,y,z) &= 1 \quad \text{if } |\{x,y,z\}| = 3 \\ m_{728}(x,y,z) &= 2 \quad \text{if } |\{x,y,z\}| = 3 \end{aligned}$$

$$m_{109}(x,y,z) = \begin{cases} 0 & \text{if } (x,y,z) \in \sigma \\ 1 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{473}(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z) \in \sigma \\ 2 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{510}(x,y,z) = \begin{cases} 2 & \text{if } (x,y,z) \in \sigma \\ 0 & \text{if } (x,y,z) \in \tau \end{cases}$$

$$m_{624}(x,y,z) = y \quad \text{if } |\{x,y,z\}| = 3$$

For the reader's sake, we summarize all maximal centralizing monoids on E_3. Recall that $M(f)$ means the centralizing monoid having f as its witness.

Maximal centralizing monoids on E_3

$$\begin{aligned}
M\,(c_0) &= \{s_1,s_2\} \cup \{j_1,j_2,j_5,u_1,u_2,u_5\} \cup \{c_0\} \\
M\,(c_1) &= \{s_1,s_6\} \cup \{j_1,j_3,j_5,v_0,v_2,v_4\} \cup \{c_1\} \\
M\,(c_2) &= \{s_1,s_3\} \cup \{u_2,u_4,u_5,v_2,v_4,v_5\} \cup \{c_2\}
\end{aligned}$$

$$\begin{aligned}
M\,(m_0) &= \{s_1,s_2\} \cup \{j_1,j_2,j_3,j_4,\ u_1,u_2,u_3,u_4\} \cup \{v_1,v_2,v_3,v_4\} \cup \{c_0,c_1,c_2\} \\
M(m_{364}) &= \{s_1,s_6\} \cup \{j_0,j_2,j_3,j_5,\ u_0,u_2,u_3,u_5\} \cup \{v_0,v_2,v_3,v_5\} \cup \{c_0,c_1,c_2\} \\
M(m_{728}) &= \{s_1,s_3\} \cup \{j_0,j_1,j_4,j_5,\ u_0,u_1,u_4,u_5\} \cup \{v_0,v_1,v_4,v_5\} \cup \{c_0,c_1,c_2\}
\end{aligned}$$

$$\begin{aligned}
M(m_{109}) &= \{s_1,s_3\} \cup \{j_2,j_3,u_2,u_3,v_2,v_3\} \cup \{c_0,c_1,c_2\} \\
M(m_{473}) &= \{s_1,s_2\} \cup \{j_0,j_5,u_0,u_5,v_0,v_5\} \cup \{c_0,c_1,c_2\} \\
M(m_{510}) &= \{s_1,s_6\} \cup \{j_1,j_4,u_1,u_4,v_1,v_4\} \cup \{c_0,c_1,c_2\}
\end{aligned}$$

$$M(m_{624}) = \{s_1,s_2,s_3,s_4,s_5,s_6\} \cup \{c_0,c_1,c_2\}$$

References

[Cs 83] Csákány, B, All minimal clones on the three element set, *Acta Cybernet.*, **6**, 1983, 227-238.

[La 84] Lau, D., Die Unterhalbgruppen von $(P_3^1;*)$, *Rostock Math. Colloq.*, **26**, 1984, 55-62.

[La 06] Lau, D., Function Algebras on Finite Sets, *Springer Monographs in Mathematics*, Springer, 2006.

[MR 04] Machida, H. and Rosenberg, I. G., On centralizers of monoids, *Novi Sad Journal of Mathematics*, Vol. **34**, No. **2**, 2004, 153-166.

[MR 09] Machida, H. and Rosenberg, I. G., On endoprimal monoids in clone theory, *Proc. 39th Int. Symp. Multiple-Valued Logic*, IEEE, 2009, 167-172.

[MR 10] Machida, H. and Rosenberg, I. G., Endoprimal monoids and witness lemma in clone theory, *Proceedings 40th International Symposium on Multiple-Valued Logic*, IEEE, 2010, 195-200.

[MR 11] Machida, H. and Rosenberg, I. G., Maximal Centralizing Monoids and their Relation to Minimal Clones, *Proceedings 41st International Symposium on Multiple-Valued Logic*, IEEE, 2011.

List of Centralizing Monoids on E_3 which have Minimal Functions as their Witnesses

In the following tables, all centralizing monoids on E_3 ($= \{0, 1, 2\}$) are shown that have minimal functions as their witnesses. Minimal functions f for all minimal clones on E_3 appear in the leftmost column. In the row starting from a minimal function f in the leftmost box, all members of the centralizing monoid $M(f)$ are shown by the circles "o". That is, the circle "o" in the crossing of row f (minimal function) and column g (unary function) indicates that g is a member of the centralizing monoid $M(f)$, i.e., $g \in M(f)$. (Equivalently, this means $f \perp g$.)

(i) Centralizing Monoids on E_3 with Unary Minimal Functions as their Witnesses

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	Const
	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1	0	0	1	1	2	2	
	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2	1	2	0	2	0	1	
	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2	2	1	2	0	1	0	
c_0		o	o			o		o	o			o							o	o					c_0
c_1		o		o		o							o		o		o		o					o	c_1
c_2									o		o	o			o		o	o	o		o				c_2
j_1		o			o				o										o						c_0 c_1
j_5	o					o									o				o						c_0 c_1
u_2		o							o	o									o						c_0 c_2
u_5							o					o					o		o						c_0 c_2
v_2						o									o	o			o						c_1 c_2
v_4												o		o			o		o						c_1 c_2
s_2																			o	o					c_0
s_3																			o		o				c_2
$s_4(s_5)$																			o			o	o		
s_6																			o					o	c_1

Table 1: Unary Minimal Functions

(ii) Centralizing Monoids on E_3 with Binary Idempotent Minimal Functions as their Witnesses: Part 1

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	Const
	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1	0	0	1	1	2	2	
	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2	1	2	0	2	0	1	
	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2	2	1	2	0	1	0	
b_0		o	o					o	o										o	o					ooo
b_{364}				o		o							o		o				o					o	ooo
b_{728}											o	o					o	o	o		o				ooo
b_8		o							o	o									o						ooo
b_{368}						o									o	o			o						ooo
b_{80}							o					o					o		o						ooo
b_{36}		o			o				o										o						ooo
b_{40}	o					o									o				o						ooo
b_{692}												o		o			o		o						ooo
b_{10}			o			o			o			o			o			o	o						ooo
b_{280}		o		o					o		o				o		o		o						ooo
b_{458}			o			o			o			o			o			o	o						ooo
b_{20}		o				o		o				o	o				o		o						ooo
b_{448}		o				o		o				o	o				o		o						ooo
b_{188}		o		o					o		o				o		o		o						ooo
b_{11}						o						o							o	o					ooo
b_{286}		o															o		o					o	ooo
b_{215}									o						o				o		o				ooo
b_{16}																			o						ooo
b_{281}																			o						ooo
b_{296}																			o						ooo
b_{47}																			o						ooo
b_{205}																			o						ooo
b_{179}																			o						ooo

Table 2: Binary Idempotent Minimal Functions: Part 1

(iii) Centralizing Monoids on E_3 with Binary Idempotent Minimal Functions as their Witnesses: Part 2

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	Const
	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1	0	0	1	1	2	2	
	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2	1	2	0	2	0	1	
	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2	2	1	2	0	1	0	
b_{17}									o	o					o	o			o						ooo
b_{287}									o	o					o	o			o						ooo
b_{53}	o					o	o					o							o						ooo
b_{38}		o			o									o			o		o						ooo
b_{43}	o					o	o					o							o						ooo
b_{206}		o			o									o			o		o						ooo
b_{26}		o															o		o					o	ooo
b_{449}						o						o							o	o					ooo
b_{37}									o						o				o		o				ooo
b_{33}																			o	o					ooo
b_{122}																			o					o	ooo
b_{557}																			o		o				ooo
b_{35}			o	o					o	o									o						ooo
b_{125}			o	o											o	o			o						ooo
b_{71}							o					o	o					o	o						ooo
b_{42}		o			o			o			o								o						ooo
b_{41}	o					o							o					o	o						ooo
b_{530}								o			o			o			o		o						ooo
b_{68}													o					o	o	o					ooo
b_{528}								o			o								o					o	ooo
b_{116}			o	o															o		o				ooo
b_{178}																			o			o	o		ooo
b_{290}																			o			o	o		ooo
b_{624}																			o	o	o	o	o	o	ooo

Table 3: Binary Idempotent Minimal Functions: Part 2

(iv) Centralizing Monoids on E_3 with Ternary Majority Minimal Functions or Ternary Minimal Semiprojections as their Witnesses

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	Const
	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1	0	0	1	1	2	2	
	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2	1	2	0	2	0	1	
	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2	2	1	2	0	1	0	
m_0		o	o	o	o			o	o	o	o			o	o	o	o		o	o					ooo
m_{364}	o		o	o		o	o		o	o		o	o		o	o		o	o					o	ooo
m_{728}	o	o			o	o	o	o			o	o	o	o			o	o	o		o				ooo
m_{109}			o	o					o	o					o	o			o		o				ooo
m_{473}	o					o	o					o	o					o	o	o					ooo
m_{510}		o			o			o			o			o			o		o					o	ooo
m_{624}																			o	o	o	o	o	o	ooo
p_0																			o	o					ooo
p_{364}																			o					o	ooo
p_{728}																			o		o				ooo
p_8			o	o					o	o					o	o			o						ooo
p_{368}			o	o					o	o					o	o			o						ooo
p_{80}	o					o	o					o	o					o	o						ooo
p_{36}		o			o			o			o			o			o		o						ooo
p_{40}	o					o	o					o	o					o	o						ooo
p_{692}		o			o			o			o			o			o		o						ooo
p_{26}																			o					o	ooo
p_{449}																			o	o					ooo
p_{37}																			o		o				ooo
p_{76}	o					o	o					o	o					o	o	o					ooo
p_{684}		o			o			o			o			o			o		o					o	ooo
p_{332}			o	o					o	o					o	o			o		o				ooo
p_{424}																			o	o	o	o	o	o	ooo

Table 4: Ternary Majority Minimal Functions and Ternary Minimal Semiprojections