<table>
<thead>
<tr>
<th>Title</th>
<th>On Automorphism Groups of Petri Nets Based on Place Connectivity (Algebraic Systems and Theoretical Computer Science)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kunimochi, Yoshiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2012, 1809: 100-109</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194468">http://hdl.handle.net/2433/194468</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On Automorphism Groups of Petri Nets Based on Place Connectivity

静岡理工科大学・総合情報学部
國持良行 (Yoshiyuki Kunimochi)
Faculty of Comprehensive Informatics,
Shizuoka Institute of Science and Technology

1 Introduction

A Petri net is a mathematical model which is applied to descriptions of parallel processing systems. So far, some types of morphisms related to Petri nets (or condition/event net) have been studied in terms of the category theory, in order to simplify the behavior of more complicated Petri nets and understand the concurrency in other computation models [4][10].

Studying how the structure of Petri nets have an effect on Petri net languages and codes, we often realize that the ratio between the number of tokens in a place and the weights of edges connected to the place is important. So we give our definition of morphisms between Petri nets focusing on the connection state/level of edges which come in or go out a place. This is an extension of an automorphism which we used to introduce to a net in [5][6].

In the second section we introduce morphisms between two Petri nets. The set of all morphisms of a Petri net forms a monoid expressed by a semi-direct product. Especially, the set of all automorphisms of a Petri net forms a group. We investigate the inclusion relations among such monoids and groups. The third section deals with a pre-order induced by a surjective morphism. Two diamond properties are proved. It is a common case that one gives some redundancy or multiple provisions to systems to improve their reliability and safety. Surjective morphism will be effective to analyze such redundant systems. In the last section we show the properties of languages generated by two Petri nets ordered by a surjective morphism. The languages generated by them and their reachability sets have close correspondence.

2 Preliminaries

Here we give our definition of morphisms of a Petri net and state the properties of some monoids composed of these morphisms.

2.1 Petri Nets and Morphisms

In this section, we give definitions and fundamental properties related to Petri nets. We denote the set of all nonnegative integers by $N_0$, that is, $N_0 = \{0, 1, 2, \ldots\}$.

First of all, a Petri net is viewed as a particular kind of directed graph, together with an initial state $\mu_0$, called the initial marking. The underlying graph $N$ of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called places and transitions, where arcs are either from a place to a transition or from a transition to a place.

**DEFINITION 2.1 (Petri net)** A Petri net is a 4-tuple $(P, T, W, \mu_0)$ where
(1) $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places,
(2) $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,
(3) $W : E(P, T) \rightarrow \{0, 1, 2, 3, \ldots\}$, i.e., $W \in N_0^{E(P, T)}$, is a weight function, where $E(P, T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \rightarrow \{0, 1, 2, 3, \ldots\}$, i.e., $\mu_0 \in N_0^P$, is the initial marking,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by $N$, a Petri net with a given initial marking $\mu_0$ is denoted by $(N, \mu_0)$.

$\square$
In the graphical representation, the places are drawn as circles and the transitions are drawn as bars or boxes. Arrows are labeled with their weights (positive integers), where a -weighted arrow can be interpreted as the set of -weighted arrows. Labels for unity weights are usually omitted. A marking (state) assigns a nonnegative integer to each place. If a marking assigns a nonnegative integer to a place , we say that is marked with tokens. Pictorially, we put black dots (tokens) in place . A marking is denoted by , an -dimensional row vector, where is the total number of places. The -th component of , denoted by , is the number of tokens in the -th place .

**EXAMPLE 2.1** Fig. 1 shows a graphical representation of a Petri net \( \mathcal{P} = (P, T, W, \mu_0) \) with \( T = \{ t \} \), \( (a, t) \) and \( (t, b) \) are arcs of weights 2 and 1 respectively. \( (t, a) \) and \( (b, t) \) are arcs of weight 0, which are not usually drawn in the picture. Note that the weight of \( (t, b) \) is omitted since it is unity. That is, \( W(a, t) = 2, W(b, t) = 1, W(t, a) = W(b, t) = 0 \). The initial marking \( \mu_0 \) with \( \mu_0(a) = 3, \mu_0(b) = 0 \) is often written like a row vector \( \mu_0 = (3, 0) \).

![Graphical representation of a Petri net](image)

**Figure 1. Graphical representation of a Petri net**

Now we introduce a Petri net morphism based on place connectivity. We denote the set of all positive rational numbers by \( \mathbb{Q}_+ \).

**DEFINITION 2.2** Let \( \mathcal{P}_1 = (P_1, T_1, W_1, \mu_1) \) and \( \mathcal{P}_2 = (P_2, T_2, W_2, \mu_2) \) be Petri nets. Then a triple \( (f, (\alpha, \beta)) \) of maps is called a morphism from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \) if the maps \( f : P_1 \rightarrow \mathbb{Q}_+ \), \( \alpha : P_1 \rightarrow P_2 \) and \( \beta : T_1 \rightarrow T_2 \) satisfy the condition that for any \( p \in P_1 \) and \( t \in T_1 \),

\[
\begin{align*}
W_2(\alpha(p), \beta(t)) &= f(p)W_1(p, t), \\
W_2(\beta(t), \alpha(p)) &= f(p)W_1(t, p), \\
\mu_2(\alpha(p)) &= f(p)\mu_1(p).
\end{align*}
\]

(2.1)

In this case we write \( (f, (\alpha, \beta)) : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \). $

The morphism \( (f, (\alpha, \beta)) : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \) is called injective (resp. surjective) if both \( \alpha \) and \( \beta \) are injective (resp. surjective). In particular, it is called an isomorphism from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \) if it is injective and surjective. Then \( \mathcal{P}_1 \) is said to be isomorphic to \( \mathcal{P}_2 \) and we write \( \mathcal{P}_1 \simeq \mathcal{P}_2 \). Moreover, in case of \( \mathcal{P}_1 = \mathcal{P}_2 \), an isomorphism is called an automorphism of \( \mathcal{P}_1 \). By \( \text{Aut}(\mathcal{P}) \) we denote the set of all the automorphisms of \( \mathcal{P} \).

For Petri nets \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), we write \( \mathcal{P}_1 \supseteq \mathcal{P}_2 \) if there exists a surjective morphism from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \). The relation \( \supseteq \) forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as an order by identifying isomorphisms.

**PROPOSITION 2.1** Let \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \) be Petri nets. Then,

(1) \( \mathcal{P}_1 \supseteq \mathcal{P}_1 \).

(2) \( \mathcal{P}_1 \supseteq \mathcal{P}_2 \text{ and } \mathcal{P}_2 \supseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2 \).

(3) \( \mathcal{P}_1 \supseteq \mathcal{P}_2 \) and \( \mathcal{P}_2 \supseteq \mathcal{P}_3 \) imply \( \mathcal{P}_1 \supseteq \mathcal{P}_3 \).

Proof: Let \( \mathcal{P}_i = (P_i, T_i, W_i, \mu_i) \) \((i = 1, 2, 3)\) through the proof. The proof complete in the order (1), (3), (2).

(1) Trivial.

(3) There exist surjective morphisms \( (f_i, (\alpha_i, \beta_i)) : \mathcal{P}_i \rightarrow \mathcal{P}_{i+1} \) \((i = 1, 2)\). We define a map \( f : \mathcal{P}_1 \rightarrow \mathbb{Q}_+ \) by \( f(p) = f_1(p) \cdot f_2(\alpha(p)) \). Then \( (f, (\alpha_1, \alpha_2, \beta_1, \beta_2)) \) is a surjective morphism from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \).
(2) \((\Rightarrow)\) There exist surjective morphisms \((f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2\) and \((g, (\alpha', \beta')) : \mathcal{P}_2 \to \mathcal{P}_1\). Since \(\alpha\alpha'\) is surjective by (3) above and \(\mathcal{P}_1\) is finite, both \(\alpha\) and \(\alpha'\) are bijections. \(\beta\) and \(\beta'\) are also. Therefore \(\mathcal{P}_1 \cong \mathcal{P}_2\).

\((\Leftarrow)\) If \((f, (\alpha, \beta))\) be a isomorphism from \(\mathcal{P}_1\) to \(\mathcal{P}_2\), then it is easily shown that \((\alpha^{-1}f^{-1}, (\alpha^{-1}, \beta^{-1}))\) is an isomorphism from \(\mathcal{P}_2\) to \(\mathcal{P}_1\), where \(f^{-1} : \mathcal{P}_2 \to Q_+, p \mapsto 1/f(p)\).

**DEFINITION 2.3 (Similar)** Let \(\mathcal{P} = (P, T, W, \mu)\) be a Petri net. Two places \(p, q \in P\) are said to be similar if there exists some positive rational number \(r\) such that \(\mu(p) = r\mu(q)\), \(W(q, t) = rW(p, t)\) and \(W(t, q) = rW(t, p)\) for all \(t \in T\). Two transitions \(s, t \in T\) are said to be similar if \(W(p, s) = W(p, t)\) and \(W(s, p) = W(t, p)\) for all \(p \in P\).

The similarity defined above is obviously an equivalence relation on \(P \cup T\). We denote this relation by \(\sim_P\) or simply \(\sim\) and the \(\sim_P\)-class of a place or a transition \(u\) by \(C(u)\). A place (resp. a transition) is said to be isolated if it has no connection to any transitions (resp. any places). Especially, a place \(p\) is 0-isolated if it is isolated and \(\mu(p) = 0\). Note that two 0-isolated places \(p\) and \(q\) are similar because for any positive rational number \(r\), \(\mu(p) = 0 = r\mu(q)\), \(W(q, t) = 0 = rW(p, t)\) and \(W(t, q) = 0 = rW(t, p)\) for all \(t \in T\).

### 2.2 Monoids \(S^1\) of Surjective Morphisms of Petri Nets

We introduce a composition of morphisms; all the morphisms between Petri nets form a monoid under this composition.

Let \(\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\) \((i = 1, 2, 3)\) be Petri nets, \((f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2\) and \((g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3\) be morphisms. Then,

\[
\begin{align*}
W_3(\gamma(\alpha(p)), \delta(\beta(t))) &= g(\alpha(p))W_2(\alpha(p), \beta(t)) \\
&= g(\alpha(p))f(p)W_1(p, t), \\
W_3(\delta(\beta(t)), \gamma(\alpha(p))) &= g(\alpha(p))W_2(\beta(t), \alpha(p)) \\
&= g(\alpha(p))f(p)W_1(t, p), \\
\mu_3(\gamma(\alpha(p))) &= g(\alpha(p))\mu_2(\alpha(p)) = g(\alpha(p))f(p)\mu_1(p)
\end{align*}
\]

hold.

In this manuscript, by writing compositions of maps like \(g \circ \alpha, \gamma \circ \alpha\) and \(\delta \circ \beta\) in the form of multiplications like \(\alpha g, \alpha \gamma\) and \(\beta \delta\) respectively, the composition of morphisms is written as \((f \otimes_{P}, (\alpha g, (\alpha \gamma, \beta \delta)))\), where \(\otimes_{P}\) is the operation in the following fundamental commutative group \((Q_+^P, \otimes_P)\).

The set \((Q_+^P, \otimes_P)\) of all maps from a set \(P\) to \(Q_+\) forms a commutative group under the operation \(\otimes_P\) defined by \(f \otimes_P g : p \mapsto f(p)g(p)\). \(1_{\otimes_P} : P \to Q_+^P : p \mapsto 1\) is the identity and \((f^{-1} : P \to Q_+ : p \mapsto 1/f(p))\) is the inverse of a \(f \in Q_+^P\). Whenever it does not cause confusion, we write \(\otimes\) instead of \(\otimes_P\). Immediately we obtain the following lemma.

**LEMMA 2.1** Let \(\alpha\) and \(\beta\) be arbitrary maps on \(P\) and \(f, g : P \to Q_+\). Then the following equations are true.

\[(1)\quad (\alpha \beta)f = \alpha(\beta f).\]
\[(2)\quad \alpha(f \otimes g) = (\alpha f) \otimes (\alpha g).\]
\[(3)\quad \alpha 1_{\otimes} = 1_{\otimes}.\]
\[(4)\quad (\alpha f) \otimes (\alpha f^{-1}) = 1_{\otimes}.\]
\[(5)\quad (\alpha f)^{-1} = \alpha f^{-1}.\]

**Proof** For each \(p \in P\), the following equations hold.

\[
\begin{align*}
((\alpha \beta)f)(p) &= f(\beta(\alpha(p))) = (\beta f)(\alpha(p)) = (\alpha(\beta f))(p), \\
(\alpha(f \otimes g))(p) &= f(\alpha(p)) \cdot g(\alpha(p)) = (\alpha f)(p) \cdot (\alpha g)(p) = ((\alpha f) \otimes (\alpha g))(p), \\
(\alpha 1_{\otimes})(p) &= 1_{\otimes}(\alpha(p)) = 1_{\otimes}(p), \\
(\alpha f)(p) \otimes (\alpha f^{-1}) &= \alpha(f \otimes f^{-1}) = \alpha 1_{\otimes} = 1_{\otimes}, \\
(\alpha f)^{-1}(p) &= 1/f(\alpha(p)) = f^{-1}(\alpha(p)) = (\alpha f^{-1})(p).
\end{align*}
\]

For a surjective morphism \(z : \mathcal{P}_1 \to \mathcal{P}_2\), \(\mathcal{P}_1\) is called the domain of \(z\), denoted by \(Dom(z)\), and \(\mathcal{P}_2\) is called the image(or range) of \(z\), denoted by \(Im(z)\). Especially \(Dom(0) = Im(0) = \emptyset\).
We denote the set of all surjective morphisms between two Petri nets and a zero element 0, by \( \mathcal{S} \). \( \mathcal{S} \) forms a semigroup, equipped with the multiplication of \( x = (f, (\alpha, \beta)) \) and \( y = (g, (\gamma, \delta)) \):

\[
x \cdot y \overset{\text{def}}{=} \begin{cases} 
(f \circ g, (\alpha \gamma, \beta \delta)) & \text{if } \text{Im}(x) = \text{Dom}(y), \\
0 & \text{otherwise}.
\end{cases}
\]

\( \mathcal{S}^1 = \mathcal{S} \cup \{1\} \) is the monoid obtained from \( \mathcal{S} \) by adjoining an (extra) identity 1, that is, \( 1 \cdot s = s \cdot 1 = s \) for all \( s \in \mathcal{S} \) and \( 1 \cdot 1 = 1 \).

**EXAMPLE 2.2** Let \( \mathcal{P}_i = (P_i, T_i, W_i, \mu_i) \) (\( 1 \leq i \leq 3 \)) be Petri nets shown in Figure 2. The four morphisms \( x_i = (f_i, (\alpha_i, \beta_i)) \) (\( 0 \leq i \leq 3 \)) are from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \), where

\[
\begin{align*}
    f_0 &= \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1 \end{pmatrix}, & \alpha_0 &= \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}, \\
    f_1 &= \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1/3 \end{pmatrix}, & \alpha_1 &= \begin{pmatrix} p_1 & p_2 \\ q_1 & q_1 \end{pmatrix}, \\
    f_2 &= \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1/3 \end{pmatrix}, & \alpha_2 &= \begin{pmatrix} p_1 & p_2 \\ q_1 & q_1 \end{pmatrix}, \\
    f_3 &= \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1 \end{pmatrix}, & \alpha_3 &= \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix},
\end{align*}
\]

and \( \beta_0 = \beta_1 = \beta_2 = \beta_3 : T_1 \rightarrow T_2, t_1 \mapsto s, t_2 \mapsto s \). Especially only \( x_0 \) and \( x_1 \) are surjective morphisms. Only one morphism \( y = (g, (\gamma, \delta)) \) exists from \( \mathcal{P}_2 \) to \( \mathcal{P}_3 \), where

\[
\begin{align*}
    g : P_2 &\rightarrow Q_+, q_1 \mapsto 1, q_2 \mapsto 1/3, \\
    \gamma : P_2 &\rightarrow P_3, q_1 \mapsto r, q_2 \mapsto r, \\
    \delta : T_2 &\rightarrow T_3, s \mapsto u.
\end{align*}
\]

This is a surjective morphism. The compositions of morphisms \( x_i \) (\( 0 \leq i \leq 3 \)) and \( y \) are the same surjective morphism \( (h, (\sigma, \tau)) \) from \( \mathcal{P}_1 \) to \( \mathcal{P}_3 \), where

\[
\begin{align*}
    h : P_1 &\rightarrow Q_+, p_1 \mapsto 1/2, p_2 \mapsto 1/3, \\
    \sigma = \alpha_i \gamma : P_1 &\rightarrow P_3, p_1 \mapsto r, p_2 \mapsto r, \\
    \tau = \beta_i \delta : T_1 &\rightarrow T_3, t_1 \mapsto u, t_2 \mapsto u.
\end{align*}
\]

for any \( i = 1, 2, 3, 4 \). Note that \( h \) is expressed as \( h = f_i \otimes (\alpha_i g) \). □

![Figure 2. Petri nets \( \mathcal{P}_1, \mathcal{P}_2 \) and \( \mathcal{P}_3 \) with \( \mathcal{P}_1 \equiv \mathcal{P}_2 \equiv \mathcal{P}_3 \).](image)

### 3 Ideals in the monoid \( \mathcal{S}^1 \)

In this section we consider ideals and Green's relations on the monoid \( \mathcal{S}^1 \).

At first, we consider some properties of the structure of the automorphism group of a Petri net \( \mathcal{P} \).
3.1 Green's equivalences on the monoid $S^1$

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a monoid $M$, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$
\begin{align*}
\mathcal{L}x &\iff Mx = My, \\
\mathcal{R}y &\iff yM = yM, \\
\mathcal{J}y &\iff MxM = MyM, \\
\mathcal{H} &\iff \mathcal{L} \cap \mathcal{R}, \\
\mathcal{D} &\iff (\mathcal{L} \cup \mathcal{R})^*,
\end{align*}
$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. $Mx$ (resp. $xM$) is called the principal left (resp. right) ideal generated by $x$ and $MxM$ the it principal (two-sided) ideal generated by $x$. Then, the following facts are generally true[2, 1].

**FACT 1** The following relations are true.

$$(1) \, \mathcal{D} = \mathcal{L} \mathcal{R} = \mathcal{R} \mathcal{L}$$

$$(2) \, \mathcal{H} \subseteq \mathcal{L} \quad (\text{resp. } \mathcal{R}) \subseteq \mathcal{D} \subseteq \mathcal{J}$$

**FACT 2** An $\mathcal{H}$-class of a monoid $M$ is a group if and only if it contains an idempotent.

Now we consider the case of $M = S^1$ in the rest of the manuscript. The following lemma is obviously true.

**LEMMA 3.1** Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in S^1$. Then,

1. $xS^1 \subset yS^1 \Rightarrow \mathcal{P}_1 = \mathcal{P}_3$ and $\mathcal{P}_2 \subseteq \mathcal{P}_4$.
2. $S^1x \subset S^1y \Rightarrow \mathcal{P}_1 \subseteq \mathcal{P}_3$ and $\mathcal{P}_2 = \mathcal{P}_4$.
3. $xS^1 = yS^1 \Rightarrow \mathcal{P}_1 = \mathcal{P}_3$ and $\mathcal{P}_2 \simeq \mathcal{P}_4$.
4. $S^1x = S^1y \Rightarrow \mathcal{P}_1 \simeq \mathcal{P}_3$ and $\mathcal{P}_2 = \mathcal{P}_4$.

Note that any reverses of the implications above are not necessarily true.

**PROPOSITION 3.1** The following conditions are equivalent.

1. $H$ is an $\mathcal{H}$-class and a group.
2. $H = \text{Aut}(\mathcal{P})$ for some Petri net $\mathcal{P}$.

Proof) (1)$\Rightarrow$(2) By FACT2, $H$ contains an idempotent $e$, that is $e^2 = e$. This implies $\text{Dom}(e) = \text{Im}(e) = \mathcal{P}$ for some Petri net $\mathcal{P}$. By (3) and (4) of LEMMA 3.1, $\text{Dom}(x) = \text{Dom}(e) = \mathcal{P}$ and $\text{Im}(x) = \text{Im}(e) = \mathcal{P}$ for any $x \in H$ because $xS^1 = eS^1$ and $S^1x = S^1e$ hold. Therefore each element of $H$ is an automorphism of $\mathcal{P}$. Conversely, for an automorphism $x$ of $\mathcal{P}$, $x \in H$ because $x$ is a surjective morphism with $\text{Dom}(x) = \text{Im}(x) = \mathcal{P}$. Hence we have $H = \text{Aut}(\mathcal{P})$.

(2)$\Rightarrow$(1) For $x, y \in H = \text{Aut}(\mathcal{P})$, there exists $z \in H$ such that $x = yz$ and $wx = y$. This implies $S^1x = S^1y$. Similarly we have $xS^1 = yS^1$. Therefore $x \mathcal{H} y$. Conversely, $x \mathcal{H} y$ and $x \in H$ implies $y \in H$ because $y$ is a surjective morphism with $\text{Dom}(y) = \text{Im}(y) = \mathcal{P}$. Hence $H$ is an $\mathcal{H}$-class and a group.

**PROPOSITION 3.2** On the monoid $S^1$, $\mathcal{J} = \mathcal{D}$.

Proof) Since $\mathcal{D} \subseteq \mathcal{J}$ holds, it is enough to show the reverse inclusion.

$$x\mathcal{J}y \iff S^1xS^1 = S^1yS^1 \iff \exists u, v, z, w \in S^1 (x = uyv, y = zyw)$$

implies that $x = uzxvw, y = zuyvw$. Setting $P = \text{Dom}(x), Q = \text{Dom}(y), R = \text{Im}(x)$ and $S = \text{Im}(y), uz : P \to P, zu : Q \to Q, wv : R \to R, vw : S \to S$ are automorphisms. This implies that $u, v, z, w$ are isomorphisms and $u^{-1} = z, v^{-1} = w$. Let $t = xw$. Then,

$$\begin{align*}
x &= x(ww^{-1}) = (xw)w^{-1} = tw^{-1} \\
y &= z(xw) = zt \\
t &= (z^{-1}z)t = z^{-1}(zt) = z^{-1}y
\end{align*}$$

Therefore $xS^1 = tS^1$ and $S^1t = S^1y$, that is, $x\mathcal{R}t\mathcal{L}y$. Thus $\mathcal{D} \subseteq \mathcal{J}$. 

\begin{itemize}
\item[\text{Lemma 3.1}] Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in S^1$. Then,
\item[\text{FACT 1}] The following relations are true.
\item[\text{FACT 2}] An $\mathcal{H}$-class of a monoid $M$ is a group if and only if it contains an idempotent.
\item[\text{PROPOSITION 3.1}] The following conditions are equivalent.
\item[\text{PROPOSITION 3.2}] On the monoid $S^1$, $\mathcal{J} = \mathcal{D}$.
\end{itemize}
3.2 Intersection of principal ideals

The aim here is that for given $x, y \in S^1$ we find a elements $z$ such that $S^1x \cap S^1y = S^1z$ (resp. $xS^1 \cap yS^1 = zS^1$). $xS^1 \cap yS^1 = \{0\}$ (resp. $S^1x \cap S^1y = \{0\}$) is a trivial case ($z = 0$). We should only consider the non-trivial case.

**Lemma 3.2** Let $P_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets, $x = (f_i, (\alpha, \beta_i)) : P_1 \rightarrow P_3$, $y = (g, (\gamma, \delta)) : P_2 \rightarrow P_3$ be elements of $S^1$. If $|\alpha^{-1}(p)| \leq |\gamma^{-1}(p)|$ and $|\beta^{-1}(t)| \leq |\delta^{-1}(s)|$ for any $p \in P_3$ and $t \in T_3$, then $S^1y \subset S^1x$.

**Proof** By the assumption, we can choose arbitrary surjective morphisms $\xi : P_2 \rightarrow P_1$ and $\eta : T_2 \rightarrow T_1$ such that $\xi(\gamma^{-1}(p)) = \alpha^{-1}(p)$ for any $p \in P_3$ and $\eta(\delta^{-1}(s)) = \beta^{-1}(t)$ for any $t \in T_3$. $h : P_2 \rightarrow Q_+$ is defined by $h(q) = g^{-1}(q)f(\xi(q))$ for each $q \in P_2$. Then, we can verify that $z = (h, (\xi, \eta))$ is a surjective morphism from $P_2$ to $P_1$ and thus $z \in S^1$, $y = zx$. Therefore $S^1y \subset S^1x$.

**Lemma 3.3** Let $P_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets, $x = (f_i, (\alpha, \beta_i)) : P_0 \rightarrow P_1$, $y = (g, (\gamma, \delta)) : P_0 \rightarrow P_2$ be elements of $S^1$. If for any $p \in P_1$ and $t \in T_1$, there exist $q \in P_2$ and $s \in T_2$ such that $\alpha^{-1}(p) \subset \gamma^{-1}(q)$ and $\beta^{-1}(t) \subset \delta^{-1}(s)$, then $yS^1 \subset xS^1$.

**Proof** Let $p$ and $t$ be arbitrary elements of $P_1$ and $T_1$, respectively. By the assumption, $q \in P_2$ and $s \in T_2$ is uniquely defined and

$$\alpha^{-1}(p) = \{p_1, p_2, \ldots, p_n\} \subset \gamma^{-1}(q),$$

$$\beta^{-1}(t) = \{t_1, t_2, \ldots, t_m\} \subset \delta^{-1}(s).$$

Then we can easily check that

$$\mu_0(q) = g(p_i)f^{-1}(p_i)\mu_1(p),$$

$$W_2(q, s) = g(p_i)f^{-1}(p_i)W_1(p, t_j)$$

for any $i (1 \leq i \leq n)$ and any $f (1 \leq f \leq m)$. Since the values of $g(p_i)f^{-1}(p_i)$ are the same rational number determined only depending on $p \in P_1$, the maps

$$\xi : P_1 \rightarrow P_2, \ p \mapsto q,$$

$$\eta : T_1 \rightarrow T_2, \ t \mapsto s,$$

$$h : P_1 \rightarrow Q_+, \ p \mapsto g(p_i)f^{-1}(p_i),$$

are well-defined. Therefore we have $z = (h, (\xi, \eta)) \in S^1$ and thus $y = zx$, that is, $yS^1 \subset xS^1$.

**Proposition 3.3 (Intersection of Principal Left Ideals)** Let $P_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets, $x = (f_1, (\alpha_1, \beta_1)) : P_1 \rightarrow P_3$, $y = (f_2, (\alpha_2, \beta_2)) : P_2 \rightarrow P_3$ be elements of $S^1$, $P_3 = \{c_1, c_2, \ldots, c_N\}$ and $T_3 = \{d_1, d_2, \ldots, d_M\}$.

$$n_i = \max(|\alpha_1^{-1}(c_i)|, |\alpha_2^{-1}(c_i)|) \text{ for each } i = 1, 2, \ldots, N,$$

$$m_j = \max(|\beta_1^{-1}(d_j)|, |\beta_2^{-1}(d_j)|) \text{ for each } j = 1, 2, \ldots, M.$$  

Taking disjoint sets $C_1, C_2, \ldots, C_N$ and $D_1, D_2, \ldots, D_M$ with their sizes $|C_i| = n_i (i = 1, 2, \ldots, N)$ and $|D_j| = m_j (j = 1, 2, \ldots, M)$, we define a Petri net $\mathcal{P} = (P, T, W, \mu)$, where $P = \bigcup_{1 \leq i \leq N} C_i$, $T = \bigcup_{1 \leq j \leq M} D_j$, and for any $p \in P$ and $t \in T$,

$$W(p, t) = W_3(c_i, d_j) \text{ if } (p, t) \in C_i \times D_j,$$

$$W(t, p) = W_3(d_j, c_i) \text{ if } (t, p) \in D_j \times C_i,$$

$$\mu(p) = \mu_3(c_i) \text{ if } p \in C_i,$$

Then, $z = (1_{\mathcal{P}}, (\gamma, \delta)) : \mathcal{P} \rightarrow \mathcal{P}_3$, where $\gamma : C_i \ni p \mapsto c_i$ and $\delta : D_j \ni t \mapsto d_j$ are surjective morphisms. Moreover, $S^1x \cap S^1y = S^1z$.

**Proof** Let we can easily check that $z = ux = vy$ for some $u, v \in S^1$. Therefore $z \in S^1x \cap S^1y$.

Conversely, we show that $w = (h, (\xi, \eta)) \in S^1x \cap S^1y$ implies $w \in S^1z$. We can write $w = u'x = v'y$ for some $u', v' \in S^1$. Let $p \in P_3$. In our construction, $|\gamma^{-1}(p)| = \max(|\alpha_1^{-1}(p)|, |\alpha_2^{-1}(p)|)$. Since $w = u'x = v'y$ holds, we have $|\alpha_1^{-1}(p)| \leq |\gamma^{-1}(p)|$ and $|\alpha_2^{-1}(p)| \leq |\gamma^{-1}(p)|$ and thus $|\gamma^{-1}(p)| \leq |\xi^{-1}(p)|$ and $|\gamma^{-1}(p)| \leq |\eta^{-1}(p)|$. Similarly, $|\delta^{-1}(p)| \leq |\gamma^{-1}(p)|$. By **Lemma 3.2**, we conclude $S^1x \cap S^1y = S^1z$.  

\[\blacksquare\]
COROLLARY 3.1 (Diamond Property I) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. Then there exists a Petri net $\mathcal{P}_0$ such that $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$.

We consider the intersection of two principal right ideals. The case of principal right ideals is rather difficult compared to that of principal left ideals. We begin with an introduction of the relation $=_{f}$.

Let $P$ be a set and $f, g$ maps whose domain is $P$. The relation $=_{f}$ on $P$ defined by $(\forall x, y \in P)\{x =_{f} y \iff f(x) = f(y)\}$. Then $(=_{f} \cup =_{g})^{*}$ is the smallest equivalence relation on $P$ which includes both $=_{f}$ and $=_{g}$, where $(=_{f} \cup =_{g})^{*}$ is the reflexive and transitive closure of $=_{f} \cup =_{g}$.

PROPOSITION 3.4 (Intersection of Principal Right Ideals) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets, $x = (f_1, (\alpha_1, \beta_1)) : P_1 \rightarrow P_3$, $y = (f_2, (\alpha_2, \beta_2)) : P_2 \rightarrow P_3$ be elements of $S^1$. Let $C_1, C_2, \ldots, C_N$ be all the $(=_{\alpha_1} \cup =_{\alpha_2})^{*}$-classes in $P_0$ and $D_1, D_2, \ldots, D_M$ be all the $(=_{\beta_1} \cup =_{\beta_2})^{*}$-classes in $T_0$.

$f \in Q_+^P$ is defined by if $p$ is 0-isolated then $f(p) = 1$ and otherwise

$$f(p) = 1/\gcd(\{\mu(p), W_0(p, t_i), W_0(t_i, p) \mid 1 \leq i \leq n\})$$

where $n = |T_0|$ and $T_0 = \{t_1, t_2, \ldots, t_n\}$ and $\gcd(S)$ denotes the greatest common divisor of all integers in a set $S$.

(1) A Petri net $\mathcal{P}_3 = (P_3, T_3, W_3, \mu_3)$ can be constructed in the following way:

$$\mathcal{P}_3 = P_0/(=_{\alpha_1} \cup =_{\alpha_2})^{*} = \{C_1, C_2, \ldots, C_N\},$$

$$\mathcal{T}_3 = T_0/(=_{\beta_1} \cup =_{\beta_2})^{*} = \{D_1, D_2, \ldots, D_M\}.$$

For $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, M\},$

$$\mu_3(C_i) = f(p)\mu_0(p) \quad \text{for any } p \in C_i,$$

$$W_3(C_i, D_j) = f(p)W_0(p, t) \quad \text{for any } p \in C_i, t \in D_j,$$

$$W_3(D_j, C_i) = f(p)W_0(t, q) \quad \text{for any } p \in C_i, t \in D_j$$

are well-defined.

(2) Let $z = (f, (\alpha, \beta)) : \mathcal{P}_0 \rightarrow \mathcal{P}_3$, where $\alpha$ is the canonical surjection from $P_0$ onto $P_3$, $\beta$ is the canonical surjection from $T_0$ onto $T_3$. Then, $z$ is a surjective morphism and $zS^1 \cap yS^1 = zS^1$.

Proof) Omitted.

The above-mentioned proposition immediately leads the following corollary.

COROLLARY 3.2 (Diamond Property II) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$.

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$.

DEFINITION 3.1 (Irreducible) A Petri net $\mathcal{P}$ is called $\sqsupseteq$-irreducible if $\mathcal{P} \sqsupseteq \mathcal{P}'$ implies $\mathcal{P} \simeq \mathcal{P}'$ for any Petri net $\mathcal{P}'$. Then $\mathcal{P}$ is called an $\sqsupseteq$-irreducible form.

COROLLARY 3.3 Let $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. Then one has: If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$.

Proof) Trivial by COROLLARY 3.2 and the definition of $\sqsupseteq$-irreducibility.
4 Surjective Morphisms and Petri Net Languages

4.1 Behavior of Petri Nets

The behavior of many systems can be described in terms of system states and their changes. In order to simulate the dynamic behavior of a system, a state or marking in a Petri net $\mathcal{P} = (P, T, W, \mu)$ is changed according to the following transition (firing) rule:

1. A transition $t \in T$ is said to be enabled (under the marking $\mu$ or under the Petri net $\mathcal{P}$) if $W(p, t) \leq \mu(p)$ for every place $p \in P$, where $W(p, t)$ is the weight of the arc from $p$ to $t$. Then each input place $p$ of $t$ is marked with at least $W(p, t)$ tokens. An enabled transition may or may not fire (depending on whether or not the event actually takes place).

2. A firing of an enabled transition $t$ removes $W(p, t)$ tokens from each input place $p$ of $t$, and adds $W(t, p)$ tokens to each output place $p$ of $t$. As a consequence of the firing, the current marking $\mu$ is replaced with the following new marking $\mu'$:

\[
\mu'(p) = \mu(p) - W(p, t) + W(t, p) \quad \text{for } \forall p \in P.
\] (4.1)

Then we define the transition function $\delta_p$ by $\delta_p(\mu, t) = \mu'$.

3. A sequence $w = t_1t_2\ldots t_n$ of transitions is said to be a firing sequence in a Petri net $\mathcal{P} = (P, T, W, \mu)$ if $\mu_0 = \mu$, $\mu_i = \delta_p(\mu_{i-1}, t_i)$ for each $i (1 \leq i \leq n)$. Then $\mu'$ is called a reachable from $\mathcal{P}$, and we extend $\delta_p$ from $T$ to $T^*$ by $\delta_p(\mu, w) = \mu'$. By assuming that $\delta_p(\mu, w) = \perp$ if $w$ is not a firing sequence from $\mathcal{P}$ or $\mu = \perp$, the transition function $\delta_p : (N_0^P \cup \\{\perp\}) \times T^* \rightarrow (N_0^P \cup \{\perp\})$ is regarded as a total function. The set of all reachable markings from $\mathcal{P}$ is called the reachability set of $\mathcal{P}$, denoted by $R(\mathcal{P})$.

**Lemma 4.1** Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i) (i = 1, 2)$ be Petri nets. $(f, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,

1. $t \in T_1$ is enable in $\mathcal{P}_1$ $\iff$ $\beta(t) \in T_2$ is enable in $\mathcal{P}_2$. More precisely,

\[
\mu'_1 = \delta_{\mathcal{P}_1}(\mu_1, t) (\neq \perp) \iff \mu'_2 = \delta_{\mathcal{P}_2}(\mu_2, \beta(t)) (\neq \perp),
\]

\[
f \otimes \mu_1 = \alpha \mu_2 \quad \text{and} \quad f \otimes \mu'_1 = \alpha \mu'_2 \quad \text{hold.}
\]

2. $w$ is a firing sequence in $\mathcal{P}_1$ $\iff$ $\beta(w)$ is a firing sequence in $\mathcal{P}_2$. More precisely,

\[
\mu'_1 = \delta_{\mathcal{P}_1}(\mu_1, w) (\neq \perp) \iff \mu'_2 = \delta_{\mathcal{P}_2}(\mu_2, \beta(w)) (\neq \perp),
\] (4.2)

\[
f \otimes \mu_1 = \alpha \mu_2 \quad \text{and} \quad f \otimes \mu'_1 = \alpha \mu'_2 \quad \text{hold.}
\]

**Proof** (1) For each $p \in P_1$,

\[
\mu_2(\alpha(p)) - W_2(\alpha(p), \beta(t)) = f(p)\{\mu_1(p) - W_1(p, t)\} \quad \text{and} \quad f(p) > 0.
\]

Therefore, if $\beta(t)$ is enabled in $\mathcal{P}_2$, then $t$ is enabled in $\mathcal{P}_1$. Conversely, since $\alpha$ is surjective, $\beta(t)$ is enabled in $\mathcal{P}_2$ if $t$ is enabled in $\mathcal{P}_1$.

In addition, the equation $\mu'_2(\alpha(p)) = \mu_2(\alpha(p)) - W_2(\alpha(p), \beta(t)) + W_2(\beta(p), \alpha(t)) = f(p)\{\mu_1(p) - W_1(p, t) + W_1(t, p)\} = f(p)\mu'_1(p)$ leads the equivalence of the two firing rules shown in (1).

(2) It is trivial by (1) and the definition of a firing sequence. □

**Lemma 4.2** Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i) (i = 1, 2)$ be Petri nets. $(f, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,

1. $\varphi : R(\mathcal{P}_1) \rightarrow R(\mathcal{P}_2), \mu'_1 \mapsto \mu'_2$, where $\mu'_1$ and $\mu'_2$ are markings satisfying (4.2), is a bijection.

2. Let $R_i \subset R(\mathcal{P}_i)$ with $\varphi(R_1) = R_2$ and $K_i = \{w \in T^*_i | \delta_{\mathcal{P}_i}(\mu_i, w) \in R_i\} (i = 1, 2)$. Then $K_2 = \beta(K_1)$. 


Proof (1) $\varphi$ is well-defined. Indeed, for any $\mu'_1 \in R(\mathcal{P}_1)$, there exists at least one marking $\mu'_2 \in R(\mathcal{P}_2)$ such that $f \otimes \mu'_1 = \alpha \mu'_2$ by LEMMA 4.1 (2). Moreover if any two marking $\mu'_2, \mu''_2 \in R(\mathcal{P}_2)$ satisfy $f \otimes \mu'_1 = \alpha \mu'_2 = \alpha \mu''_2$ then we have $\mu'_2 = \mu''_2$ because $\alpha$ is surjective.

Next we show that $\varphi$ is surjective. Let $\mu'_2 \in R(\mathcal{P}_2)$. Since $\beta$ is surjective, by LEMMA 4.1 (2), there exists $w \in T^*_1$ such that $\mu'_1 = \delta_{\mathcal{P}_1}(\mu_1, w)$ and $\mu'_2 = \delta_{\mathcal{P}_2}(\mu_2, \beta(w))$. Then $\varphi(\mu'_1) = \mu'_2$.

Finally we show that $\varphi$ is injective. Suppose that $\varphi(\mu'_1) = \varphi(\mu''_1) = \mu'_2$. $f \otimes \mu'_1 = f \otimes \mu''_1 = \alpha \mu'_2$. By LEMMA 2.1, $\mu'_1 = (f^{-1} f) \otimes \mu'_1 = (f^{-1} f) \otimes \mu''_1 = \mu'_2$.

(2) Let $w \in K_1$ with $\delta_{\mathcal{P}_1}(\mu_1, w) = \mu'_1 \in R_1$. Then $\delta_{\mathcal{P}_2}(\mu_2, \beta(w)) = \mu'_2 = \varphi(\mu'_1) \in R_2$. Therefore $\beta(w) \in K_2$.

Conversely let $w \in K_2$ with $\delta_{\mathcal{P}_2}(\mu_2, w) = \mu'_2 \in R_2$. Since $\beta$ is surjective, $w = \beta(u)$ for some $u \in T^*_1$. $\delta_{\mathcal{P}_1}(\mu_1, u) = \mu'_1 = \varphi^{-1}(\mu'_2) \in R_1$. Therefore $w = \beta(u) \in \beta(K_1)$.

\[
\square
\]

4.2 Petri net Languages

Let $\mathcal{P} = (P, T, W, \mu_0)$ be a Petri net, $\Sigma$ be an alphabet, $\sigma : T \to \Sigma$ be a labeling of the transitions and $F \subseteq N_0^P$ be a finite set of final markings. Then we define the languages $\mathcal{L}_L(\mathcal{P}, \sigma, F)$, $\mathcal{L}_G(\mathcal{P}, \sigma, F)$, $\mathcal{L}_T(\mathcal{P}, \sigma)$ and $\mathcal{L}_P(\mathcal{P}, \sigma)$ as follows:

\[
\begin{align*}
\mathcal{L}_L(\mathcal{P}, \sigma, F) &= \{ \sigma(w) | w \in T^*, \mu = \delta_{\mathcal{P}}(\mu_0, w) \text{ and } \mu \in F \}, \\
\mathcal{L}_G(\mathcal{P}, \sigma, F) &= \{ \sigma(w) | w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \geq \mu_f \text{ for some } \mu_f \in F \}, \\
\mathcal{L}_T(\mathcal{P}, \sigma) &= \{ \sigma(w) | w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp \} \\
\mathcal{L}_P(\mathcal{P}, \sigma) &= \{ \sigma(w) | w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp \}.
\end{align*}
\]

Languages $\mathcal{L}_L(\mathcal{P}, \sigma, F)$, $\mathcal{L}_G(\mathcal{P}, \sigma, F)$, $\mathcal{L}_T(\mathcal{P}, \sigma)$ and $\mathcal{L}_P(\mathcal{P}, \sigma)$ for some Petri net $\mathcal{P}$, some labeling $\sigma'$ and some set $F$ of markings are called $L$-type, $G$-type, $T$-type and $P$-type Petri net languages respectively.

PROPOSITION 4.1 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_0)$ $(i = 1, 2)$ be Petri nets. $(f, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$.

For any $L_1 = \mathcal{L}_X(\mathcal{P}_1, \sigma_1, F_1), X \in \{ L, G \}$ (resp. $L_1 = \mathcal{L}_X(\mathcal{P}_1, \sigma_1), X \in \{ T, P \}$), there exists some $L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2, F_2)$ (resp. $L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2)$) such that $L_1 = \sigma_1(\beta^{-1}(\sigma_2^{-1}(L_2)))$. Then $L_1$ is regular (resp. linear, context-free) if and only if $L_2$ is regular (resp. linear, context-free).

Proof We only show the case of $X = L$. The remainder of proof is done in a similar way.

Putting $\sigma_2 = 1_{T_2}, R_1 = F_1 \cap R(\mathcal{P}_1), F_2 = \varphi(R_1)$ and $K_i = \{ w \in T_i | \delta_{\mathcal{P}_i}(\mu_i, w) \in R_i \}$ $(i = 1, 2)$, where $1_{T_2}$ is the identity map on $T_2$ and $\varphi$ is the bijection defined in LEMMA 4.2. Then we have $L_1 = \sigma_1(K_1), L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2, F_2) = \sigma_2(K_2)$, and by LEMMA 4.2 (2) $K_2 = \beta(K_1)$. Therefore $L_1 = \sigma_1(\beta^{-1}(\sigma_2^{-1}(L_2)))$.

Regarding operations with languages, the families of regular, linear and context-free languages are closed under the morphism and inverse morphism operations respectively. This leads to the equivalence condition.

\[
\square
\]

5 Conclusions

In this paper we introduced Petri net morphisms/automorphism based on similarity of places and transition. Some algebraic properties related to them were investigated. We first considered Green's relations and ideals in the monoids $S^1$ of morphisms of Petri nets, which is adjoined the extra zero $0$ and the extra identity 1. For two given monoids, the principal left (resp. right) ideal of them is also a principal left (resp. right) ideal. This implies two kinds of diamond properties (confluencies) with respect to that the pre-order induced by surjective morphisms. It is technically interesting to construct such two kinds of synthesis of Petri nets. Next, the automorphism group $G = \text{Aut}(\mathcal{P})$ of a givne Petri net $\mathcal{P}$ was investigated. It is closely related to the symmetric groups preserves the partition determined by the equivalence relation of similiarity on $\mathcal{P}$. By using this property, we can achieve the decomposition of $G$ into a redundant part $N$ and the other
The similarity can be described in term of a surjective morphism onto an irreducible Petri net. Finally two Petri nets ordered by a surjective morphism have isomorphic reachability sets. Thus, the languages generated by them have a close correspondence.

Here we did not investigate problems, for example, whether the principal (two-sided) ideal of them is also a principal ideal in $S^1$, whether an arbitrary left (resp. right, two-sided) ideal is principal in $S^1$. Also we wonder whether the Petri nets with the same irreducible form constitute a lattice with respect to the order or not. In addition to these problems, we started investigating the application of Petri net morphism/automorphism to formal languages and codes. We will apply these results to famous and basic decision problems related to Petri nets.

References