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Filter theory of non-commutative residuated lattices

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概 要

In this paper we show that for any non-commutative residuated lattice X and a filter F of X,

(a) F is an implicative filter $\Longleftrightarrow$ X/F is a Heyting algebra;

(b) F is a positive implicative filter $\Longleftrightarrow$ X/F X/F is a Boolean algebra;

(c) If F is normal, then it is a fantastic filter $\Longleftrightarrow$ X/F is a pseudo-MV algebra.

Moreover, every implicative filter is normal.

1 はじめに

In research of logics, theory of filters plays a very important role in proving completeness with respect to algebraic semantics. For example, in the case of the classical propositional logic (CPL), we can show the completeness theorem of the logic by Boolean algebras. To do so we use the Lindenbaum-Tarski algebra of CPL, which is a quotient algebra by theories, or equivalently, by filters. The Lindenbaum-Tarski method can be applied to many logics including many-valued logics and fuzzy logics BL, MV, MTL etc.

Recently, the research of filter theory heads to non-commutative residuated algebras, for example, pseudo-MV algebras ([1]), pseudo-BL algebras ([6, 7]), pseudo-Rℓ monoid ([8]) and so on. Here we extend such filter theory to more general case, namely, we develop the filter theory of non-commutative residuated lattices. We note that the class of all non-commutative residuated lattices properly contains the classes of all algebras above. Thus our results hold in all algebras above.

In this short paper we prove that for any non-commutative residuated lattice X and a filter F of X,

(a) F is an implicative filter f and $\Longleftrightarrow$ X/F is a Heyting algebra;

(b) F is a positive implicative filter $\Longleftrightarrow$ X/F is a Boolean algebra;

(c) If F is a normal filter, then F is a fantastic filter $\Longleftrightarrow$ X/F is a pseudo-MV algebra.

This generalizes the result proved in [8], where it is verified that the similar results hold under the extra condition of normality on Rℓ-monoids (which are axiomatic extension of residuated lattices).

2 準備

We define non-commutative residuated lattices and some types of filters according to [4, 5, 8]. An algebraic structure $(X, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ is called a *non-commutative residuated lattice* (simply called residuated lattice or RL in this paper) if it satisfies the following conditions

(C1) $(X, \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(X, \odot, 1)$ is a monoid with a unit 1;

(C3) For all $x, y, z \in X$, we have

$$x \odot y \leq z \iff x \leq y \rightarrow z \iff y \leq x \rightsquigarrow z.$$

By a Rℓ-monoid ([8]), we mean a residuated lattice which satisfies the *divisibility* condition

$$\text{(div)} : (x \rightarrow y) \odot x = x \wedge y = x \odot (x \rightsquigarrow y).$$

The following result is well known ([3, 8]).

命題 1. *For all $x, y, z \in X$, we have*

(1) $x \leq y \iff x \rightarrow y = 1 \iff x \rightsquigarrow y = 1$

(2) $x \odot (x \rightsquigarrow y) \leq y$, $(x \rightarrow y) \odot x \leq y$

(3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$

(4) $x \le y \implies x \odot z \le y \odot z,\ z \odot x \le z \odot y$

(5) $x \le y \implies z \to x \le z \to y,\ y \to z \le x \to z$

(6) $x \le y \implies z \rightsquigarrow x \le z \rightsquigarrow y,\ y \rightsquigarrow z \le x \rightsquigarrow z$

(7) $x \to y \le (y \to z) \rightsquigarrow (x \to z),\ x \rightsquigarrow y \le (y \rightsquigarrow z) \to (x \rightsquigarrow z)$

(8) $x \to y \le (z \to x) \to (z \to y),\ x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$

A subset F of A is called a *filter* of a residuated lattice X ([4, 5, 8]) if it satisfies

(F0) $1 \in F$.

(F1) $x, y \in F$ implies $x \odot y \in F$.

(F2) $x \in F$ and $x \le y$ imply $y \in F$.

It is easy to prove that for a subset $F \subseteq X$, F is a filter if and only it is a *deductive system* defined in [9], that is,

(DS1) $1 \in F$ and

(DS2) If $x \in F$ and $x \to y \in F$ then $y \in F$

(DS2)' If $x \in F$ and $x \rightsquigarrow y \in F$ then $y \in F$

For a filter F, it is called *normal* when $x \to y \in F$ if and only if $x \rightsquigarrow y \in F$ for all $x, y \in X$. For any normal filter F of X, a relation $\equiv_F$ on X defined by

$$x \equiv_F y \iff x \to y, y \to x \in F \text{ (or equivalently } x \rightsquigarrow y, y \rightsquigarrow x \in F)$$

is a congruence on X and, since the class $\mathcal{RL}$ of all (non-commutative) residuated lattices is a variety, a quotient structure $X/F = \{x/F \mid x \in X\}$ by $\equiv_F$ is also a residuated lattice.

We use the same terminology about definitions of some types of filters according to [8]. Let X be a residuated lattice. A subset $F \subseteq X$ is called an *implicative filter* if it satisfies

(F0) $1 \in F$,

(I) $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$ imply $x \to z \in F$, and

(I)' $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$ imply $x \rightsquigarrow z \in F$.

Also a subset F is called a *positive implicative filter* if

(F0) $1 \in F$,

(PI) $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$ imply $y \in F$, and

(PI)' $x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F$ and $x \in F$ imply $y \in F$.

Lastly, a filter F is said to be *fantastic* if

(FF) $y \to x \in F$ implies $((x \to y) \rightsquigarrow y) \to x \in F$ and

(FF)' $y \rightsquigarrow x \in F$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F$.

3 各種 filter の特徴付け

In this section we give simple characterizations of filters defined in the previous section. First of all we treat implicative filters of residuated lattices. For any Rℓ-monoid M, it is proved that

命題 2 (Theorem 3.4, 3.5 in [8]). *Let F be a* normal *filter of an Rℓ-monoid M. Then F is implicative if and only if $x \to x^2 \in F$ for any $x \in M$.*

If F is a normal *filter of an Rℓ-monoid M then F is an implicative filter of M if and only if the quotient Rℓ-monoid M/F is a Heyting algebra.*

We should note that every Rℓ-monoid is an residuated lattice with meeting the divisibility condition $(x \to y) \odot x = x \wedge y = x \odot (x \rightsquigarrow y)$. Further the result above requires that the filter is normal.

命題 3. *Let X be a residuated lattice and F be a filter of X. If F satisfies the condition*

$$(I) : x \to x^2 \in F \text{ and } x \rightsquigarrow x^2 \in F \text{ for all } x \in X,$$

then we have $x \odot y \to y \odot x \in F$ and $x \odot y \rightsquigarrow y \odot x \in F$ for all $x, y \in X$.

We see that the following result holds (c.f.[4, 5, 8]).

定理 1. *For any residuated lattice X and filter F, F is an implicative filter if and only if $x \to x^2 \in F$ and $x \rightsquigarrow x^2 \in F$ for all $x \in X$.*

It follows from the above that every implicative filter is normal.

定理 2. *Every implicative filter is normal in any residuated lattice.*

Proof. Let F be an implicative filter and $x \to y \in F$. Since $x \odot (x \to y) \rightsquigarrow (x \to y) \odot x \in F$ and $(x \to y) \odot x \rightsquigarrow y = 1 \in F$ by $(x \to y) \odot x \leq y$, we get $x \odot (x \to y) \rightsquigarrow y = (x \to y) \rightsquigarrow (x \rightsquigarrow y) \in F$. It follows from $x \to y \in F$ that $x \rightsquigarrow y \in F$. The converse is similar. □

It also follows from the results in [4, 5, 8] that

定理 3. *Let X be a residuated lattice and F a filter of X. Then F is an implicative filter if and only if X/F is an Heyting algebra.*

We note that this is a generalization of the result in [8], where the similar result is proved under the conditions of divisibility $(x \to y) \odot x = x \wedge y = x \odot (x \rightsquigarrow y)$ and of normality.

Next, we consider the case of positive implicative filters. It is clear that

命題 4. *Every positive implicative filter is an implicative filter, thus positive implicative filters are normal.*

The following gives a characterization of positive implicative filters.

定理 4. *For a filter F, the following conditions are equivalent.*

(1) F is a positive implicative filter.
(2) If $(x \to y) \rightsquigarrow x \in F$ then $x \in F$ and if $(x \rightsquigarrow y) \to x \in F$ then $x \in F$.
(3) $((x \to y) \rightsquigarrow x) \to x \in F$ and $((x \rightsquigarrow y) \to x) \rightsquigarrow x \in F$.
(4) $(x^- \rightsquigarrow x) \to x \in F$ and $(x^\sim \to x) \rightsquigarrow x \in F$.
(5) F is a Boolean filter, that is, $x \vee x^- \in F$ and $x \vee x^\sim \in F$.

Proof. See [4, 5, 8]. □

Since every positive implicative filter is normal, we can construct the quotient residuated lattice X/F by the positive implicative filter F.

命題 5. *Let F be a filter of a residuated lattice X. F is a positive implicative filter if and only if X/F is a Boolean algebra.*

Lastly, according to [8], we define fantastic filters. A filter F of a residuated lattice X is called a *fantastic filter* if it satisfies

(FF) $y \to x \in F$ implies $((x \to y) \rightsquigarrow y) \to x \in F$ and

(FF)' $y \rightsquigarrow x \in F$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F$.

We give a simple characterization of fantastic filters.

補題 1. *Let F be a filter. Then F is a fantastic filter if and only if*

$$((x \to y) \rightsquigarrow y) \to x \vee y \in F \text{ and } ((x \rightsquigarrow y) \to y) \rightsquigarrow x \vee y \in F.$$

Proof. It follows from $y \to x \vee y = 1 \in F$ that $((x \vee y \to y) \rightsquigarrow y) \to x \vee y \in F$. Since $x \vee y \to y = (x \to y) \wedge (y \to y) = x \to y$, we have $((x \to y) \rightsquigarrow y) \to x \vee y \in F$. Similarly, $((x \rightsquigarrow y) \to y) \rightsquigarrow x \vee y \in F$.

Conversely, suppose that $y \to x \in F$. Since $x \vee y \to x = y \to x \in F$ and $((x \to y) \rightsquigarrow y) \to x \vee y \in F$, we get that $((x \to y) \rightsquigarrow y) \to x \in F$. The other case can be proved similarly. □

定理 5. *For a filter F, F is a fantastic filter if and only if X/F is a pseudo-MV algebra.*

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