数理解析研究所講究録
第 1810 巻 2012 年 113-138

The symmetry breaking of the non-critical Caffarelli-Kohn-Nirenberg type inequalities by a linearization method

Toshio Horiuchi

Faculty of Science Ibaraki University
310-8512 Bunkyou 2-1-1 Mito Ibaraki Japan
E-mail: horiuchi@mx.ibaraki.ac.jp

Abstract

この講究録では、Caffarelli-Khon-Nirenberg 型の不等式においてパラメータ $\gamma \to \pm\infty$ のとき対称性の破れが実際におこる事の解説をする。ここで用いられる手法は、非線形退化型作用素に対する線形化法である。ここでは論文 [HK3] において発表予定である最新の Caffarelli-Khon-Nirenberg 型の不等式に関する結果を紹介し、それらに基づき対称性の破れを主要定理の証明の概略を含め考察する。

The main purpose of this article is to show that the symmetry breaking actually occurs in the CKN-type inequalities provided that the parameter $|\gamma|$ is large enough. In the argument we employ the so-called linearization method for the variational problems of the CKN type inequalities. First we shall explain recent results on the CKN-type inequalities for all $\gamma \in \mathbf{R}$ in the fore-coming paper [HK3] as a necessary back-ground for this research and we shall give a sketch of proof of the main result.

Contents

1 Introduction

We start with introducing the CKN-type inequalities according to the paper [HK3]. In the CKN-type inequalities, we work with parameters p, q and γ whose ranges consist of

$$1 \le p \le q < \infty, \quad (0 \le)\, \tau_{p,q} = \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}, \quad \gamma \in \mathbf{R} \setminus \{0\}. \tag{1.1}$$

From these conditions we obtain for a fixed p

$$p \le q \le p^* = \frac{np}{n-p} \ \text{ if } 1 \le p < n \ ; \ p \le q < p^* = \infty \ \text{ if } n \le p < \infty. \tag{1.2}$$

Here

$$p' = \frac{p}{p-1}, \ p^* = \frac{np}{(n-p)_+} \ \text{ for } 1 \le p < \infty. \tag{1.3}$$

Here we set $t_+ = \max\{0, t\}$ and $1/0 = \infty$.

The ranges $\gamma > 0$ and $\gamma < 0$ are said to be subcritical and supercritical respectively. The case of $\gamma = 0$ is called critical, and we do not treat it in the present article

Definition 1.1. *For $\alpha \in \mathbf{R}$ and $R \ge 1$ we set*

$$I_\alpha(x) = I_\alpha(|x|) = \frac{1}{|x|^{n-\alpha}} \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}, \tag{1.4}$$

When $0 < \alpha < n$ holds, I_α is called a Riesz kernel of order α.

Under these notations the CKN-type inequality in the non-critical case ($\gamma \neq 0$) has the following form with $S^{p,q;\gamma}$ being the best constant: For any $u \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$,

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \geq S^{p,q;\gamma}\Big(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx\Big)^{p/q}. \tag{1.5}$$

We note that if $\gamma = n/p - 1 = n/q$ holds, then this is called the Sobolev inequality, and if $p = q, \gamma = n/p - 1$, then this is called the Hardy inequality. Here the best constant $S^{p,q;\gamma}$ is given by the variational problem;

$$\inf_{u \in C_c^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}} \frac{\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx}{\Big(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx\Big)^{p/q}}. \tag{1.6}$$

By $S_{\mathrm{rad}}^{p,q;\gamma}$ we denote the best constant in the radially symmetric function space $C_c^\infty(\mathbf{R}^n)_{\mathrm{rad}}$ instead of $C_c^\infty(\mathbf{R}^n)$. For the precise definition, see §1.1. In [HK3] we established the symmetry of the best constants in $\gamma \in \mathbf{R}$ and the radial symmetry of the extremals for small γ among many results. As a necessary back-ground let us pick out them from [HK3] below.

Proposition 1.1. (The symmetry) *Assume that $n \geq 1$, $1 < p \leq q < \infty$ and $\tau_{p,q} \leq 1/n$. Then it holds that:*

1. $S^{p,q;\gamma} = S^{p,q;-\gamma}$, $S_{\mathrm{rad}}^{p,q;\gamma} = S_{\mathrm{rad}}^{p,q;-\gamma}$ *for* $\gamma \neq 0$.
2. $S_{\mathrm{rad}}^{p,q;\gamma} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $\gamma \neq 0$.
3. $S^{p,q;\gamma} = S_{\mathrm{rad}}^{p,q;\gamma} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $0 < |\gamma| \leq \gamma_{p,q}$.
4. $\dfrac{1}{(2-\gamma_{p,p^*}/\gamma)^p} S^{p,p^*;\gamma_{p,p^*}} \leq S^{p,p^*;\gamma} \leq S^{p,p^*;\gamma_{p,p^*}} = S_{\mathrm{rad}}^{p,p^*;\gamma_{p,p^*}}$ *for* $|\gamma| \geq \gamma_{p,p^*} = \dfrac{n-p}{p}$ *if* $p < n$.
5. $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}} = S_{\mathrm{rad}}^{2,2^*;\gamma_{2,2^*}}$ *for* $|\gamma| \geq \gamma_{2,2^*} = \dfrac{n-2}{2}$ *if* $p = 2 < n$.

Here $\gamma_{p,q}$ and $S_{p,q}$ are defined in Definition 3.2.

The assertion 3 means that the best constant $S^{p,q;\gamma}$ is attained by radially symmetric functions if $|\gamma|$ is small. Now we state our main result below which will cover the case that $|\gamma|$ is large. See also Corollary 5.1 in §4.

Theorem 1.1. **(The symmetry breaking)** *Assume that* $1 < p < n$. *Assume that* q *is fixed such as* $p < q < p^*$. *Then for sufficiently large* $|\gamma|$, *the best constant* $S^{p,q;\gamma}$ *is not attained in the radial function space* $W^{1,p}_{\alpha,0}(\mathbf{R}^n)_{\mathrm{rad}}$. *Here the space* $W^{1,p}_{\alpha,0}(\mathbf{R}^n)_{\mathrm{rad}}$ *is defined in Definition 2.1.*

Remark 1.1. *Since it was shown that in [HK3] the best constant* $S^{p,q;\gamma}$ *is attained in* $W^{1,p}_{\alpha,0}(\mathbf{R}^n)$ *for* $p < q < p^*$, *we can conclude by this result that the symmetry breaking actually occurs if* $|\gamma|$ *is large.*

For $p = 2$ and $\gamma > 0$ it was shown in [CW1] that the symmetry breaking occurs by a method of perturbation using eigenfunctions of the linearized operator. When $p \neq 2$ and $\gamma > 0$, this phenomenon was also shown in [BW] by constructing a clever non-symmetric perturbation to the radial extremal function which is supposed to attain the best constant. Our method in the present paper is making effective use of the linearization of quasilinear elliptic operator at a radial extremal. For the semilinear operator, this method was employed in [CW1]. Since in our case the operator is quasilinear, the linearized operator at a radial extremal is degenerated at the origin. We shall overcome this difficulty by using weighted Hardy's inequalities and effective changes of variables. We note that by virtue of this method, a lower estimate of $|\gamma|$ for the symmetry breaking is also given in terms of the first eigenvalues of the linearized operators.

2 The non-critical CKN-type inequalities

In this subsection we shall prepare a general setting for the precise description of the CKN-type inequalities. First we introduce function spaces and relating norms.

Definition 2.1. *Let $1 \le p \le q < \infty$ and $\gamma \in \mathbf{R}$. Let Ω be a domain of $\mathbf{R}^n$ and let $u : \Omega \to \mathbf{R}$.*

1. *For $\delta : \Omega \to \mathbf{R}$ satisfying $\delta \ge 0$ a.e. on Ω, we set*

$$\|u\|_{L^q(\Omega;\delta)} = \left(\int_\Omega |u(x)|^q \delta(x)\,dx\right)^{1/q}. \tag{2.1}$$

2. *Under the above notation we set*

$$\|u\|_{L^q_\gamma(\Omega)} = \|u\|_{L^q(\Omega;I_{q\gamma})}, \quad \|\nabla u\|_{L^p_{1+\gamma}(\Omega)} = \||\nabla u|\|_{L^p_{1+\gamma}(\Omega)} \tag{2.2}$$

and

$$L^q_\gamma(\Omega) = \{u : \Omega \to \mathbf{R} \mid \|u\|_{L^q_\gamma(\Omega)} < \infty\}. \tag{2.3}$$

3. *By $W^{1,p}_{\gamma,0}(\Omega)$ we denote the completion of $C^\infty_{\mathrm{c}}(\Omega\backslash\{0\})$ with respect to the norm*

$$u \mapsto \|\nabla u\|_{L^p_{1+\gamma}(\Omega)}.$$

4. *Let Ω be a radially symmetric domain. For any function space $V(\Omega)$ on Ω, we set*

$$V(\Omega)_{\mathrm{rad}} = \{u \in V(\Omega) \mid u \text{ is radial}\}. \tag{2.4}$$

Then the noncritical CKN-type inequalities are simply represented as follows:

For $\gamma \ne 0$,

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \ge S^{p,q;\gamma} \|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n). \tag{2.5}$$

Remark 2.1. 1. *For $1 < p < \infty$ and $\gamma > 0$, $C^\infty_{\mathrm{c}}(\mathbf{R}^n) \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ and $C^\infty_{\mathrm{c}}(\mathbf{R}^n)$ is densely contained in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)$. When $\gamma < 0$ holds, $C^\infty_{\mathrm{c}}(\mathbf{R}^n) \not\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.*

2. *When $p = q$ holds, this inequality is called the Hardy-Sobolev inequalitiy. It is known that the best constant $S^{p,p;\gamma}$ of (2.5) coincides with the one restricted in the radial functional space $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$, and hence we have*

$$S^{p,p;\gamma} = \gamma^p. \tag{2.6}$$

3. *It follows from the Hardy-Sobolev inequalities that if $\gamma \neq 0$, then the space $W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ coincides with the completion of $C^\infty_c(\mathbf{R}^n \setminus \{0\})$ with respect to the norm*

$$\|u\|_{W^{1,p}_\gamma(\mathbf{R}^n)} = \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u\|_{L^p_\gamma(\mathbf{R}^n)}. \tag{2.7}$$

4. *The classical CKN-type inequalities are often represented in the following way:*

$$\int_{\mathbf{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq C \left(\int_{\mathbf{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{\frac{p}{q}} \text{ for any } u \in C^\infty_c(\mathbf{R}^n),$$

where $1 \leq p \leq q < +\infty, 0 \leq 1/p - 1/q = (1 - a + \beta)/n$ *and* $-n/q < \beta \leq \alpha$. *If we set*

$$\gamma = \alpha - 1 + \frac{n}{p} = \beta + \frac{n}{q},$$

then we have the representations (1.5) and (2.5).

3 Some known results on the noncritical CKN-type inequalities

In this section we describe the results when $\gamma \neq 0$.

Definition 3.1. *Let* $1 \leq p \leq q < \infty$ *and* $\gamma \neq 0$.

1.

$$E^{p,q;\gamma}[u] = \left(\frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\|u\|_{L^q_\gamma(\mathbf{R}^n)}} \right)^p \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}. \tag{3.1}$$

2.

$$\begin{aligned} S^{p,q;\gamma} &= \inf\{ E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}\} \qquad (3.2) \\ &= \inf\{ E^{p,q;\gamma}[u] \mid u \in C^\infty_c(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}\}, \end{aligned}$$

$$\begin{aligned} S^{p,q;\gamma}_{\mathrm{rad}} &= \inf\{ E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}} \setminus \{0\}\} \qquad (3.3) \\ &= \inf\{ E^{p,q;\gamma}[u] \mid u \in C^\infty_c(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}\}. \end{aligned}$$

First of all we state the CKN-type inequalities in the noncritical case.

Theorem 3.1. *Assume that* $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$ *and* $\gamma \neq 0$. *Then, we have* $S^{p,q;\gamma}_{\rm rad} \geq S^{p,q;\gamma} > 0$ *and the follwing inequalities.*

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq S^{p,q;\gamma}\|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n), \tag{3.4}$$

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq S^{p,q;\gamma}_{\rm rad}\|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\rm rad}. \tag{3.5}$$

This follows from the assertions 1-4 of Theorem 3.2. Let us introduce more notations.

Definition 3.2. *For* $1 < p \leq q < \infty$, *we set*

$$\gamma_{p,q} = \frac{n-1}{1+q/p'}, \tag{3.6}$$

$$S_{p,q} = \begin{cases} (p')^{p-2+p/q}q^{p/q}\left(\dfrac{\omega_n}{\tau_{p,q}}\mathrm{B}\left(\dfrac{1}{p\tau_{p,q}}, \dfrac{1}{p'\tau_{p,q}}\right)\right)^{1-p/q} & \text{if } p < q, \\ 1 & \text{if } p = q \end{cases}$$

Here $\mathrm{B}(\cdot,\cdot)$ *is the beta function.*

Remark 3.1. *1. It holds that*

$$\mathrm{B}\left(\frac{1}{p\tau}, \frac{1}{p'\tau}\right)^\tau \to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0. \tag{3.7}$$

In fact for $0 < \tau < \min\{1/p, 1/p'\}$, *we see that*

$$t^{1/p-\tau}(1-t)^{1/p'-\tau} \leq \frac{1}{(1-2\tau)^{1-2\tau}}\left(\frac{1}{p}-\tau\right)^{1/p-\tau}\left(\frac{1}{p'}-\tau\right)^{1/p'-\tau} \quad \text{for } 0 \leq t \leq 1, \tag{3.8}$$

hence we have

$$\begin{aligned} \mathrm{B}\left(\frac{1}{p\tau}, \frac{1}{p'\tau}\right)^\tau &= \left(\int_0^1 (t^{1/p-\tau}(1-t)^{1/p'-\tau})^{1/\tau}dt\right)^\tau \\ &\leq \frac{1}{(1-2\tau)^{1-2\tau}}\left(\frac{1}{p}-\tau\right)^{1/p-\tau}\left(\frac{1}{p'}-\tau\right)^{1/p'-\tau} \to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0, \\ \mathrm{B}\left(\frac{1}{p\tau}, \frac{1}{p'\tau}\right)^\tau &\geq \left(\int_0^1 (t^{1/p}(1-t)^{1/p'})^{1/\tau}dt\right)^\tau \\ &\to \max_{0\leq t\leq 1} t^{1/p}(1-t)^{1/p'} = \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0. \end{aligned}$$

2. Since $\tau_{p,q} \to 0$ *as* $q \to p$, *it follows from the argument of 1. that we have*

$$S_{p,q} = \frac{(p')^{p-1-p\tau_{p,q}}}{(1/p-\tau_{p,q})^{1-p\tau_{p,q}}}\left(\frac{\omega_n}{\tau_{p,q}}\mathrm{B}\left(\frac{1}{p\,\tau_{p,q}},\frac{1}{p'\tau_{p,q}}\right)\right)^{p\tau_{p,q}} \to 1 = S_{p,p} \quad \text{as } q \to p. \tag{3.9}$$

Under these preparation we can compute the best constant $S_{\mathrm{rad}}^{p,q;\gamma}$ of the CKN-type inequality in the radial function space to obtain the exact representation. In the next we describe important relations among the best constants $S_{\mathrm{rad}}^{p,q;\gamma}$ and $S^{p,q;\gamma}$.

Theorem 3.2. *Assume that* $1 < p \le q \le \overline{q} < \infty$ *and* $\tau_{p,q} \le 1/n$. *Then it holds that:*

1. $S^{p,q;\gamma} = S^{p,q;-\gamma}$, $S_{\mathrm{rad}}^{p,q;\gamma} = S_{\mathrm{rad}}^{p,q;-\gamma}$ *for* $\gamma \neq 0$.
2. $S_{\mathrm{rad}}^{p,q;\gamma} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $\gamma \neq 0$.
3. $S^{p,q;\gamma} = S_{\mathrm{rad}}^{p,q;\gamma} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $0 < |\gamma| \le \gamma_{p,q}$.
4. $\left|\dfrac{\gamma}{\overline{\gamma}}\right|^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}} \le S^{p,q;\gamma} \le \left|\dfrac{\overline{\gamma}}{\gamma}\right|^{p\tau_{p,q}} S^{p,q;\overline{\gamma}}$ *for* $0 < |\gamma| \le |\overline{\gamma}|$.
5. $\dfrac{1}{(2-\gamma_{p,p^*}/\gamma)^p} S^{p,p^*;\gamma_{p,p^*}} \le S^{p,p^*;\gamma} \le S^{p,p^*;\gamma_{p,p^*}} = S_{\mathrm{rad}}^{p,p^*;\gamma_{p,p^*}}$
   for $|\gamma| \ge \gamma_{p,p^*} = \dfrac{n-p}{p}$ *if* $p < n$.
6. $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}} = S_{\mathrm{rad}}^{2,2^*;\gamma_{2,2^*}}$ *for* $|\gamma| \ge \gamma_{2,2^*} = \dfrac{n-2}{2}$ *if* $p = 2 < n$.
7. $S^{p,q;\gamma} \ge (|\gamma|^{p\tau_{q,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}})^{1/\tau_{p,\overline{q}}}$ *for* $\gamma \neq 0$.
   In particular,
   $S^{p,q;\gamma} \ge |\gamma|^{p(1-n\tau_{p,q})}(S^{p,p^*;\gamma})^{n\tau_{p,q}}$ *for* $\gamma \neq 0$ *if* $p < n$.

Remark 3.2. *1. It follows from Remark 2.1 and Theorem 3.2,1 that we have*

$$S^{p,p;\gamma} = S_{\mathrm{rad}}^{p,p;\gamma} = |\gamma|^p \quad \text{for } \gamma \neq 0. \tag{3.10}$$

2. For $1 < p < n$, *the number;*

$$S^{p,p^*;\gamma_{p,p^*}} = S_{\mathrm{rad}}^{p,p^*;\gamma_{p,p^*}} = n\left(\frac{n-p}{p-1}\right)^{p-1}\left(\frac{\omega_n}{p'}\mathrm{B}\left(\frac{n}{p},\frac{n}{p'}\right)\right)^{p/n} \tag{3.11}$$

coincides with the classical best constant of the Sobolev inequality;

$$\|\nabla u\|^p_{L^p(\mathbf{R}^n)} = \|\nabla u\|^p_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} \geq S\|u\|^p_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)} = S\|u\|^p_{L^{p^*}(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma_{p,p^*},0}(\mathbf{R}^n).$$

In particular for $n \geq 3$, $p = 2$, *we see that*

$$S^{2,2^*;\gamma_{2,2^*}} = S^{2,2^*;\gamma_{2,2^*}}_{\mathrm{rad}} = n(n-2)\left(\frac{\omega_n}{2}\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)\right)^{2/n} = n(n-2)\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2/n}\pi \tag{3.12}$$

Here, $\Gamma(\cdot)$ *is the gamma function.*

Moreover the best constant $S^{p,q;\gamma}$ is a continuous function of the parameters q and γ. Namely we have the following.

Theorem 3.3. *For* $1 < p < \infty$, *the maps*

$$([p,p^*]\setminus\{\infty\})\times(\mathbf{R}\setminus\{0\}) \ni (q;\gamma) \mapsto S^{p,q;\gamma},\ S^{p,q;\gamma}_{\mathrm{rad}} \in \mathbf{R} \tag{3.13}$$

are continuous. In particular, it holds that

$$S^{p,q;\gamma} \to S^{p,p;\gamma} = |\gamma|^p \quad \text{as } q \to p. \tag{3.14}$$

In the next we describe results on the existence and non-existence of extremal functions which attain the best constants of the CKN-type inequalities. Shortly speaking, the best constant $S^{p,q;\gamma}$ is attained by some element in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ provided that $p < q < p^*$ is satisfied. On the other hand if $q = p$, then the corresponding CKN-type inequalities are reduced to the Hardy-Sobolev inequalities and therefore no extremal function exists. When $q = p^*$ holds, then $S^{p,p^*;\gamma}$ is attained provided that $0 < |\gamma| \leq (n-p)/p = \gamma_{p,p^*}$, but in the case that $|\gamma| > (n-p)/p$, it is unkown in general except for the case $p = 2$, whether $S^{p,p^*;\gamma}$ is achieved by some element or not. If $p = 2$ is assumed, then it is shown that no extremal exists provided that $|\gamma| > (n-2)/2$ holds.

Theorem 3.4. *Assum that* $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$ *and* $\gamma \neq 0$. *Then we have the followings.*

1. *If* $p < q$, *then* $S^{p,q;\gamma}_{\mathrm{rad}}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$.
2. *If* $p < q < p^*$, *then* $S^{p,q;\gamma}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.

3. *If* $p < n$, $q = p^*$ *and* $|\gamma| \le (n-p)/p = \gamma_{p,p^*}$, *then* $S^{p,p^*;\gamma} = S^{p,p^*;\gamma}_{\rm rad}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\rm rad}\backslash\{0\}$.
4. *If* $p = 2 < n$, $q = 2^* = 2n/(n-2)$ *and* $|\gamma| > (n-2)/2 = \gamma_{2,2^*}$, *then* $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}}_{\rm rad}$ *holds and* $S^{2,2^*;\gamma}$ *is not achieved in* $W^{1,2}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}$.

Proposition 3.1. *If* $1 < p = q < \infty$, $\gamma \neq 0$, *then* $S^{p,p;\gamma}$ *and* $S^{p,p;\gamma}_{\rm rad}$ *are not achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}$ *and* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\rm rad}\backslash\{0\}$ *respectively.*

Lastly let us explain the radial case more precisely which is rather fundamental in this work.

Theorem 3.5. (The radial case) *Assume that* $1 < p < q < +\infty$ *and* $\gamma > 0$. *Then we have the followings:*

1. $S^{p,q;\gamma}_{\rm rad}$ *is achieved by the function* u *below*

$$u(r) = \lambda^{\frac{1}{q-p}}[1 + r^{\frac{ph}{p-1}}]^{-\frac{p}{q-p}} \qquad (r = |x|), \tag{3.15}$$

$$\left\{\begin{array}{l} h = q\gamma\tau_{p,q} > 0, \\ \lambda = \left(\frac{p}{p-1}\right)^{p-1}\gamma^p q. \end{array}\right. \tag{3.16}$$

Moreover u *satisfies the Euler-Lagrange equation:*

$$-\mathrm{div}\,(I_{p(1+\gamma)}(x)|\nabla u|^{p-2}\nabla u) = I_{q\gamma}(x)|u|^{q-2}u. \tag{3.17}$$

4 A linearization method

From Proposition 1.1 we see that the best constants $S^{p,q;\gamma}$ are symmetric with respect to γ. Therefore, in the subsequent argument it suffices to consider the subcritical case that $\gamma > 0$.

Definition 4.1. *For* $\gamma > 0$ *we set for* $r = |x|$

$$L_{p,\gamma}(u) = -\mathrm{div}\,(I_{p(\gamma+1)}(r)|\nabla u|^{p-2}\nabla u), \tag{4.1}$$
$$M_{p,\gamma}(u) = L_{p,\gamma}(u) - I_{q\gamma}(r)|u|^{q-2}u. \tag{4.2}$$

We will study a linearization of these operator in a precise way. First, by a linearization at u we formally have

$$L'_{p,\gamma}(u)\varphi = -\mathrm{div}\left(I_{p(\gamma+1)}(r)|\nabla u|^{p-2}\left(\nabla\varphi + (p-2)\frac{(\nabla u, \nabla\varphi)}{|\nabla u|^2}\nabla u\right)\right), \tag{4.3}$$

$$M'_{p,\gamma}(u)\varphi = L'_{p,\gamma}(u)\varphi - (q-1)I_{q\gamma}(r)u^{q-2}\varphi \tag{4.4}$$

for any $\varphi \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$.

Definition 4.2. *For $\gamma > 0$, $r = |x|$ and u is defined by (3.15) in Theorem 3.5 we set*

$$\omega(r) = \omega(r; p, q, \gamma) = I_{p(\gamma+1)}(r)|\partial_r u|^{p-2}. \tag{4.5}$$

By a polar coordinate system $x = (r, \omega), r > 0, \omega \in S^{n-1}$, the Laplacian Δ is represented by $r^{1-n}\partial_r(r^{n-1}\partial_r\cdot) + \Delta_{S^{n-1}}/r^2$. Here $\Delta_{S^{n-1}}$ is the Laplace Beltrami operator on the unit sphere. Then we have

Lemma 4.1. *We assume that u is a spherically symmetric function on $\mathbf{R}^n$. Then*

$$L'_{p,\gamma}(u)\varphi = -(p-1)r^{1-n}\partial_r\left(r^{n-1}\omega(r)\partial_r\varphi\right) - r^{-2}\omega(r)\Delta_{S^{n-1}}\varphi. \tag{4.6}$$

Proof: Let u be a radial smooth function. Then we have

$$\begin{aligned} L'_{p,\gamma}(u)\varphi &= -\mathrm{div}\left(\omega\left(\nabla\varphi + (p-2)\frac{(\nabla u, \nabla\varphi)}{|\nabla u|^2}\nabla u\right)\right) \\ &= -\omega\Delta\varphi - \partial_r\omega\partial_r\varphi - (p-2)\left(\partial_r\omega\partial_r\varphi + \omega\mathrm{div}\left(\frac{x}{r}\partial_r\varphi\right)\right) \\ &= -(p-1)r^{1-n}\partial_r\left(r^{n-1}\omega\partial_r\varphi\right) - \frac{\omega}{r^2}\Delta_{S^{n-1}}\varphi. \end{aligned}$$

Here we used

$$\mathrm{div}\left(\frac{x}{r}\partial_r\varphi\right) = \partial_r^2\varphi + \frac{n-1}{r}\partial_r\varphi. \tag{4.7}$$

□

For $\omega(r) = \omega(r; p, q, \gamma)$ $(\gamma > 0)$ we employ the spaces $L^2(\mathbf{R}^n; \omega)$ and $L^2(\mathbf{R}^n; r^{-2}\omega)$ according to Definition 2.1. In a similar way, by $L^2(\mathbf{R}_+;\omega r^{n-3})$ we denote the space of all Lebesgue measurable functions on $\mathbf{R}_+ = (0,\infty)$ for which

$$||\varphi||_{L^2(\mathbf{R}_+;\omega r^{n-3})} = \left(\int_0^\infty |\varphi(r)|^2\omega(r)r^{n-3}\,dr\right)^{\frac{1}{2}} < +\infty. \tag{4.8}$$

To study the eigenvalue problem for the operator $M'_{p,\gamma}(u)$, we need more preparations. Let us define the following Hilbert spaces.

Definition 4.3. *By $W^{1,2}(\mathbf{R}^n;\omega)$ we denote the completion of $C_c^\infty(\mathbf{R}^n\setminus\{0\})$ with respect to the norm*

$$\varphi \to ||\varphi||_{W^{1,2}(\mathbf{R}^n;\omega)} = \left(||\nabla\varphi||^2_{L^2(\mathbf{R}^n;\omega)} + ||\varphi||^2_{L^2(\mathbf{R}^n;r^{-2}\omega)}\right)^{\frac{1}{2}}. \tag{4.9}$$

In a similar way, by $W^{1,2}(\mathbf{R}_+;\omega r^{n-1})$ we denote the completion of $C_c^\infty(\mathbf{R}_+)$ with respect to the norm

$$\varphi \to ||\varphi||_{W^{1,2}(\mathbf{R}_+;r^{n-1}\omega)} = \left(||\varphi'||^2_{L^2(\mathbf{R}_+;\omega r^{n-1})} + ||\varphi||^2_{L^2(\mathbf{R}_+;\omega r^{n-3})}\right)^{\frac{1}{2}}. \tag{4.10}$$

Then we see

Lemma 4.2. *$L^2(\mathbf{R}^n;r^{-2}\omega)$, $W^{1,2}(\mathbf{R}^n;\omega)$, $L^2(\mathbf{R}_+;\omega r^{n-3})$ and $W^{1,2}(\mathbf{R}_+;\omega r^{n-1})$ become Hilbert spaces with the canonical inner products.*

By separation of variables, the linearization of (4.2) at the radial solution u decomposes into infinitely many ordinary differential operators. Denote by

$$\nu_k = k(n-2+k), \qquad (k=0,1,2,\ldots) \tag{4.11}$$

the k^{th} eigenvalue of the Laplace Beltrami operator $\Delta_{S^{n-1}}$ on S^{n-1}. We denote by μ_k and f_k the first eigenvalue and the corresponding positive eigenfunction in the k^{th} eigenvalue problem of μ, defined by

$$\begin{cases} -(p-1)r^{1-n}\partial_r\left(r^{n-1}\omega\partial_r f\right) + \frac{\nu_k\omega}{r^2}f - (q-1)I_{q\gamma}u^{q-2}f = \mu\frac{\omega}{r^2}f \ \text{ in } \mathbf{R}_+ = (0,\infty), \\ f \in W^{1,2}(\mathbf{R}_+;\omega r^{n-1})\setminus\{0\}, \end{cases} \tag{4.12}$$

where differentiations are taken in the distribution sense. If there exists the first eigenfunction $f_k \in W^{1,2}(\mathbf{R}_+; \omega r^{n-1})$ with the first eigenvalue μ_k, then f_k becomes a solution to the variational problem (E_k):

$$(E_k) \qquad \mu_k = \inf_{f \in W^{1,2}(\mathbf{R}_+;\omega r^{n-1}), f \neq 0} E_k(f), \tag{4.13}$$

where

$$\begin{cases} E_k(f) = E_0(f) + \nu_k \\ E_0(f) = \frac{(p-1)\int_0^\infty |\partial_r f|^2 \omega(r) r^{n-1}\,dr - (q-1)\int_0^\infty r^{n-1} I_{q\gamma}(r) u^{q-2} f^2\,dr}{\int_0^\infty f^2 \omega(r) r^{n-3}\,dr}. \end{cases} \tag{4.14}$$

By the definition we clearly see that

$$\mu_k = \nu_k + \mu_0 \quad \text{and} \quad f_k = f_0 \quad \text{for } k = 0, 1, 2, \ldots.$$

Remark 4.1. *It is easy to see that $\mu_0 < 0$. Moreover it will be shown that for any $k > 0$ the eigenvalue μ_k is negative provided that γ is sufficiently large. In fact, the negativity of μ_k for a large $\gamma > 0$ readily follows from the elementary argument below, provided that $p > \frac{2q}{q+1}$ and $q > p$ hold. Using the solution u as a test function, μ_k should satisfy*

$$\mu_k = E_k(f_0) \leq \nu_k + (p-q)\frac{\int_{\mathbf{R}^n} |\partial_r u|^2 \omega(r)\,dx}{\int_{\mathbf{R}^n} u^2 \frac{\omega(r)}{r^2}\,dx} = \nu_k - \frac{p'\gamma^2 \tau_{p,q}(p(q+1)-2q)}{(1-\tau_{p,q})(1-2\tau_{p,q})}.$$

Noting that $0 < \tau_{p,q} < 1/2$ and $\nu_0 = 0$, $\mu_0 < 0$ immediately follows. Further we see that

$$\mu_k \to -\infty, \qquad \text{as } \gamma \to \infty \qquad (k = 0, 1, 2, \ldots). \tag{4.15}$$

Here we note that the condition $p > \frac{2q}{q+1}$ is automatically satisfied if $p \geq 2$.

In the rest of this subsection we shall establish the Hardy type inequalities. By virtue of them and the fact $u^{q-2} I_{q\gamma}(x) \to 0$ as $r \to \infty$ we shall see that the variational problem (E_k) or equivalently the eigenvalue problem (4.12) is well-posed.

Let us recall a fundamental lemma. For the proof one can employ an obvious modification of Theorem 2 in [Ma; §1.3.1].

Lemma 4.3. *Let $\gamma > 0$ and let u be the function defined in Theorem 3.5 and let $\omega(r) = I_{p(\gamma+1)}(r)|\partial_r u|^{p-2}$. In order that there exists a* *constant* C*, independent of each* $\varphi \in C_c^\infty((0,\infty))$ *such that*

$$\int_0^\infty \varphi(r)^2 u(r)^{q-2} r^{n-1} I_{q\gamma}(r)\,dr \le C \int_0^\infty \varphi'(r)^2 \omega(r) r^{n-1}\,dr, \quad (4.16)$$

it is necessary and sufficient that

$$B = \sup_{r\in(0,+\infty)} B(r) < +\infty, \quad (4.17)$$

where

$$B(r) = \int_0^r u(r)^{q-2} r^{n-1} I_{q\gamma}(r)\,dr \int_r^\infty \left(\omega(r) r^{n-1}\right)^{-1} dr. \quad (4.18)$$

In order to check the condition (4.17), we prepare fundamental lemmas that are given by direct calculations. By the notation $u(r) = O(r^k)$ as $r \to \infty$ $(r \to 0)$, we mean that there are some positive numbers C_1 and C_2 such that

$$C_1 \le \frac{|u(r)|}{r^k} \le C_2, \qquad \text{as } r \to \infty\,(r \to 0).$$

On the other hand by the notation $u(r) = o(r^k)$ as $r \to \infty$ $(r \to 0)$, we mean that $u(r)/r^k \to 0$ as $r \to \infty$ $(r \to 0)$.

Lemma 4.4. *Let* $\gamma > 0$ *and let* u *be the function defined in Theorem 3.5.Then we have*

$$u(r) = \begin{cases} O\left(r^{-p'\gamma}\right) & \text{as } r \to +\infty, \\ O(1) & \text{as } r \to +0. \end{cases}$$

$$u'(r) = \begin{cases} O\left(r^{-p'\gamma-1}\right) & \text{as } r \to +\infty, \\ O\left(r^{p'h-1}\right) & \text{as } r \to +0. \end{cases}$$

Lemma 4.5. *Let* $\gamma > 0$ *and let* u *be the function defined in Theorem 3.5.Then we have*

$$\int_0^r u(r)^{q-2} r^{n-1} I_{q\gamma}(r)\,dr = \begin{cases} O\left(r^{p'\gamma(2-\frac{q}{p})}\right) & (\text{ if } p > \frac{q}{2}) \text{ as } r \to +\infty, \\ O(\log r) & (\text{ if } p = \frac{q}{2}) \text{ as } r \to +\infty, \\ O(1) & (\text{ if } p < \frac{q}{2}) \text{ as } r \to +\infty, \\ O(r^{q\gamma}) & \text{as } r \to +0. \end{cases}$$

$$\int_r^\infty \left(\omega(r)r^{n-1}\right)^{-1} dr = \begin{cases} O\left(r^{-p'\gamma}\right) & \textit{as } r \to +\infty, \\ O\left(r^{-p'\gamma(q+1-2\frac{q}{p})}\right) & (\textit{ if } p > \frac{2q}{q+1}) \ \textit{as } r \to +0. \\ O\left(\log \frac{1}{r}\right) & (\textit{ if } p = \frac{2q}{q+1}) \ \textit{as } r \to +0. \\ O(1) & (\textit{ if } 1 < p < \frac{2q}{q+1}) \ \textit{as } r \to +0. \end{cases}$$

Then we have the following.

Lemma 4.6. (*Hardy type inequality in* $\mathbf{R}_+$) *The inequality 4.16 holds for any* $\varphi \in C_c^\infty((0,\infty))$.

Proof: It suffices to check the condition (4.17). Then we see as $r \to \infty$

$$B(r) = \begin{cases} O\left(r^{-p'\gamma(\frac{q}{p}-1)}\right), & \text{if } p > \frac{q}{2} \\ O\left(r^{-p'\gamma}\log r\right), & \text{if } p = \frac{q}{2} \\ O\left(r^{-p'\gamma}\right), & \text{if } p < \frac{q}{2}. \end{cases} \tag{4.19}$$

Thus we see $B(r)$ is finite as $r \to \infty$. On the other hand, we see as $r \to 0$

$$B(r) = \begin{cases} O\left(r^{p'\gamma(\frac{q}{p}-1)}\right), & \text{if } p > \frac{2q}{q+1}, \\ O\left(r^{q\gamma}\log \frac{1}{r}\right), & \text{if } p = \frac{2q}{q+1}, \\ O\left(r^{q\gamma}\right), & \text{if } 1 < p < \frac{2q}{q+1}. \end{cases} \tag{4.20}$$

Therefore the assertion is now clear. □

Then we immediately have

Lemma 4.7. (*Hardy type inequality in* $\mathbf{R}^n$) *Let* $\gamma > 0$ *and let* u *be the function defined in Theorem 3.5. Then, there is a positive number* C *independent of each* $\varphi \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$ *such that we have for* $r = |x|$

$$\int_{\mathbf{R}^n} \varphi(x)^2 u(r)^{q-2} I_{q\gamma}(r)\, dx \le C \int_{\mathbf{R}^n} |\nabla\varphi(x)|^2 \omega(r)\, dx, \tag{4.21}$$
$$\omega(r) = |u'(r)|^{p-2} I_{p(\gamma+1)}(r).$$

Remark 4.2. *1. The left-hand side is always finite for any* $\varphi \in C_c^\infty(\mathbf{R}^n)$. *If* $p > \frac{2q}{q+1}$, *then for any* $\gamma > 0$ *the weight function* ω *is locally integrable as well. In fact we see that*

$$\omega = O(r^{\frac{p(q+1)-2q}{p-1}\gamma+2-n}), \qquad \textit{as } r \to 0.$$

2. In a similar way, if $p > \frac{2q}{q+1}$, then we are able to show that

$$\int_0^\infty \varphi^2 \omega r^{n-3}\, dr \le C \int_0^\infty |\varphi'|^2 \omega r^{n-1}\, dr \qquad (4.22)$$

for any $\varphi \in C_c^\infty([0,\infty))$. Here C is apositive number independent of each φ. As a result, the norm $||\varphi||_{W^{1,2}(\mathbf{R}_+;r^{n-1}\omega)}$ is equivatent to the single norm $||\nabla\varphi||_{L^2(\mathbf{R}_+;\omega r^{n-3})}$ provided that $p > \frac{2q}{q+1}$.

5 Main Theorem

Let us restate our main result, which is equivalent to Theorem 1.1.

Theorem 5.1. (The symmetry breaking) *Assume that $1 < p < n$. Assume that q is fixed such as $p < q < p^*$. Then for sufficiently large $|\gamma|$, the best constant $S^{p,q;\gamma}$ is not attained in the radial function space $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$.*

From this theorem and Proposition 1.1 together with the continuity of the best constants on parameters, we immediately have the following:

Corollary 5.1. *Assume that $1 < p < n$. Then there exists a symmetry-breaking function $S_b(\gamma)$ for $|\gamma| \ge \gamma_{p,p^*}$ satisfying $S_b(\gamma_{p.p^*}) = p^*$, $S_b(\gamma) \in (p,p^*)$ for $|\gamma| > \gamma_{p,p^*}$ and $\lim_{|\gamma|\to\infty} S_b(\gamma) = p$ such that we have $S^{p,q;\gamma} < S^{p,q;\gamma}_{\mathrm{rad}}$ for any $q \in (S_b(\gamma), p^*)$ with $|\gamma| > \gamma_{p,p^*}$.*

Proof of Corollary: From Theorem 5.1 the existence of a symmetry-breaking function $S_b(\gamma)$ is clear if γ is sufficiently large. On the other hand, for each γ with $|\gamma| > \gamma_{p,p^*}$, $S^{p,q;\gamma} < S^{p,q;\gamma}_{\mathrm{rad}}$ holds provided that q is sufficiently close to p^*. In fact, it follows from the assertions 2 and 5 of Proposition 1.1 that we have $S^{p,p^*;\gamma} \le S^{p,p^*;\gamma_{p,p^*}}_{\mathrm{rad}} < S^{p,p^*;\gamma}_{\mathrm{rad}}$ for $|\gamma| > \gamma_{p,p^*}$. Here we note that $S^{p,p^*;\gamma}_{\mathrm{rad}}$ is strictly increasing in $|\gamma|$. Since the best constants are continuously dependent on parameters, if q is sufficiently close to p^*, then $S^{p,q;\gamma} < S^{p,q;\gamma}_{\mathrm{rad}}$ holds for each γ with $|\gamma| > \gamma_{p,p^*}$. □

In order to prove Theorem 5.1 we need to employ the followings which are of interest by themselves, and we shall sketch the proofs in the last section.

Theorem 5.2. *The eigenvalue problem (4.12) is well-posed. For an arbitrary number $k \in \mathbf{N}$, there is a positive number M such that if $|\gamma| > M$, then the k^{th} eigenvalue problem (4.12) (or equivalently the variational problem (E_k)) has a negative first eigenvalue $\mu_k = \nu_k + \mu_0$ and a corresponding first eigenfunction $f_k = f_0$ in $W^{1,2}(\mathbf{R}_+; \omega r^{n-1})$.*

Proposition 5.1. *Let $f_0 \geq 0$ be the first eigenfunction to (4.12) with $k = 0$. Let $\phi_0(> 0), \phi_1$ be the first and second spherical harmonic functions. By $\varphi(x)$ we denote an arbitrary linear combination of functions $\{f_0(|x|)\phi_k(x/|x|)\}_{k=0}^1$ on $\mathbf{R}^n$, namely $\varphi = c_0 f_0(r)\phi_0(\theta) + c_1 f_0(r)\phi_1(\theta)$ with $r = |x|$, $\theta = x/|x|$ and $c_0, c_1 \in \mathbf{R}$. Then, if $\gamma > 0$ is sufficiently large, then we have*

$$\sup_{s\in[0,1]} \int_{\mathbf{R}^n} |\nabla(u(x) + s\varphi(x))|^{p-2} |\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\, dx < \infty.$$

In the rest of this section we shall establish the symmetric breaking result Theorem 5.1 admitting Theorem 5.2 and Proposition 4.1. The argument below is similar to the one used in [CW1] when $p = 2$.

Proof of Theorem 5.1:

By the symmetry with respect to γ it suffices to consider the case when $\gamma > 0$. We shall show the symmetry breaking actually happens for a sufficiently large $\gamma > 0$. To this end we assume that

$$S^{p,q;\gamma} = \inf\{\, E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}\} \tag{5.1}$$

is attained by a radial function u defined by (3.15) and (3.16) in Theorem 3.5. Now we set $w_k(x) = f_0(|x|)\phi_k(x/|x|)$ for $k = 0, 1$ which are defined in Proposition 5.1, and we set

$$G(\eta, s) = \int_{\mathbf{R}^n} |u(r) + \eta w_0(x) + s w_1(x)|^q I_{q\gamma}(r)\, dx \qquad (r = |x|). \tag{5.2}$$

Here we note that $w_0 = f_0\phi_0 > 0$ and ϕ_0 is a constant function by the definition. Then we shall show that $E^{p,q;\gamma}[u]$ can be smaller by replacing u by a suitable perturbation using w_0 and w_1. Note that

$$G(0, 0) = 1.$$

By differentiating G we also have for small η and s

$$\begin{cases} \frac{\partial G}{\partial \eta} = q \int_{\mathbf{R}^n} |u(r) + \eta w_0(x) + s w_1(x)|^{q-1} w_0(x) I_{q\gamma}(r)\, dx, \\ \frac{\partial G}{\partial \eta}(0,0) = q \int_{\mathbf{R}^n} |u(r)|^{q-1} w_0(x) I_{q\gamma}(r)\, dx < \infty, \\ \frac{\partial G}{\partial s}(0,0) = q \int_{\mathbf{R}^n} |u(r)|^{q-1} w_1(x) I_{q\gamma}(r)\, dx = 0, \\ \frac{\partial^2 G}{\partial s^2}(0,0) = q(q-1) \int_{\mathbf{R}^n} |u(r)|^{q-2} w_1(x)^2 I_{q\gamma}(r)\, dx > 0, \\ \frac{\partial^2 G}{\partial \eta \partial s}(0,0) = q(q-1) \int_{\mathbf{R}^n} |u(r)|^{q-2} w_0(x) w_1(x) I_{q\gamma}(r)\, dx = 0. \end{cases} \tag{5.3}$$

We remark the following fact. The eigenfunction f_k satisfies

$$(p-1)\int_0^\infty f_0'(r)^2 \omega(r) r^{n-1}\, dr + \nu_k \int_0^\infty f_0(r)^2 \omega(r) r^{n-3}\, dr$$
$$= (q-1)\int_0^\infty f_0(r)^2 |u(r)|^{q-2} r^{n-1} I_{q\gamma}(x)\, dr + \mu_k \int_0^\infty f_0(r)^2 \omega(r) r^{n-3}\, dr$$

Hence we see that

$$\int_{\mathbf{R}^n} |u(r)|^{q-2} w_k(x)^2 r^{n-1} I_{q\gamma}(r)\, dx$$
$$= Const. \int_0^\infty |u(r)|^{q-2} f_0(r)^2 r^{n-1} I_{q\gamma}(r)\, dx < \infty.$$

Since

$$\frac{\partial G}{\partial \eta}(0,0) = q \int_{\mathbf{R}^n} |u(r)|^{q-1} w_0(x) I_{q\gamma}(r)\, dx > 0,$$

it follows from the implicit function theorem that there are $\delta > 0$ and $\eta(s)$ such that for $|s| < \delta$

$$\begin{cases} G(\eta(s), s) = 1, \quad \eta(0) = 0, \\ \frac{\partial G}{\partial \eta}(\eta(s), s)\eta'(s) + \frac{\partial G}{\partial s}(\eta(s), s) = 0, \\ \frac{\partial^2 G}{\partial \eta^2}(\eta(s), s)\eta'(s)^2 + 2\frac{\partial^2 G}{\partial \eta \partial s}(\eta(s), s)\eta'(s) \\ + \frac{\partial G}{\partial \eta}(\eta(s), s)\eta''(s) + \frac{\partial^2 G}{\partial s^2}(\eta(s), s) = 0. \end{cases} \tag{5.4}$$

Since $\frac{\partial G}{\partial \eta}(\eta(0), 0) = \frac{\partial G}{\partial \eta}(0,0) > 0$ and $\frac{\partial G}{\partial s}(\eta(0), 0) = 0$, we have $\eta'(0) = 0$. Moreover from $\frac{\partial G}{\partial \eta} 0, 0)\eta''(0) + \frac{\partial^2 G}{\partial s^2}(0,0) = 0$, we have

$$\eta''(0) = -\frac{(q-1)\int_{\mathbf{R}^n} u(r)^{q-2} w_1(x)^2 I_{q\gamma}(r)\, dx}{\int_{\mathbf{R}^n} u(r)^{q-1} w_0(x) I_{q\gamma}(r)\, dx} < 0 \tag{5.5}$$

and then

$$\eta(s) = \frac{s^2}{2}\eta''(0) + o(s^2). \tag{5.6}$$

Now we put

$$f(t) = \int_{\mathbf{R}^n} |\nabla(u(r) + t\varphi(x))|^p I_{p(\gamma+1)}(r)\, dx.$$

By Taylor's expansion formula, we have

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + t^2 \int_0^1 (1-z)\,(f''(tz) - f''(0))\, dz.$$

By a direct calculation we have

$$\begin{cases} f(0) = \int_{\mathbf{R}^n} |\nabla u(r)|^p I_{p(\gamma+1)}(r)\, dx = \int_{\mathbf{R}^n} u(r)^q I_{q\gamma}(r)\, dx, \\ f'(t) = p\int_{\mathbf{R}^n} |\nabla(u(r)+t\varphi(x))|^{p-2}(\nabla(u(r)+t\varphi(x)), \nabla\varphi(x)) I_{p(\gamma+1)}(r)\, dx, \\ f'(0) = p\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}(\nabla u(r), \nabla\varphi(x)) I_{p(\gamma+1)}(r)\, dx, \\ f''(t) = p(p-2)\int_{\mathbf{R}^n} |\nabla(u(r)+t\varphi(x))|^{p-4}(\nabla(u(r)+t\varphi(x)), \nabla\varphi(x))^2 I_{p(\gamma+1)}(r)\, dx \\ \quad +p\int_{\mathbf{R}^n} |\nabla(u(r)+t\varphi(x)|^{p-2}|\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\, dx, \\ f''(0) = p\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}\left(|\nabla\varphi(x)|^2 + (p-2)\frac{(\nabla u(r), \nabla\varphi(x))^2}{|\nabla u(r)|^2}\right) I_{p(\gamma+1)}(r)\, dx. \end{cases} \tag{5.7}$$

Using a dual form, we can rewrite $f''(0)$ to have

$$f''(0) = p\langle L_p'(u)\varphi, \varphi\rangle_{(W^{1,2})'\times W^{1,2}}.$$

Putting $t=1$ we get

$$\begin{aligned} &\int_{\mathbf{R}^n} |\nabla(u(r)+\varphi(x))|^p I_{p(\gamma+1)}(r)\, dx \\ &= \int_{\mathbf{R}^n} |\nabla u(r)|^p I_{p(\gamma+1)}(r)\, dx + \frac{p}{2}\langle L_p'(u)\varphi, \varphi\rangle_{(W^{1,2})'\times W^{1,2}} \\ &+ p\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}(\nabla u(r), \nabla\varphi(x)) I_{p(\gamma+1)}(r)\, dx + \int_0^1 (1-z) R_z(u,\varphi)\, dz, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} &R_z(u,\varphi) = f''(z) - f''(0) \\ &= p\int_{\mathbf{R}^n} \left(|\nabla(u+z\varphi)|^{p-2} - |\nabla u|^{p-2}\right) |\nabla\varphi|^2 I_{p(\gamma+1)}(r)\, dx \\ &+ (p-2)\int_{\mathbf{R}^n} \left(|\nabla(u+z\varphi)|^{p-4}(\nabla(u+z\varphi), \nabla\varphi)^2 - |\nabla u|^{p-4}(\nabla u, \nabla\varphi)^2\right) \end{aligned} \tag{5.9}$$

$\times I_{p(\gamma+1)}(r)\,dx.$

Now we put

$$\varphi(x) = \eta(s)w_0(x) + sw_1(x).$$

Then it follows from Proposition 5.1 that we have

$$|R_z(u,\varphi)| < \infty \qquad \text{and} \qquad \lim_{z\to 0} R_z(u,\varphi) = 0.$$

Here we note that (6.10), (6.14) and $\varphi = O(s)$ as $s \to 0$. Then from the dominated convergence theorem we have

$$\int_0^1 R_z(u,\varphi)dz = o(s^2).$$

Now we look at the each terms in (5.9) precisely. First we see

$$p\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}(\nabla u(r), \nabla\varphi(x))I_{p(\gamma+1)}(r)\,dx =$$
$$p\int_{\mathbf{R}^n} L_{p,\gamma}(u(r))(\eta(s)w_0(x) + sw_1(x))\,dx$$
$$= p\int_{\mathbf{R}^n} L_{p,\gamma}(u(r))\eta(s)w_0(x)\,dx = p\eta(s)\int_{\mathbf{R}^n} u(r)^{q-1}w_0(x)I_{q\gamma}(r)\,dx.$$

Noting that $L'_{p,\gamma}(u)w_1 = (q-1)I_{q\gamma}^{q-2}w_1 + \mu_1|\nabla u|^{p-2}I_{p(\gamma+1)}r^{-2}w_1$, we have

$$\frac{p}{2}\langle L'_{p,\gamma}(u)\varphi,\varphi\rangle = \frac{p}{2}\langle L'_{p,\gamma}(u)(\eta(s)w_0 + sw_1), \eta(s)w_0 + sw_1\rangle$$
$$= \frac{p}{2}\left[\eta(s)^2\langle L'_{p,\gamma}(u)w_0, w_0\rangle + 2s\eta(s)\langle L'_{p,\gamma}(u)w_0, w_1\rangle + s^2\langle L'_{p,\gamma}(u)w_1, w_1\rangle\right]$$
$$= \frac{ps^2}{2}\langle L'_{p,\gamma}(u)w_1, w_1\rangle + o(s^2)$$
$$= \frac{ps^2}{2}\left[(q-1)\int_{\mathbf{R}^n} u(r)^{q-2}w_1(x)^2 I_{q\gamma}(r)\,dx + \mu_1\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}w_1(x)^2 r^{-2}I_{p(\gamma+1)}(r)\,dx\right] + o(s^2)$$

Using (4.5) and (5.6), we have

$$\int_{\mathbf{R}^n} |\nabla(u(r) + \eta(s)w_0(x) + sw_1(x))|^p I_{p(\gamma+1)}(r)\,dx = \int_{\mathbf{R}^n} |\nabla u(r)|^p I_{p(\gamma+1)}(r)\,dx + o(s^2) \tag{5.10}$$
$$+ p\eta(s)\int_{\mathbf{R}^n} u(r)^{q-1}w_0(x)I_{q\gamma}(r)\,dx$$
$$+ \frac{p}{2}s^2\left[(q-1)\int_{\mathbf{R}^n} u(r)^{q-2}w_1(x)^2 I_{q\gamma}(r)\,dx + \mu_1\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}w_1(x)^2 r^{-2}I_{p(\gamma+1)}(r)\,dx\right].$$

$$= \int_{\mathbf{R}^n} |\nabla u(r)|^p I_{p(\gamma+1)}(r)\,dx + \frac{ps^2}{2}\mu_1 \int_{\mathbf{R}^n} |\nabla u(r)|^{p-2} w_1(x)^2 r^{-2} I_{p(\gamma+1)}(r)\,dx + o(s^2)$$
$$< \int_{\mathbf{R}^n} |\nabla u(r)|^p I_{p(\gamma+1)}(r)\,dx \qquad \text{for small } s.$$

Thus the assertion is proved. □

6 Sketch of proofs of Theorem 5.2 and Proposition 5.1

Proof of Theorem 5.2: Since $E_k(f) = \nu_k + E_0(f)$, it suffices to consider the variational problem (E_0);

$$E_0(f) =$$
$$\frac{(p-1)\int_0^\infty |\partial_r f(r)|^2 \omega(r) r^{n-1}\,dr - (q-1)\int_0^\infty r^{n-1} I_{q\gamma}(r) u(r)^{q-2} f(r)^2\,dr}{\int_0^\infty f(r)^2 \omega(r) r^{n-3}\,dr}.$$

Now we put

$$\begin{cases} g(r) = \omega(r) r^{n-2} = |u'(r)|^{p-2} r^{n-2} I_{p(\gamma+1)}(r), \\ \xi(r) = g(r)^{-\frac{1}{2}}, \\ f(r) = v(r)\xi(r). \end{cases} \tag{6.1}$$

Then we have the equivalent functional as follows:

Lemma 6.1. *Assume that $\gamma > 0$. Then we have*

$$E_0(f) = E_0(v\xi) = \frac{(p-1)\left(\int_0^\infty |\partial_r v(r)|^2 r\,dr + \int_0^\infty v(r)^2 G(r)\,dr\right)}{\int_0^\infty v(r)^2 \frac{1}{r}\,dr}, \tag{6.2}$$

where

$$G(r) = \frac{pq^2\gamma^2}{4(p-1)^2}\frac{1}{r}\left[\frac{p}{q^2} - 2(2\tau_{p,q}(p-1)+1)\frac{\rho}{(1+\rho)^2}\right. \tag{6.3}$$
$$\left. + (p-2)(1-2\tau_{p,q})\frac{1}{(1+\rho)^2}\right]$$

$$= \frac{A}{r} - \frac{B}{r}\frac{\rho}{(1+\rho)^2} + \frac{C}{r}\frac{1}{(1+\rho)^2},$$

where $\rho = r^{\frac{p}{p-1}h}$, $h = q\gamma\tau_{p,q}$ *and*

$$\begin{aligned} A &= \frac{p^2\gamma^2}{4(p-1)^2}, \quad B = \frac{pq^2\gamma^2(2\tau_{p,q}(p-1)+1)}{2(p-1)^2}, \qquad (6.4) \\ C &= \frac{pq^2\gamma^2(p-2)(1-2\tau_{p,q})}{4(p-1)^2}. \end{aligned}$$

Now we change a variable using $r = e^{-t}$, ($t = \log\frac{1}{r}$). Then we have for $\tilde{v}(t) = v(e^{-t})$ and $\tilde{G}(t) = G(e^{-t})e^{-t}$

$$E_0(f) = E_0(v\xi) = \frac{(p-1)\int_{-\infty}^{\infty}\left((\partial_t\tilde{v}(t))^2 + \tilde{G}(t)\tilde{v}(t)^2\right)dt}{\int_{-\infty}^{\infty}\tilde{v}(t)^2\,dt}.$$

Here $\tilde{v}$ satisfies $\tilde{v}(\pm\infty) = 0$ for any $v = f\sqrt{g}$ with any $f \in W^{1,2}(\mathbf{R}_+, \omega r^{n-1})$. For the sake of simplicity we set

$$\mathcal{E}_0(\varphi) = \frac{(p-1)\int_{-\infty}^{\infty}\left((\partial_t\varphi(t))^2 + \tilde{G}(t)\varphi(t)^2\right)dt}{\int_{-\infty}^{\infty}\varphi(t)^2\,dt}. \qquad (6.5)$$

Then we have

$$\begin{aligned} \mu_0 &= \inf \mathcal{E}_0(\varphi) : \varphi \in H^1(\mathbf{R})\}, \qquad (6.6) \\ &\text{where} \quad H^1(\mathbf{R}) = \{\varphi \in L^2(\mathbf{R}) : \varphi' \in L^2(\mathbf{R})\}. \end{aligned}$$

The potential function $\tilde{G}(t)$ is simply given by

$$\left\{ \begin{aligned} &\tilde{G}(t) = G(e^{-t})e^{-t} = A - B\cdot Q(t) + C\cdot R(t), \\ &Q(t) = \frac{e^{-t\frac{p}{p-1}h}}{(1+e^{-t\frac{p}{p-1}h})^2}, \qquad R(t) = \frac{1}{(1+e^{-t\frac{p}{p-1}h})^2}. \end{aligned} \right. \qquad (6.7)$$

Under the condition $\varphi \in H^1(\mathbf{R})$ and $\int_{\mathbf{R}}\varphi^2\,dt = 1$ we shall minimize the functional $\mathcal{E}_0(\varphi)$.

In the next we shall show the negativity of the first eigenvalue to this problem.

Lemma 6.2. *For an arbitrary number* $l > 0$*, there is a positive number* M *such that if* $\gamma > M$*, then the eigenvalue problem (6.6) has the negative first eigenvalue* μ *such as* $\mu < -l$ *and the corresponding first eigenfunction in* $H^1(\mathbf{R})$.

Proof: First we show that $\tilde{G}(t)$ has a negative minimum provided that γ is large enough. Setting $t = -\frac{p-1}{ph}\log s$ $(0 < s < \infty)$ in $\tilde{G}(t)$ we have

$$\tilde{G}\left(-\frac{p-1}{ph}\log s\right) = A - B\cdot\frac{s}{1+s^2} + C\cdot\frac{1}{1+s^2} = \left(\frac{q\gamma}{2n(p-1)}\right)^2\cdot S(s).$$

Here

$$S(s) = \left(\frac{pn}{q}\right)^2 + \frac{n^2p(p-2)(1-2\tau_{p,q})}{(1+s)^2} + \frac{2n^2p(-1-2\tau_{p,q}(p-1))s}{(1+s)^2}.$$

Now we study the minimum of $S(s)$ in $(0,\infty)$. By differentiating $S(s)$ we have

$$S'(s) = \frac{2n^2p(s(1+2\tau_{p,q}p - 2\tau_{p,q}) + 1 - p - 2\tau_{p,q})}{(1+s)^3}.$$

Therefore $S(s)$ takes its minimum when

$$s_0 = \frac{p-1+2\tau_{p,q}}{1+2\tau_{p,q}(p-1)} > 0.$$

We note that s_0 is independent of γ and that the minimum is given by

$$S(s_0) = -\frac{n^2\tau_{p,q}(3-2\tau_{p,q})(2(p-1)+p^2\tau_{p,q})}{1+2\tau_{p,q}} < 0.$$

After all we see that $\tilde{G}(t)$ takes its minimum at $t_0 = -\frac{p-1}{ph}\log s_0$, and the value is given by

$$\tilde{G}(t_0) = C(p,q)\gamma^2,$$

with

$$C(p,q) = -\frac{q^2\tau_{p,q}(3-2\tau_{p,q})(2(p-1)+p^2\tau_{p,q})}{4(p-1)^2(1+2\tau_{p,q})} < 0. \tag{6.8}$$

Clearly this minumum $\tilde{G}(t_0)$ goes to $-\infty$ as $\gamma\to\infty$. Then it is not difficult to show the assertion holds provided that γ is large enough. The existence of the first eigenfunction is also proved by a standard argument. □

Let w be the first eigenfunction of (6.6) with $||w||_{L^2} = 1$. Let us set $v_0(r) = w(-\log r)$ and $f_0(r) = v_0(r)\xi(r)$. If we check f_0 in $W^{1,2}(\mathbf{R}_+; \omega r^{n-1})$, then f_0 is the first eigenfunction to the problem (4.12). To this end we prepare the next lemma which finishes the proof of Theorem 5.2. (The proof is omitted.)

Lemma 6.3. *Let $f_0 = v_0(r)\xi(r)$ with $v(r) = w(-\log r)$. Then we have $f_0 \in W^{1,2}(\mathbf{R}_+; \omega r^{n-1})$. Further we have $f_0(|x|) \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.*

Proof of Proposition 5.1: Let us set

$$\varphi = c_0 f_0(\phi_0 + c_1\phi_1)$$

with c_0, c_1 and $\phi_0 > 0$ being constants. Since $f_0 \in W^{1,2}(\mathbf{R}_+, \omega r^{n-1})$ and $f_0(|x|) \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$, we have

$$\varphi \in W^{1,2}(\mathbf{R^n}, \omega) \cap W^{1,p}_{\gamma,0}(\mathbf{R}^n). \tag{6.9}$$

Now we establish Proposition 4.1. By the definition of μ_k we see $\mu_k = \nu_k + \mu_0$ with $k = 1, 2$ and for a sufficiently large $\gamma > 0$ we can assume that $\mu_k < 0$ with $k = 1, 2$. If $p \geq 2$, then by Hörder's inequality and (6.9) we have the next estimate which clearly verifies the assertion.

$$\int_{\mathbf{R}^n} |\nabla(u(r) + s\varphi(x))|^{p-2}|\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\,dx \leq \tag{6.10}$$
$$C\left(\int_{\mathbf{R}^n} |\nabla u(r)|^{p-2}|\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\,dx + \int_{\mathbf{R}^n} s^{p-2}|\nabla\varphi(x)|^p I_{p(\gamma+1)}(r)\,dx\right) < +\infty.$$

Now we proceed to the case that $1 < p < 2$. By Λ we denote a spherical gradient operator on a unit sphere S^{n-1} satisfying $\Lambda^*\Lambda = \Delta_{S^{n-1}}$. Then we immediately have for $c_2 > 0$

$$|\nabla\varphi| \leq c_2\left(\frac{|f_0|^2}{r^2} + |f_0'|^2\right)^{1/2}, \tag{6.11}$$

and

$$\begin{aligned} |\nabla(u + s\varphi)|^2 &= (u' + s(c_0 f_0'\phi_0 + c_1 f_0'\phi_1))^2 + (sc_1 f_0 \Lambda\phi_1)^2 r^{-2} \quad (6.12)\\ &\geq \max\{(u' + s(c_0 f_0'\phi_0 + c_1 f_0'\phi_1))^2, (sc_1 f_0 \Lambda\phi_1)^2 r^{-2}\}. \end{aligned}$$

Since $f_0(|x|) \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$, we have $\int_{\mathbf{R}^n} {f_0}^p r^{-p} I_{p(\gamma+1)}\,dx < \infty$ by the Hardy inequality. Moreover we note that the term ${f_0'}^2 {f_0}^{p-2} r^{2-p}$ has the same asymptotic behavior as $f_0'^p$ as $r \to \infty$ and as $r \to 0$, respectively. Then we have for $c_3 > 0$

$$\int_{\mathbf{R}^n} ((f_0(r)\Lambda\phi_1(x))^2 r^{-2})^{\frac{p-2}{2}} |\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\, dx \le C_3 \times \tag{6.13}$$
$$\int_{S^{n-1}} |\Lambda\phi_1(x)|^{p-2}\, dS \int_0^\infty \left(f_0'(r)^2 + \frac{f_0(r)^2}{r^2}\right) f_0(r)^{p-2} r^{n-p+1} I_{p(\gamma+1)}(r)\, dr$$
$$< \infty.$$

Combining (6.13) with (6.12) we immediately have for any $s > 0$

$$\int_{\mathbf{R}^n} |\nabla(u(r) + s\varphi(x))|^{p-2} |\nabla\varphi(x)|^2 I_{p(\gamma+1)}(r)\, dx < +\infty. \tag{6.14}$$

We shall see that (6.14) is valid for all $s \in [0,1]$ and $|\nabla(u + s\varphi)|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ converges to $|\nabla u|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ in $L^1(\mathbf{R}^n)$ as $s \to +0$. Noting that $\varphi \in W^{1,2}(\mathbf{R}^n, \omega)$, (6.14) remains valid for $s = 0$. By $B_\rho(0)$ we denote a ball centered at the origin with a radius $\rho > 0$. Let us set for $\varepsilon > 0$

$$\mathbf{R}^n = B_\varepsilon(0) \cup (\overline{B_{\varepsilon^{-1}}(0)})^c \cup K_\varepsilon, \quad K_\varepsilon = \mathbf{R}^n \setminus (B_\varepsilon(0) \cup (\overline{B_{\varepsilon^{-1}}(0)})^c).$$

Since $\nabla u(x) \ne 0$ on a compact set K_ε for any $\varepsilon > 0$, we see that $|\nabla(u + s\varphi)|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ converges to $|\nabla u|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ in $L^1(K_\varepsilon)$ as $s \to +0$. In $B_\varepsilon(0)$, f_0 have a regular singularity only at the origin by the theory of ordinary differential equations of the Bessel type. Since $-1 < p-2 < 0$ and $\varphi \in W^{1,2}(\mathbf{R}^n, \omega) \cap W^{1,p}_{\gamma,0}(\mathbf{R}^n)$, we see that the family of functions $|\nabla(u+s\varphi)|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ are integrable on $\overline{B_\varepsilon(0)}$ uniformly in $s \in [0,1]$ for a sufficiently small $\varepsilon > 0$. Therefore $|\nabla(u + s\varphi)|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ converges to $|\nabla u|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ in $L^1(B_\varepsilon(0))$ as $s \to +0$. In a similar way, from the asymptotic estimate of u and f_0 as $r \to \infty$ we see that $|\nabla(u + s\varphi)|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ converges to $|\nabla u|^{p-2}|\nabla\varphi|^2 I_{p(\gamma+1)}$ in $L^1(\mathbf{R}^n \setminus B_{\varepsilon^{-1}}(0))$ for a sufficiently small $\varepsilon > 0$ as $s \to +0$. This proves the assertion.

□

References

[BW] J. Byeon, Z.-Q. Wang, Symmetry breaking of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities, *Communications in Contemporary Mathematics*, **Vol. 4**, 2002, no. 3, pp. 457-465.

[CKN] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.*, **Vol. 53**, 1984, No. 3, pp259-275.

[CW1] F. Catrina, Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence(and nonexistence), and symmetry of extremal functions, *Communications on Pure and Applied Mathematics*, **Vol. 54**, 2001, pp229-258.

[CW2] F. Catrina, Z.-Q. Wang, Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in $\mathbf{R}^n$, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **Vol. 18**, 2001, no. 2, pp157-178.

[Ho2] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, *Journal of Inequality and Application*, **Vol. 1**, 1997, pp 275-292.

[HK1] T. Horiuchi, P. Kumlin, On the minimal solution for quasilinear degenerate elliptic equation and its blow-up, *Journal of Mathematics of Kyoto University*, **Vol. 44**, No.2, 2004, pp 381-439.

[HK2] T. Horiuchi, P. Kumlin, Erratum to "On the minimal solution for quasilinear degenerate elliptic equation and its blow-up", *Journal of Mathematics of Kyoto University*, **Vol. 46**, No.1, 2006, pp 231-234.

[HK3] T. Horiuchi, P. Kumlin, "On the Caffarelli-Kohn-Nirenberg type inequalities involving Critical and Supercritical Weights", *to appear in Kyoto Journal of Mathematics.*