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Stochastic Power Law Fluid Equations [1]

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Abstract

We consider a SPDE (stochastic partial differential equation) which describes the motion of a viscous, incompressible non-Newtonian fluid subject to a random force. Here, the extra stress tensor of the fluid is given by a polynomial of degree $p-1$ of the rate of strain tensor, while the colored noise is considered as a random force. We investigate the existence and the uniqueness of weak solutions to this SPDE.

1 Introduction

This article is based on a joint work with Professor Nobuo Yoshida from Kyoto University ([10]).

A lot of researches on the Navier-Stokes equations which describes the motion of *Newtonian incompressible fluids* have been done since a famous work of Jean Leray ([6]) in the mid 30's in the last century. As a model of the turbulent motion of viscous *Newtonian fluids*, the Stochastic Navier-Stokes equations, which is the Navier-Stokes equations with random force, has been extensively studied in recent years (cf. [3]).

Concerning the motion of *Non-Newtonian fluids*, several models were proposed. One such model is power law fluids, where the viscosity depends polynomially on the symmetric gradient of the velocity of fluids. The equations describing such motions are called the power law fluid equations. The studies of such equations have extensively been done by the group around Nečas and recently by Bothe-Prüss ([1]). For the references on it, see the references in Bothe-Prüss ([1]), for example. Concerning the turbulent flow for power law fluids, no study is done in the mathematical community, to the best our knowledge. We will consider here the stochastic power law fluid equations for studying the turbulent model of power law fluid. We present the existence and uniqueness result for the stochastic power law fluid equations.

Let us be more precise to state the main result. We consider a viscous, incompressible fluid whose motion is subject to a random force. The container of the fluid is supposed to be the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1)^d$ as a part of idealization. For a differentiable vector field $v : \mathbb{T}^d \to \mathbb{R}^d$, which is interpreted

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as the velocity field of the fluid, we denote the *rate of strain tensor* by:

$$e(v) = \left(\frac{\partial_i v_j + \partial_j v_i}{2} \right) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d. \tag{1.1}$$

We assume that the extra stress tensor

$$\tau(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

depends on $e(v)$ polynomially. More precisely, for $\nu > 0$ (the kinematic viscosity) and $p > 1$,

$$\tau(v) = 2\nu(1 + |e(v)|^2)^{\frac{p-2}{2}} e(v). \tag{1.2}$$

The linearly dependent case $p = 2$ is the *Newtonian fluid*, which is described by the Navier-Stokes equations, the special case of (1.3)–(1.4) below. On the other hand, both the *shear thinning* ($p < 2$) and the *shear thickening* ($p > 2$) cases are considered in many fields in science and engineering. For example, shear thinning fluids are used for automobile engine oil and pipeline for crude oil transportation, while applications of shear thickening fluids can be found in modeling of body armors and automobile four wheels driving systems.

Given an initial velocity $u_0 : \mathbb{T}^d \to \mathbb{R}^d$, the dynamics of the fluid is described by the following SPDE:

$$\text{div } u = 0, \tag{1.3}$$

$$\partial_t u + (u \cdot \nabla) u = -\nabla \Pi + \text{div } \tau(u) + \partial_t W, \tag{1.4}$$

where

$$u \cdot \nabla = \sum_{j=1}^{d} u_j \partial_j \quad \text{and} \quad \text{div } \tau(u) = \left(\sum_{j=1}^{d} \partial_j \tau_{ij}(u) \right)_{i=1}^{d}. \tag{1.5}$$

The unknown process in the SPDE are the velocity field $u = u(t,x) = (u_i(t,x))_{i=1}^d$ and the pressure $\Pi = \Pi(t,x)$. The Brownian motion $W = W(t,x) = (W_i(t,x))_{i=1}^d$ with values in $L_2(\mathbb{T}^d \to \mathbb{R}^d)$ (the set of vector fields on $\mathbb{T}^d$ with L_2 components) is added as the random force. Physical interpretation of (1.3) and (1.4) are the mass conservation, and the motion equation, respectively. We note that the SPDE (1.3)–(1.4) for the case $p = 2$ is the stochastic Navier-Stokes equations [3, 4].

Our motivation comes from works by J. Málek, J. Nečas, M. Rokyta, and M. Růžička [7], where the deterministic equation (the colored noise $\partial_t W$ in (1.3)–(1.4) is replaced by a non-random external force) is investigated. Let:

$$p_1(d) = \frac{3d}{d+2} \vee \frac{3d-4}{d} = \begin{cases} \frac{3d}{d+2} & \text{for } d \leq 4, \\ \frac{3d-4}{d} & \text{for } d \geq 4. \end{cases}, \tag{1.6}$$

$$p_2(d) = \frac{2d}{d-2}, \quad p_3(d) = \frac{3d - 8 + \sqrt{9d^2 + 64}}{2d} \tag{1.7}$$

and

$$p \in \begin{cases} (p_1(d), \infty) & \text{if } 2 \le d \le 8, \\ (p_1(9) \ \ p_2(9)) \cup (p_3(9), \ \infty) & \text{if } d = 9, \\ (p_3(d), \infty) & \text{if } d \ge 10, \end{cases} \tag{1.8}$$

For example, $p_1(d) = \frac{3}{2}, \frac{9}{5}, 2, \frac{11}{5}$ for $d = 2, 3, 4, 5$. A basic existence theorem [7, p.222, Theorem 3.4] states that the deterministic equation has a weak solution if (1.8) is satisfied, while a weak solution is unique if $p \ge 1 + \frac{d}{2}$ [7, p.254, Theorem 4.29].

The results in the present paper (Theorem 2.1.3 and Theorem 2.2.1 below) confirm that the above mentioned deterministic results are stable under the random perturbtation we consider.

Let us briefly sketch the outline of the proof of our existence result:

Step 1: Set up a finite dimensional subspace of a smooth, divergence-free vector fields, say $\mathcal{V}_n$, and an approximating equation to the SPDE (1.3)–(1.4) in $\mathcal{V}_n$. A good news here is that the approximating equation is a well posed SDE, admitting a unique strong solution $u^n \in \mathcal{V}_n$. See Theorem 3.1.1 below for detail.

Step 2: Establish some a priori bounds for the solution $u^n \in \mathcal{V}_n$ of the approximating SDE (e.g.,(3.10), (3.13), (3.14), (3.15) below). The point here is that the bounds should be *uniform in n* for them to be useful. Martingale inequalities (e.g., the Burkholder-Davis-Gundy inequality) are effectively used here, working in team with the Sobolev imbedding theorem. See for example the proof of (3.10) below for details.

Step 3: Show that the solutions $u^n \in \mathcal{V}_n$ to the approximating SDE are tight as $n \to \infty$. This is where the a priori bounds in Step 2 play their roles as the moment estimates to ensure that the tails of the solutions are thin enough in certain Sobolev norms. This tightness argument is implemented in section 3.4.

Step 4: By Step 3, u^n $(n \to \infty)$ converges in law along a subsequence to a limit. We verify that the limit is a weak solution to the SPDE (1.3)–(1.4). These will be the subjects of section 4.1.

Here are some comments concerning the technical difference between the Navier-Stokes equations ($p = 2$) and the power law fluid equations. For the Navier-Stokes equations (both stochastic [3, 4] and deterministic [9]), it is reasonable to discuss solutions in the L_2-space. On the other hand, for the power law fluids given by (1.2), it is the L_p-space and its dual space that become relevant. Also, due to the extra non-linearity introduced by (1.2), some of the arguments for $p \neq 2$ become considerably more involved than the case of $p = 2$, especially for $p < 2$. (See for example, proof of Lemma 3.2.2 below.) We will overcome this difficulty by carrying the ideas in [7] over to the framework of Itô's calculus.

1.1 A weak formulation

Let $\mathcal{V}$ be the set of $\mathbb{R}^d$-valued divergence free, mean-zero trigonometric polynomials, i.e., the set of $v : \mathbb{T}^d \to \mathbb{R}^d$ of the following form:

$$v(x) = \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \widehat{v}_z \psi_z(x), \quad x \in \mathbb{T}^d, \tag{1.9}$$

where $\psi_z(x) = \exp(2\pi \mathbf{i} z \cdot x)$ and the coefficients $\widehat{v}_z \in \mathbb{C}^d$, $z \in \mathbb{Z}^d$ satisfy

$$\widehat{v}_z = 0 \quad \text{except for finitely many } z, \tag{1.10}$$
$$\overline{\widehat{v}_z} = \widehat{v}_{-z} \quad \text{for all } z, \tag{1.11}$$
$$z \cdot \widehat{v}_z = 0 \quad \text{for all } z. \tag{1.12}$$

Note that (1.12) implies that:

$$\text{div } v = 0 \quad \text{for all } v \in \mathcal{V}.$$

For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define

$$(1 - \Delta)^{\alpha/2} v = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2 |z|^2)^{\alpha/2} \widehat{v}_z \psi_z.$$

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure. For $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$, we introduce:

$$V_{p,\alpha} = \text{the completion of } \mathcal{V} \text{ with respect to the norm } \|\cdot\|_{p,\alpha}, \tag{1.13}$$

where

$$\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^p. \tag{1.14}$$

Then,

$$V_{p,\alpha+\beta} \subset V_{p,\alpha}, \quad \text{for } 1 \leq p < \infty, \ \alpha \in \mathbb{R} \text{ and } \beta > 0 \tag{1.15}$$

and the inclusion $V_{p,\alpha+\beta} \to V_{p,\alpha}$ is compact if $1 < p < \infty$ [8, p.23, (6.9)].

For $v, w : \mathbb{T}^d \to \mathbb{R}^d$, with w supposed to be differentiable (for a moment), we define a vector field:

$$(v \cdot \nabla)w = \sum_j v_j \partial_j w, \tag{1.16}$$

which is bilinear in (v, w). Later on, we will generalize the definition of the above vector field (cf. (1.30)).

Here are integration-by-parts formulae with which we reformulate (1.3)–(1.4) into its weak formulation. We omit its proof. In what follows, the bracket $\langle u, v \rangle$ stands for the inner product of $L_2(\mathbb{T}^d \to \mathbb{R}^d)$, or its appropriate generalization, e.g., the pairing of $u \in V_{p,\alpha}$ and $u \in V_{p',-\alpha}$ ($p \in (1,\infty)$, $p' = \frac{p}{p-1}$, $\alpha \geq 0$). We let $C^r(\mathbb{T}^d \to \mathbb{R}^d)$ ($r = 1, ..., \infty$) denote the set of vector fields on $\mathbb{T}^d$ with C^r components.

Lemma 1.1.1 *For $v \in \mathcal{V}$ and $w, \varphi \in C^1(\mathbb{T}^d \to \mathbb{R}^d)$,*

$$\langle \varphi, (v \cdot \nabla)w \rangle = -\langle w, (v \cdot \nabla)\varphi \rangle, \tag{1.17}$$

In particular,

$$\langle w, (v \cdot \nabla)w \rangle = 0. \tag{1.18}$$

Furthermore,

$$\langle \varphi, \operatorname{div} \tau(v) \rangle = -\langle \tau(v), e(\varphi) \rangle. \tag{1.19}$$

Let us explain formally how the transformation of the problem (1.3)–(1.4) into its weak formulation. Suppose that u, Π and "$\partial_t W$"in (1.3)–(1.4) are regular enough. Then, for a test function $\varphi \in \mathcal{V}$,

$$*) \qquad \partial_t\langle \varphi, u \rangle = -\underbrace{\langle \varphi, (u \cdot \nabla)u \rangle}_{(1)} + \underbrace{\langle \varphi, \operatorname{div} \tau(u) \rangle}_{(2)} - \underbrace{\langle \varphi, \nabla \Pi \rangle}_{(3)} + \langle \partial_t W, \varphi \rangle.$$

$$(1) \overset{(1.17)}{=} -\langle (u \cdot \nabla)\varphi, u \rangle, \quad (2) \overset{(1.19)}{=} -\langle e(\varphi), \tau(u) \rangle, \quad (3) = -\langle \operatorname{div} \varphi, \Pi \rangle = 0.$$

Thus, $*)$ becomes

$$\partial_t\langle \varphi, u \rangle = \langle (u \cdot \nabla)\varphi, u \rangle - \langle e(\varphi), \tau(u) \rangle + \partial_t\langle \varphi, W \rangle.$$

By integration, we arrive at:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t (\langle (u_s \cdot \nabla)\varphi, u_s \rangle - \langle e(\varphi), \tau(u_s) \rangle)\, ds + \langle \varphi, W_t \rangle. \tag{1.20}$$

Here, $u_t = u(t, \cdot)$ and $W_t = W(t, \cdot)$. This is a standard weak formulation of (1.3)–(1.4).

1.2 Bounds on the non-linear terms

Let us prepare a couple of L_p-bounds on the non-linear terms, some of whose proofs we will omit.

Lemma 1.2.1 *Let $\alpha_i \in [0,\infty)$, $p_i \in [1,\infty)$, $i = 1,2,3$, be such that:*

$$A \geq Bd, \quad \text{where } A = \textstyle\sum_i \alpha_i \text{ and } B = \sum_i \frac{1}{p_i} - 1. \tag{1.21}$$

a) *Suppose (1.21) and that $\frac{\alpha_i B}{A} < \frac{1}{p_i}$ for all $i = 1,2,3$. Then, there exists $C_1 \in (0,\infty)$ such that:*

$$|\langle\, w, (v\cdot\nabla)\varphi\,\rangle| \leq C_1 \|v\|_{p_1,\alpha_1}\|w\|_{p_2,\alpha_2}\|\varphi\|_{p_3,1+\alpha_3}. \tag{1.22}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

b) *Suppose (1.21), $\alpha_1 + \alpha_2 > 0$, and that $B \leq \frac{1}{p_i}$ for all $i = 1,2,3$. Then, for any $\theta \in (0,1)$, there exists $C_2 \in (0,\infty)$ such that:*

$$|\langle\, w, (v\cdot\nabla)\varphi\,\rangle| \leq C_2 \|v\|^{\theta}_{p_1,\alpha_1}\|v\|^{1-\theta}_{p_1,\alpha_2}\|w\|^{1-\theta}_{p_2,\alpha_1}\|w\|^{\theta}_{p_2,\alpha_2}\|\varphi\|_{p_3,1+\alpha_3}. \tag{1.23}$$

Lemma 1.2.2 *Let: $\alpha \in (0,1]$ and $p \in (\frac{2d}{d+2\alpha}, \infty)$.*

a) *Suppose that $(d,p,\alpha) \neq (2,2,1)$. Then, there exists $C_1 \in (0,\infty)$ such that:*

$$|\langle\, w, (v\cdot\nabla)\varphi\,\rangle| \leq C_1 \|v\|_{p,\alpha}\|w\|_2\|\varphi\|_{p,\beta(p,\alpha)}. \tag{1.24}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$, where

$$\beta(p,\alpha) = \begin{cases} 1 + (\frac{2}{p} - \frac{1}{2})d - \alpha > 1, & \text{if } p < \frac{4d}{d+2\alpha}, \\ 1, & \text{if } p \geq \frac{4d}{d+2\alpha}. \end{cases} \tag{1.25}$$

b) *Suppose that $d = 2$. Then, for any $\theta \in (0,1)$, there exists $C_2 \in (0,\infty)$ such that:*

$$|\langle\, w, (v\cdot\nabla)\varphi\,\rangle| \leq C_2 \|v\|^{\theta}_{2,1}\|v\|^{1-\theta}_{2}\|w\|^{1-\theta}_{p,1}\|w\|^{\theta}_{2}\|\varphi\|_{2,1}, \tag{1.26}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

Remark: We note that the following variant of (1.24) is also true:

$$|\langle\, w, (v\cdot\nabla)\varphi\,\rangle| \leq C_1 \|v\|_2\|w\|_{p,\alpha}\|\varphi\|_{p,\beta(p,\alpha)}. \tag{1.27}$$

This can be seen by interchanging the role of (p_1,α_1) and (p_2,α_2) in the above proof.

Lemma 1.2.3 *For $p \in (1,\infty)$, there exists $C_1 \in (0,\infty)$ such that:*

$$|\langle\, e(\varphi), \tau(v)\,\rangle| \le C_1(1+\|e(v)\|_p)^{p-1}\|e(\varphi)\|_p \text{ for all } v \in V_{p,1} \text{ and } \varphi \in \mathcal{V}. \quad (1.28)$$

Proof: Since

$$|\tau(v)| \le C(1+|e(v)|)^{p-1},$$

we have that

$$\begin{aligned} |\langle\, e(\varphi), \tau(v)\,\rangle| \le C\int_{\mathbb{T}^d}(1+|e(v)|)^{p-1}|e(\varphi)| &\overset{\frac{p-1}{p}+\frac{1}{p}=1}{\le} C\|1+|e(v)|\|_p^{p-1}\|e(\varphi)\|_p \\ &\le C(1+\|e(v)\|_p)^{p-1}\|e(\varphi)\|_p, \end{aligned}$$

which proves (1.28). □

Let $p \in (\frac{2d}{d+2},\infty)$, $v,w \in V_{p,1}\cap V_{2,0}$ and $u \in V_{p,1}$. In view of Lemma 1.1.1, we think of $(v\cdot\nabla)w$ and div $\tau(u)$, respectively as the following linear functionals on $\mathcal{V}$:

$$\varphi \mapsto \langle\, \varphi, (v\cdot\nabla)w\,\rangle \overset{\text{def.}}{=} -\langle\, w, (v\cdot\nabla)\varphi\,\rangle,$$
$$\varphi \mapsto \langle\, \varphi, \operatorname{div}\tau(u)\,\rangle \overset{\text{def.}}{=} -\langle\, e(\varphi), \tau(u)\,\rangle.$$

Then, by Lemma 1.2.2 and Lemma 1.2.3, they extend continuously, respectively on $V_{p,\beta(p,1)}$, and on $V_{p,1}$, where:

$$\beta(p,1) = \begin{cases} (\frac{2}{p}-\frac{1}{2})d > 1, & \text{if } p < \frac{4d}{d+2}, \\ 1, & \text{if } p \ge \frac{4d}{d+2}, \end{cases} \quad (1.29)$$

(cf. (1.25)). This way, we regard $(v\cdot\nabla)w \in V_{p',-\beta(p,1)}$ ($p' = \frac{p}{p-1}$) with:

$$\|(v\cdot\nabla)w\|_{p',-\beta(p,1)} \le \begin{cases} C\|v\|_{2,1}^{\theta}\|v\|_2^{1-\theta}\|w\|_{2,1}^{1-\theta}\|w\|_2^{\theta}, & \text{if } p=d=2 \\ C\|v\|_{p,1}\|w\|_2, & \text{if otherwise}, \end{cases} \quad (1.30)$$

and div $\tau(u) \in V_{p',-1}$ with:

$$\|\operatorname{div}\tau(u)\|_{p',-1} \le C(1+\|e(u)\|_p)^{p-1}. \quad (1.31)$$

Finally, for $v \in V_{p,1}\cap V_{2,0}$, we define:

$$b(v) = -(v\cdot\nabla)v + \operatorname{div}\tau(v) \in V_{p',-\beta(p,1)}. \quad (1.32)$$

With this notation, (1.20) takes the form:

$$\langle\, \varphi, u_t\,\rangle = \langle\, \varphi, u_0\,\rangle + \int_0^t \langle\, \varphi, b(u_s)\,\rangle ds + \langle\, \varphi, W_t\,\rangle.$$

i.e.,

$$u_t = u_0 + \int_0^t b(u_s)ds + W_t \quad (1.33)$$

as linear functionals on $\mathcal{V}$.

2 The stochastic power law fluids

2.1 The existence theorem

We need the following definition.

Definition 2.1.1 Let H be a Hilbert space, and $\Gamma : H \to H$ be a self-adjoint, non-negative definite operator of trace class. A random variable $(W_t)_{t\geq 0}$ with values in $C([0,\infty) \to H)$ is called a *H-valued Brownian motion* with the covariance operator Γ (abbreviated by BM(H,Γ) below) if, for each $\varphi \in H$ and $0 \leq s < t$,

$$E\left[\exp\left(\mathbf{i}\langle \varphi, W_t - W_s \rangle\right) | (W_u)_{u\leq s}\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right), \quad \text{a.s.}$$

To introduce the notion of weak solution (Definition 2.1.2 below), we agree on the following standard notation and convention. For a Banach space X, we let $L_{q,\mathrm{loc}}([0,\infty) \to X)$ $(1 \leq q \leq \infty)$ denote the set of locally L_q-functions $u : [0,\infty) \to X$, with the Fréchet space metric induced by the semi-norms $\|u\|_{L_q([0,T]\to X)}$, $0 < T < \infty$, where $\|u\|_{L_q([0,T]\to X)}$ stands for the standard L_q-norm for $u|_{[0,T]} : [0,T] \to X$. We also regard $C([0,\infty) \to X)$, the set of continuous functions $u : [0,\infty) \to X$, as the Fréchet space induced by the semi-norms $\sup_{0\leq t\leq T} \|u(t)\|_X$, $0 < T < \infty$.

We recall that the number p is from (1.2) and that $b(v) \in V_{p',-\beta(p,1)}$ for $v \in V_{p,1} \cap V_{2,0}$ is defined by (1.32).

Definition 2.1.2 Suppose that

▶ $\Gamma : V_{2,0} \to V_{2,0}$ is a bounded self-adjoint, non-negative definite operator of trace class;

▶ μ_0 is a Borel probability measure on $V_{2,0}$.

▶ $(X,Y) = ((X_t,Y_t))_{t\geq 0}$ is a process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that:

$$X \in L_{p,\mathrm{loc}}([0,\infty) \to V_{p,1}) \cap L_{\infty,\mathrm{loc}}([0,\infty) \to V_{2,0}) \cap C([0,\infty) \to V_{2\wedge p',-\beta}), \quad (2.1)$$

for some $\beta > 0$, and $(Y_t)_{t\geq 0}$ is a BM$(V_{2,0},\Gamma)$ (cf. Definition 2.1.1).

Then, the process (X,Y) is said to be a *weak solution* to the SDE (stochastic differential equation)

$$X_t = X_0 + \int_0^t b(X_s)ds + Y_t \quad (2.2)$$

with the initial law μ_0 if the following conditions are satisfied;

$$P(X_0 \in \cdot) = \mu_0; \tag{2.3}$$

$$Y_{t+\cdot} - Y_t \text{ and } \{\langle \varphi, X_s \rangle \,;\, s \le t, \varphi \in \mathcal{V}\} \text{ are independent for any } t \ge 0; \tag{2.4}$$

$$\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle ds + \langle \varphi, Y_t \rangle, \quad \text{for all } \varphi \in \mathcal{V} \text{ and } t \ge 0. \tag{2.5}$$

We can now state our existence result.

Theorem 2.1.3 *Let Γ and μ_0 be as in Definition 2.1.2 and suppose additionally that*

- *(1.8) holds;*
- *$\Delta\Gamma = \Gamma\Delta$ and both Γ, $\Delta\Gamma$ are of trace class;*
- *μ_0 is a probability measure on $V_{2,1}$ and*

$$m_\alpha = \int \|\xi\|_{2,\alpha}^2 \mu_0(d\xi) < \infty \ \textit{for } \alpha = 0, 1. \tag{2.6}$$

Then, there exists a weak solution to the SDE (2.2) with the initial law μ_0 (cf. Definition 2.1.2) such that (2.1) holds with $\beta = \beta(p, 1)$ (cf. (1.29)). Moreover, for any $T > 0$,

$$E\left[\sup_{t \le T} \|X_t\|_2^2 + \int_0^T \|X_t\|_{p,1}^p dt\right] \le (1 + T)C < \infty, \tag{2.7}$$

where $C = C(d, p, \Gamma, m_0) < \infty$.

Remark: It would be worthwhile to mention that Theorem 2.1.3 with $p = 2$ is valid for *all* d, although it is not covered by the condition (1.8) if $d \ge 4$. In fact, Lemma 3.2.2 below is the only place we need condition (1.8). For $p = 2$, however, we can avoid the use of that lemma, cf. remarks at the end of section 3.4 and after Lemma 4.1.1.

2.2 The uniqueness theorem

As in the case of deterministic equation [7, p.254, Theorem 4.29], we have the following uniqueness result:

Theorem 2.2.1 *Suppose that:*

$$p \geq 1 + \frac{d}{2}. \tag{2.8}$$

Then, the weak solution to the SDE (2.2) subject to the a priori bound (2.7) is pathwise unique in the following sense: if (X, Y) *and* $(\widetilde{X}, Y)$ *are two solutions on a common probability space* $(\Omega, \mathcal{F}, P)$ *with a common BM*$(V_{2,0}, \Gamma)$ Y *such that* $X_0 = \widetilde{X}_0$ *a.s., then,*

$$P(X_t = \widetilde{X}_t \text{ for all } t \geq 0) = 1.$$

This can be proved as similarly as the deterministic case, and we omit its proof. The above uniqueness theorem, together with the Yamada-Watanabe theorem provides us with the so called *strong solution in the stochastic sense* to the SDE (2.2).

Corollary 2.2.2 *Suppose (2.8) in addition to all the assumptions in Theorem 2.1.3, and let* ξ *be a given* $V_{2,0}$*-valued random variable with the law* μ_0*, and* Y *be a given BM*$(V_{2,0}, \Gamma)$*, independent of* ξ*. Then, there exists a process* X *obtained as a function of* (ξ, Y)*, such that* (X, Y) *is weak solution to the SDE (2.2) with* $X_0 = \xi$ *and with all the properties stated in Theorem 2.1.3. Moreover, the law of the above process* X *is unique.*

Proof: Corollary 2.2.2 is a direct consequence of Theorem 2.1.3 and Theorem 2.2.1 via the Yamada-Watanabe theorem [2, p.163, Theorem 1.1]. The Yamada-Watanabe theorem is usually stated for SDE's in finite dimensions. However, as is obvious from its proof, it applies to the present setting. □

Remark: For $p \in [1 + \frac{d}{2}, \frac{2d}{d-2})$, an even stronger version of Corollary 2.2.2 is shown in [11] as a consequence of strong convergence of the Galerkin approximation (cf. section 3 below).

3 The Galerkin approximation

3.1 The exsitence theorem for the approximations

For each $z \in \mathbb{Z}^d \backslash \{0\}$, let $\{e_{z,j}\}_{j=1}^{d-1}$ be an orthonormal basis of the hyperplane: $\{x \in \mathbb{R}^d \; ; \; z \cdot x = 0\}$ and let:

$$\psi_{z,j}(x) = \begin{cases} \sqrt{2} e_{z,j} \cos(2\pi z \cdot x), & j = 1, ..., d-1, \\ \sqrt{2} e_{z,j-d+1} \sin(2\pi z \cdot x), & j = d, ..., 2d-2, \end{cases} \quad x \in \mathbb{T}^d. \tag{3.1}$$

Then,

$$\{\psi_{z,j}\ ;\ \ (z,j)\in(\mathbb{Z}^d\backslash\{0\})\times\{1,...,2d-2\}\}$$

is an orthonormal basis of $V_{2,0}$. We also introduce:

$$\begin{aligned}\mathcal{V}_n &= \text{the linear span of } \{\psi_{z,j}\ ;\ (z,j) \text{ with } z\in[-n,n]^d\},\\ \mathcal{P}_n &= \text{the orthogonal projection} : V_{2,0}\to\mathcal{V}_n.\end{aligned}\tag{3.2}$$

Using the orthonormal basis (3.1), we identify $\mathcal{V}_n$ with $\mathbb{R}^N$, $N=\dim\mathcal{V}_n$ Let μ_0 and $\Gamma : V_{2,0}\to V_{2,0}$ be as in Theorem 2.1.3. Let also ξ be a random variable such that $P(\xi\in\cdot)=\mu_0$. Finally, let W_t be a $\mathrm{BM}(V_{2,0},\Gamma)$ defined on a probability space $(\Omega^W,\mathcal{F}^W,P^W)$. Then, $\mathcal{P}_nW_t$ is identified with an N-dimensional Brownian motion with covariance matrix $\Gamma\mathcal{P}_n$. Then, we consider the following approximation of (2.5)

$$X_t^n = X_0^n+\int_0^t \mathcal{P}_n b(X_s^n)ds+\mathcal{P}_nW_t,\quad t\geq 0,\tag{3.3}$$

where $X_0^n=\mathcal{P}_n\xi$. Let:

$$X_t^{n,z,j}=\langle\, X_t^n,\psi_{z,j}\,\rangle\tag{3.4}$$

be the (z,j)-coordinate of X_t^n. Then, (3.3) reads:

$$X_t^{n,z,j}=X_0^{n,z,j}+\int_0^t b^{z,j}(X_s^n)ds+W_t^{z,j},\tag{3.5}$$

where

$$b^{z,j}(X_s^n)=\langle\, X_s^n,(X_s^n\cdot\nabla)\psi_{z,j}\,\rangle-\langle\,\tau(X_s^n),e(\psi_{z,j})\,\rangle,\quad W_t^{z,j}=\langle\, W_t,\psi_{z,j}\,\rangle.\tag{3.6}$$

Let $W_\cdot$ and ξ as above. We then define

$$\begin{aligned}\mathcal{G}_t^{\xi,W} &= \sigma(\xi,W_s,\ s\leq t),\ \ 0\leq t<\infty,\quad \mathcal{G}_\infty^{\xi,W}=\sigma\left(\cup_{t\geq 0}\mathcal{G}_t^{\xi,W}\right),\\ \mathcal{N}^{\xi,W} &= \{N\subset\Omega,\ ;\ \exists\widetilde{N}\in\mathcal{G}_\infty^{\xi,W},\ N\subset\widetilde{N},\ P^W(\widetilde{N})=0\},\end{aligned}$$

and

$$\mathcal{F}_t^{\xi,W}=\sigma\left(\mathcal{G}_t^{\xi,W}\cup\mathcal{N}^{\xi,W}\right),\ \ 0\leq t<\infty.\tag{3.7}$$

In what follows, expectation with respect to the measure P^W will be denoted by $E^W[\,\cdot\,]$.

Theorem 3.1.1 *Let $W_\cdot$, ξ, and $\mathcal{F}_t^{\xi,W}$ be as above. Then, for each $n=1,2,...$ there exists a unique process $X_\cdot^n$ such that:*

a) *X_t^n is $\mathcal{F}_t^{\xi,W}$-measurable for all $t \geq 0$;*

b) *(3.3) is satisfied;*

c) *For any $T > 0$,*

$$E^W\left[\|X_T^n\|_2^2 + 2\int_0^T \langle\, e(X_t^n), \tau(X_t^n)\,\rangle dt\right] = E^W[\|X_0^n\|_2^2] + \mathrm{tr}(\Gamma\mathcal{P}_n)T, \tag{3.8}$$

$$E^W\left[\|X_T^n\|_2^2 + \frac{1}{C}\int_0^T \|X_t^n\|_{p,1}^p dt\right] \leq m_0 + (C + \mathrm{tr}(\Gamma))T < \infty, \tag{3.9}$$

where $C = C(d,p) \in (0,\infty)$.

Suppose in addition that $p \geq \frac{2d}{d+2}$, where p is from (1.2). Then, for any $T > 0$,

$$E^W\left[\sup_{t\leq T}\|X_t^n\|_2^2 + \int_0^T \|X_t^n\|_{p,1}^p dt\right] \leq (1+T)C' < \infty, \tag{3.10}$$

where $C' = C'(d,p,\Gamma,m_0) \in (0,\infty)$.

Proof: We fix the accuracy n of the approximation introduced above, and suppress the superscript "n " from the notation: $X = X^n$. We write the summation over $z \in [-n,n]^d$ and $j = 1,..,2d-2$ simply by $\sum_{z,j}$. Since $v \mapsto \mathcal{P}_n b(v) : \mathcal{V}_n \to \mathcal{V}_n$ is locally Lipschitz continuous (see (3.6)) and

1) $$\langle\, v, b(v)\,\rangle \stackrel{(1.18)}{=} -\langle\, e(v), \tau(v)\,\rangle \leq C - \frac{1}{C}\|v\|_{p,1}^p,$$

where we have used [7, (1.11) on p.196,and $(1.20)_2$ on p.198] to see the second inequality. This implies that there exists a unique process $X.$ with the properties a)–b) above, as can be seen from standard existence and uniqueness results for the SDE, e.g. [2, Theorem 2.4 on p.177, Theorem 3.1 on pp.178–179] (cf. the remark after the proof). Note that for $\alpha = 0,1,2,...$:

$$\|\nabla^\alpha v\|_2^2 = \langle\, v, (-\Delta)^\alpha v\,\rangle = \sum_{z,j}(-4\pi^2|z|^2)^\alpha\langle\, v, \psi_{z,j}\,\rangle^2, \quad v \in \mathcal{V}_n.$$

On the other hand, we have by Itô's formula that:

$$|X_t^{z,j}|^2 = |X_0^{z,j}|^2 + 2\int_0^t X_s^{z,j}dW_s^{z,j} + 2\int_0^t X_s^{z,j}b_s^{z,j}(X_s)ds + \langle\, \psi_{z,j}, \Gamma\psi_{z,j}\,\rangle t,$$

Therefore,

$$\|\nabla^\alpha X_t\|_2^2 = \|\nabla^\alpha X_0\|_2^2 + 2M_t + 2\int_0^t \langle\, (-\Delta)^\alpha X_s, b(X_s)\,\rangle ds + \mathrm{tr}(\Gamma(-\Delta)^\alpha \mathcal{P}_n)t, \tag{3.11}$$

where

$$M_t = \sum_{z,j} \int_0^t (-\Delta)^\alpha X_s^{z,j} dW_s^{z,j}. \tag{3.12}$$

Here, we will use (3.11) only for $\alpha = 0$. The case $\alpha = 1$ will be used in the proof of Lemma 3.2.3 later on. By (3.11) with $\alpha = 0$,

2) $$\|X_t\|_2^2 + \frac{2}{C}\int_0^t \|X_s\|_{p,1}^p ds \le \|X_0\|_2^2 + 2M_t + (C + \mathrm{tr}(\Gamma))t,$$

where M_t in 2) is defined by (3.12) with $\alpha = 0$. Since it is not difficult to see that the above M_t is a martingale (cf. [3, p.60, Proof of (10)]), we get (3.8) by taking expectation of the equality (3.11). Similarly, we obtain (3.9) by taking expectation of the inequality 2). To see (3.10), it is enough to show that there exists $\delta \in (0,1]$ such that:

3) $$E^W\left[\sup_{t\le T}\|X_t\|_2^2\right] \le (1+T)C + CE^W\left[\left(\int_0^T \|X_t\|_{p,1}^p dt\right)^\delta\right].$$

To see this, we start with a bound on the quadratic variation of the martingale $M_\cdot$:

4) $$\langle\, M\,\rangle_t = \int_0^t \langle\, \Gamma X_s, X_s\,\rangle ds \le \|\Gamma\|_{2\to 2}\int_0^t \|X_s\|_2^2 ds,$$

where $\|\Gamma\|_{2\to 2}$ denotes the operator norm of $\Gamma : V_{2,0} \to V_{2,0}$. We now recall the Burkholder-Davis-Gundy inequality [2, p.110, Theorem 3.1]:

5) $$E^W\left[\sup_{t\le T}|M_t|^q\right] \le CE^W\left[\langle\, M\,\rangle_T^{q/2}\right] \quad \text{for } q \in (0,\infty)$$

We then observe that:

6) $$\begin{aligned} E^W\left[\sup_{t\le T}\|X_t\|_2^2\right] &\overset{2)}{\le} (1+T)C + 2E^W\left[\sup_{t\le T}|M_t|\right] \\ &\overset{4)-5)}{\le} (1+T)C + C'E^W\left[\left(\int_0^T \|X_s\|_2^2 ds\right)^{1/2}\right]. \end{aligned}$$

This proves 3) for $p \geq 2$. We assume $p < 2$ in what follows. We have

$$e_\ell \stackrel{\text{def}}{=} \inf\{t \,;\, \|X_t\|_2 \geq \ell\} \nearrow \infty, \quad \text{as } \ell \nearrow \infty,$$

since the process X_t does not explode. On the other hand, it is clear that the following variant of 6) is true:

6') $$E^W\left[\sup_{t \leq T\wedge e_\ell} \|X_t\|_2^2\right] \leq (1+T)C + CE^W\left[\left(\int_0^{T\wedge e_\ell} \|X_s\|_2^2 ds\right)^{1/2}\right].$$

We have by Sobolev embedding that for $v \in V_{p,1}$:

7) $\|v\|_2 \leq C\|v\|_{p,1}$, since $p \geq \frac{2d}{d+2}$.

Let $\varepsilon > 0$, $r = \frac{4}{2-p} \in (4,\infty)$ and $r' = \frac{r}{r-1} = \frac{4}{2+p} \in (1, 4/3)$. Then,

8) $$\begin{aligned}\left(\int_0^{T\wedge e_\ell} \|X_s\|_2^2 ds\right)^{1/2} &\leq \sup_{s\leq T\wedge e_\ell} \|X_s\|_2^{\frac{2-p}{2}} \left(\int_0^{T\wedge e_\ell} \|X_s\|_2^p ds\right)^{1/2} \\ &\stackrel{7)}{\leq} C \sup_{s\leq T\wedge e_\ell} \|X_s\|_2^{\frac{2-p}{2}} \left(\int_0^{T\wedge e_\ell} \|X_s\|_{p,1}^p ds\right)^{1/2} \\ &\stackrel{\text{Young}}{\leq} \frac{\varepsilon^r C}{r} \sup_{s\leq T\wedge e_\ell} \|X_s\|_2^2 + \frac{\varepsilon^{-r'} C}{r'} \left(\int_0^{T\wedge e_\ell} \|X_s\|_{p,1}^p ds\right)^{\frac{2}{2+p}}.\end{aligned}$$

Since $E^W\left[\sup_{t\leq T\wedge e_\ell} \|X_t\|_2^2\right] \leq \ell^2 < \infty$, we have by 6') and 8) that:

$$E^W\left[\sup_{t \leq T\wedge e_\ell} \|X_t\|_2^2\right] \leq (1+T)C + CE^W\left[\left(\int_0^{T\wedge e_\ell} \|X_t\|_{p,1}^p dt\right)^{\frac{2}{2+p}}\right].$$

Letting $\ell \nearrow \infty$, we obtain 3). □

Remark: Unfortunately, the SDE (3.3) does not satisfy the condition (2.18) imposed in the existence theorem [2, p.177, Theorem 2.4]. However, we easily see from the proof of the existence theorem that (2.18) there can be replaced by:

$$\|\sigma(x)\|^2 + x \cdot b(x) \leq K(1+|x|^2).$$

We have applied [2, p.177, Theorem 2.4] with this modification.

3.2 Further a priori bounds

We first prove the following general estimates, which apply both to the weak solution X to (2.2), and to the unique solution to (3.3).

Lemma 3.2.1 *Let $T > 0$ and $X = (X_t)_{t\geq 0}$ be a process on a probability space $(\Omega, \mathcal{F}, P)$ such that:*

$$X \in L_p([0,T] \to V_{p,1}) \cap L_\infty([0,T] \to V_{2,0}), \quad a.s.$$

and

$$A_T = E\left[\int_0^T \|X_s\|_{p,1}^p ds\right] < \infty, \quad B_T = E\left[\sup_{s\in[0,T]} \|X_s\|_2^2\right] < \infty.$$

a) *For $p \in [\frac{2d}{d+2}, \infty)$,*

$$E\left[\left(\int_0^T \|(X_s \cdot \nabla)X_s\|_{p',-\beta(p,1)}^p ds\right)^\delta\right] \leq C A_T^\delta B_T^{1-\delta} < \infty, \tag{3.13}$$

where $\delta = \frac{p}{p+2}$, $p' = \frac{p}{p-1}$, $\beta(p,1)$ is defined by (1.29), and $C = C(d,p) \in (0,\infty)$.

b)

$$E\left[\int_0^T \|\mathrm{div}\ \tau(X_s)\|_{p',-1}^{p'} ds\right] \leq (T + A_T)C' < \infty, \tag{3.14}$$

where $C' = C'(p,\nu) \in (0,\infty)$.

Proof: a): We have by (1.30) that

1) $\quad \|(v\cdot\nabla)v\|_{p',-\beta(p,1)} \leq C\|v\|_{p,1}\|v\|_2 \quad$ for $v \in V_{p,1} \cap V_{2,0}$

We then use 1) to see that

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_0^T \|(X_s\cdot\nabla)X_s\|_{p',-\beta(p,1)}^p ds \stackrel{1)}{\leq} C\int_0^T \|X_s\|_{p,1}^p \|X_s\|_2^p ds \\ &\leq C \sup_{s\in[0,T]} \|X_s\|_2^p \int_0^T \|X_s\|_{p,1}^p ds. \end{aligned}$$

Finally, noting that $\frac{p\delta}{1-\delta} = 2$, we conclude that

$$\begin{aligned} E[I^\delta] &\leq CE\left[\sup_{s\in[0,T]} \|X_s\|_2^{p\delta} \left(\int_0^T \|X_s\|_{p,1}^p ds\right)^\delta\right] \\ &\leq CE\left[\sup_{s\in[0,T]} \|X_s\|_2^2\right]^{1-\delta} E\left[\int_0^T \|X_s\|_{p,1}^p ds\right]^\delta = CB_T^{1-\delta} A_T^\delta. \end{aligned}$$

b):

$$\|\text{div}\,\tau(X_s)\|_{p',-1} \overset{(1.28)}{\leq} C(1+\|e(X_s)\|_p)^{p-1}.$$

which implies that:

$$\|\text{div}\,\tau(X_s)\|_{p',-1}^{p'} \leq C + C\|e(X_s)\|_p^p$$

and hence that:

$$\begin{aligned} &E\left[\int_0^T \|\text{div}\,\tau(X_s)\|_{p',-1}^{p'} ds\right] \\ &\leq \; CT + CE\left[\int_0^T \|e(X_s)\|_p^p ds\right] \leq (T+A_T)C. \end{aligned}$$

□

Let $X^n = (X_t^n)_{t\geq 0} \in \mathcal{V}$ be the unique solution of (3.3) for the Galerkin approximation.

Lemma 3.2.2 *Suppose (1.8). Then, there exist* $\widetilde{p} \in (1,p)$ *and* $\widetilde{\alpha} \in (1,\infty)$ *such that for each* $T > 0$*:*

$$E^W\left[\int_0^T \|X_t^n\|_{\widetilde{p},\widetilde{\alpha}}^{\widetilde{p}} dt\right] \leq C_T < \infty, \tag{3.15}$$

where the constant C_T *is independent of* n.

We will have slightly the better results than the results stated in Lemma 3.2.2 in the course of the proof. For i) $d=2$ and $p \geq 2$ and ii) $d \geq 3$ and $p > p_3(d)$, we have that:

$$E^W\left[\int_0^T \|\Delta X_t^n\|_2^{\frac{2p}{p+2\lambda}} dt\right] \leq C_T < \infty, \tag{3.16}$$

where $\lambda \geq 0$ is defined by (3.18) below. For $p < \frac{2d}{d-2}$, we have that:

$$E^W\left[\int_0^T \|X_t^n\|_{p,\widetilde{\alpha}}^{\widetilde{p}} dt\right] \leq C_T < \infty, \tag{3.17}$$

for *any* $\widetilde{p} \in (1,p)$ with some $\widetilde{\alpha} = \widetilde{\alpha}(\widetilde{p}) > 1$.

The rest of this section is devoted to the proof of Lemma 3.2.2. We suppress the superscript n from the notations. We write the summation over $z \in [-n,n]^d$ and $j = 1,..,2d-2$ simply by $\sum_{z,j}$. We first establish the following bounds.

Lemma 3.2.3 *Suppose that $p \in (\frac{3d-4}{d}, \infty)$ if $d \geq 3$ and let:*

$$\lambda = \begin{cases} 0 & \text{if } d = 2, \\ \dfrac{2(3-p)^+;}{dp - 3d + 4} & \text{if } d \geq 3 \end{cases} \quad \textit{cf. [7, p.236, (3.47)]}, \tag{3.18}$$

$$\mathcal{J}_t = \begin{cases} \dfrac{\|\Delta X_t\|_2^2}{(1 + \|\nabla X_t\|_2^2)^\lambda}, & \text{if } p \geq 2, \\ \dfrac{\|\Delta X_t\|_p^2}{(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^{2-p}}, & \text{if } 1 < p < 2. \end{cases} \tag{3.19}$$

Then, for any $T > 0$,

$$E^W \left[\int_0^T \mathcal{J}_t dt \right] \leq C_T < \infty, \tag{3.20}$$

where $C_T = C(T, d, p, \Gamma, m_1)$.

Proof: By (3.11) with $\alpha = 1$,

1) $\quad \frac{1}{2}\|\nabla X_t\|_2^2 = \frac{1}{2}\|\nabla X_0\|_2^2 + M_t + \int_0^t K_s ds,$

where

$$M_t = -\sum_{z,j} \int_0^t \Delta X_s^{z,j} dW_s^{z,j}, \quad K_s = \langle -\Delta X_s, b(X_s) \rangle + \tfrac{1}{2}\mathrm{tr}(-\Gamma \Delta \mathcal{P}_n).$$

Step 1: We will prove that:

2) $\quad K_s + c_1 \mathcal{I}_s \leq \begin{cases} 0 & \text{if } d = 2, \\ C_1(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p, & \text{if } d \geq 3 \end{cases}$

where $c_1, C_1 \in (0, \infty)$ are constants and

$$\mathcal{I}_s = \int_{\mathbb{T}^d} (1 + |e(X_s)|^2)^{\frac{p-2}{2}} |\nabla e(X_s)|^2.$$

To show 2), note that:

$$\langle -\Delta X_s, b(X_s) \rangle = \langle -\Delta X_s, (X_s \cdot \nabla) X_s \rangle - \langle \tau(X_s), e(-\Delta X_s) \rangle.$$

We see from the argument in [7, p.225, proof of (3.19)] that:

3) $\quad \langle \tau(X_s), e(-\Delta X_s) \rangle \geq 2c_1 \mathcal{I}_s.$

On the other hand, we have by integration by parts and Hölder's inequality that:

$$\langle -\Delta X_s, (X_s \cdot \nabla) X_s \rangle = \sum_{i,j,k} \int_{\mathbb{T}^d} \partial_k X_s^j \partial_j X_s^i \partial_k X_s^i \le \|\nabla X_s\|_3^3,$$

where $X_s^j = \sum_{z \in [-n,n]^d} X_s^{z,j} \psi_{z,j}$. It is also well known that the inner pdoduct on the LHS vanishes if $d = 2$ [7, p.225,(3.20)]. By the argument in [7, pp.234–235, proof of (3.46)] (This is where the choice of λ is used), we get:

$$\|\nabla X_s\|_3^3 \le C_1 (1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p + c_1 \mathcal{I}_s.$$

These imply that:

$$4) \ \langle -\Delta X_s, (X_s \cdot \nabla) X_s \rangle \begin{cases} = 0, & \text{if } d = 2, \\ \le C_1 (1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p + c_1 \mathcal{I}_s, & \text{if } d \ge 3. \end{cases}$$

We get 2) by 3)–4).
Step 2, Proof of (3.20): By [7, p.227, (3.25)–(3.26)], $\mathcal{J}_t$ and $\mathcal{I}_t$ are related as:

$$\mathcal{J}_t \le C \frac{\mathcal{I}_t}{(1 + \|\nabla X_t\|_2^2)^\lambda}.$$

Therefore, it is enough to prove that:

$$5) \qquad E^W \left[\int_0^t \frac{\mathcal{I}_s ds}{(1 + \|\nabla X_s\|_2^2)^\lambda} \right] \le C_T < \infty$$

where $C_T = C(T, d, p, \Gamma, m_0, m_1) \in (0, \infty)$.

To see this, we introduce the following concave function of $x \ge 0$:

$$f(x) = \begin{cases} \frac{1}{1-\lambda}(1+x)^{1-\lambda} & \text{if } \lambda \ne 1, \\ \ln(1+x) & \text{if } \lambda = 1 \end{cases}$$

Then, we have by 1) and Itô's formula that:

$$f(\|\nabla X_t\|_2^2) \le f(\|\nabla X_0\|_2^2) + \int_0^t \frac{dM_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} + 2 \int_0^t \frac{K_s ds}{(1 + \|\nabla X_s\|_2^2)^\lambda},$$

where we have omitted the term with $f'' \le 0$. Moreover, by 2)

$$\frac{K_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} \le -\frac{c_1 \mathcal{I}_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} + C_1 (1 + \|\nabla X_s\|_p)^p,$$

$$0 \le f(x) \le C_2(1+x) \ \text{ if } \lambda \in [0,1], \quad \text{and} \quad -\frac{1}{\lambda - 1} \le f(x) \le 0 \text{ if } \lambda > 1.$$

Putting these together, we get:

$$\begin{aligned}-C_3 \quad + \quad & 2c_1 E^W \left[\int_0^t \frac{\mathcal{I}_s ds}{(1+\|\nabla X_s\|_2^2)^\lambda} \right] \\ \leq \quad & C_2(1+E[\|\nabla X_0\|_2^2]) + C_1 E^W \left[\int_0^t (1+\|\nabla X_s\|_p)^p ds \right] \\ \overset{(3.10)}{\leq} \quad & C(T,d,p,\Gamma,m_0,m_1) < \infty,\end{aligned}$$

where $C_3 = 0$ if $\lambda \in (0,1]$, and $C_3 = \frac{1}{\lambda-1}$ if $\lambda > 1$. This proves 5). □

Proof of Lemma 3.2.2: We note that:

$$\begin{aligned}& p_1(d) < p_3(d) < p_2(d) \quad \text{for } d \leq 8, \\ & p_1(9) = 2.555... < p_2(9) = 2.5714... < p_3(9) = 2.620... \\ & p_2(d) < p_1(d) \quad \text{for } d \geq 10.\end{aligned}$$

Thus, the condition (1.8) takes the following form in any $d \geq 2$:

$$p \in (p_1(d), p_2(d)) \cup (p_3(d), \infty). \tag{3.21}$$

We consider the following four cases separately:

Case 1: $d = 2$ and $p \geq 2$;

Case 2: $d \geq 3$ and $p > p_3(d)$;

Case 3: $p \in (p_1(d), p_2(d))$ and $p \geq 2$;

Case 4: $p \in (p_1(d), 2)$ (This case appears only if $d = 2, 3$).

The first two cases cover the interval $(p_3(d), \infty)$ in (3.21) (Note that $p_3(2) = 2$), while the last two cases cover the interval $(p_1(d), p_2(d))$.

Case 1: By (3.20), (3.15) has already been shown with $\widetilde{p} = \widetilde{\alpha} = 2$.

Case 2: Note that $p > p_3(d) > 2$ and that $\beta \overset{\text{def}}{=} \frac{p}{p+2\lambda} > 1/2$. We prove (3.16). Since $\lambda\beta = \frac{p}{2}(1-\beta)$,

$$\begin{aligned}E^W \left[\int_0^T \|\Delta X_s\|_2^{2\beta} ds \right] \quad = \quad & E^W \left[\int_0^T \mathcal{J}_s^\beta (1+\|\nabla X_s\|_2^2)^{\lambda\beta} ds \right] \\ \overset{\beta+(1-\beta)=1}{\leq} \quad & E^W \left[\int_0^T \mathcal{J}_s ds \right]^\beta E^W \left[\int_0^T (1+\|\nabla X_s\|_2^2)^{\frac{p}{2}} ds \right]^{1-\beta} \\ \overset{(3.10),(3.20)}{\leq} \quad & C_T < \infty,\end{aligned} \tag{1)}$$

where, we used (3.20) for $p \geq 2$.

Case 3: We prove (3.17) for given $\widetilde{p} \in (1, p)$ with some $\widetilde{\alpha} = \widetilde{\alpha}(\widetilde{p}) \in (1, 2)$. Let

$\beta = \frac{p}{p+2\lambda} \in (0,1)$. Then, the bound 1) from Case 2 is still valid, although it may no longer be the case that $2\beta > 1$ here. On the other hand, it is not difficult to see via the interpolation and the Sobolev imbedding that for any $\widetilde{p} \in (1,p)$, there exist $\widetilde{\alpha} \in (1,2)$ and $\theta \in (0,1)$ such that:

$$\int_0^T \|X_s\|_{p,\widetilde{\alpha}}^{\widetilde{p}} ds \le C \left(\int_0^T \|X_s\|_{p,1}^{p} ds\right)^{\theta} \left(\int_0^T \|X_s\|_{2,2}^{2\beta} ds\right)^{1-\theta}.$$

(cf. [7, p.238, proof of (3.58)]. This is where the restriction $p < \frac{2d}{d-2}$ is necessary.) Thus,

$$\begin{aligned} E^W\left[\int_0^T \|X_s\|_{p,\widetilde{\alpha}}^{\widetilde{p}} ds\right] &\le CE^W\left[\int_0^T \|X_s\|_{p,1}^{p} ds\right]^{\theta} E^W\left[\int_0^T \|X_s\|_{2,2}^{2\beta} ds\right]^{1-\theta} \\ &\overset{(3.10),1)}{\le} C_T < \infty. \end{aligned} \tag{3.22}$$

Case 4: We prove (3.17) for given $\widetilde{p} \in (1,p)$ and with some $\widetilde{\alpha} = \widetilde{\alpha}(\widetilde{p}) \in (1,2)$. We recall that $p > \frac{3d}{d+2}$ and set:

$$\beta = \frac{((d+2)p - 3d)p}{2((d+5)p - 3d - p^2)} \in (0, \tfrac{1}{2}).$$

Then,

$$\textbf{2)} \qquad \rho \overset{\text{def}}{=} \frac{(2-p)d\lambda}{2(1-\beta)p} \in [0,1), \text{ and } \frac{(2-p)\beta}{1-\beta} \in (0,p).$$

As a result of applications of Hölder's inequality, the interpolation and the Sobolev imbedding (cf. [7, pp.239–240, (3.60)–(3.63)]), we arrive at the following bound:

$$\textbf{3)} \qquad \int_0^T \|\Delta X_s\|_p^{2\beta} ds \le C\left(\int_0^T \mathcal{J}_s ds\right)^{\beta} (I_1 + I_2)^{1-\beta},$$

where

$$I_1 = \int_0^T (1 + \|\nabla X_s\|_p)^{\frac{(2-p)\beta}{1-\beta}} ds, \quad I_2 = \left(\int_0^T \|\Delta X_s\|_p^{2\beta} ds\right)^{\rho} \left(\int_0^T \|\nabla X_s\|_p^{p} ds\right)^{1-\rho}.$$

We first prove that:

$$\textbf{4)} \qquad E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} ds\right] \le C_T < \infty.$$

We first assume $d = 3$, where $\rho > 0$. Let $r = \frac{1}{\rho} \in (1, \infty)$ and $r' = \frac{r}{r-1} = \frac{1}{1-\rho} \in (1, \infty)$. Then, for $\varepsilon > 0$,

$$\begin{aligned}
E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} ds\right] &\overset{3)}{\leq} CE^W\left[\left(\int_0^T \mathcal{J}_s ds\right)^{\beta} (I_1 + I_2)^{1-\beta}\right] \\
&\overset{\beta+(1-\beta)=1}{\leq} CE^W\left[\int_0^T \mathcal{J}_s ds\right]^{\beta} E^W\left[I_1 + I_2\right]^{1-\beta} \\
&\overset{(3.20)}{\leq} C_T E\left[1 + I_1 + I_2\right], \\
E^W\left[I_1\right] &\overset{(3.10),2)}{\leq} C_T < \infty, \\
E^W\left[I_2\right] &\overset{\text{Young}}{\leq} \frac{\varepsilon^r}{r} E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} ds\right] + \frac{\varepsilon^{-r'}}{r'} E^W\left[\int_0^T \|\nabla X_s\|_p^p ds\right] \\
&\overset{(3.10)}{\leq} \frac{\varepsilon^r}{r} E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} ds\right] + C_T.
\end{aligned}$$

Putting things together, with ε small enough, we arrive at 4) for $d = 3$. If $d = 2$ and hence $\rho = 0$, then, we have $E^W\left[I_2\right] \leq C_T$ directly from (3.10). Therefore, the proof of 4) is even easier than the above.

We finally turn to (3.15). It is not difficult to see via the interpolation (cf. [7, pp.240–241, proof of (3.65)]) that for any $\widetilde{p} \in (1, p)$, there exist $\widetilde{\alpha} \in (1, 2)$ and $\theta \in (0, 1)$ such that:

$$\int_0^T \|X_s\|_{p,\widetilde{\alpha}}^{\widetilde{p}} ds \leq C\left(\int_0^T \|X_s\|_{p,1}^p ds\right)^{\theta} \left(\int_0^T \|X_s\|_{p,2}^{2\beta} ds\right)^{1-\theta}.$$

Thus,

$$\begin{aligned}
E^W\left[\int_0^T \|X_s\|_{p,\widetilde{\alpha}}^{\widetilde{p}} ds\right] &\leq CE^W\left[\int_0^T \|X_s\|_{p,1}^p ds\right]^{\theta} E^W\left[\int_0^T \|X_s\|_{p,2}^{2\beta} ds\right]^{1-\theta} \\
&\overset{(3.10),4)}{\leq} C_T < \infty.
\end{aligned}$$

□

3.3 Compact imbedding lemmas

We will need some compact imbedding lemmas from [4]. We first introduce:

Definition 3.3.1 Let $p \in [1, \infty)$, $T \in (0, \infty)$, and E be a Banach space.

a) We let $L_{p,1}([0,T] \to E)$ denote the Sobolev space of all $u \in L_p([0,T] \to E)$ such that:

$$u(t) = u(0) + \int_0^t u'(s)ds, \text{ for almost all } t \in [0,T]$$

with some $u(0) \in E$ and $u'(\cdot) \in L_p([0,T] \to E)$. We endow the space $L_{p,1}([0,T] \to E)$ with the norm $\|u\|_{L_{p,1}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,1}([0,T]\to E)} = \int_0^T (|u(t)|^p_E + |u'(t)|^p_E)dt.$$

b) For $\alpha \in (0,1)$, we let $L_{p,\alpha}([0,T] \to E)$ denote the Sobolev space of all $u \in L_p([0,T] \to E)$ such that:

$$\int_{0<s<t<T} \frac{|u(t) - u(s)|^p_E}{|t-s|^{1+\alpha p}} dsdt < \infty.$$

We endow the space $L_{p,\alpha}([0,T] \to E)$ with the norm $\|u\|_{L_{p,\alpha}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,\alpha}([0,T]\to E)} = \int_0^T |u(t)|^p dt + \int_{0<s<t<T} \frac{|u(t) - u(s)|^p_E}{|t-s|^{1+\alpha p}} dsdt.$$

To introduce the compact imbedding lemmas, we agree on the following standard convention. Let X be a vector space and $X_i \subset X$ be a subspace with the norm $\|\cdot\|_i$ $(i = 1,2)$. Then, we equip $X_0 \cap X_1$ and $X_0 + X_1$ respectively with the norms:

$$\begin{aligned} \|u\|_{X_0 \cap X_1} &= \|u\|_0 + \|u\|_1, \\ \|u\|_{X_0 + X_1} &= \inf\{\|u_0\|_0 + \|u_1\|_1 \ ; \ u = u_0 + u_1, \ u_i \in X_i\}. \end{aligned}$$

The following lemmas will be used in section 3.4.

Lemma 3.3.2 *[4, p.370, Theorem 2.2] Let:*

► $E_1, ..., E_n$ *and* E *be Banach spaces such that each* $E_i \overset{\text{compact}}{\hookrightarrow} E$, $i = 1, ..., n$.

► $p_1, ..., p_n \in (1, \infty)$, $\alpha_1, ..., \alpha_n > 0$ *are such that* $p_i\alpha_i > 1$, $i = 1, ..., n$.

Then, for any $T > 0$,

$$L_{p_1,\alpha_1}([0,T] \to E_1) + ... + L_{p_n,\alpha_n}([0,T] \to E_n) \overset{\text{compact}}{\hookrightarrow} C([0,T] \to E).$$

Lemma 3.3.3 *[4, p.372, Theorem 2.1] Let:*

$$E_0 \overset{\text{compact}}{\hookrightarrow} E \hookrightarrow E_1$$

be Banach spaces such that the first embedding is compact, and E_0, E_1 are reflexive. Then, for any $p \in (1, \infty)$, $\alpha \in (0,1)$ and $T > 0$,

$$L_p([0,T] \to E_0) \cap L_{p,\alpha}([0,T] \to E_1) \overset{\text{compact}}{\hookrightarrow} L_p([0,T] \to E).$$

3.4 Convergence of the approximations

Let $X^n = (X_t^n)_{t \geq 0} \in \mathcal{V}$ be the unique solution to (3.3) for the Galerkin approximation. We write:

$$p' = \frac{p}{p-1}, \quad p'' = p \wedge p'. \tag{3.23}$$

Let $\beta(p,1)$ be defined by (1.29) and let $\widetilde{p} > 1$ be the one from Lemma 3.2.2. We may assume that $\widetilde{p} \in (1, p'']$. We also agree on the following standard convention. Let S be a set and ρ_i be a metric on $S_i \subset S$ $(i = 1, 2)$. Then, we tacitly consider the metric $\rho_1 + \rho_2$ on the set $S_1 \cap S_2$ (cf. (3.24) below). Then we have the following proposition, using the various estimates proved before and the lemmas concerning the compact embedding.

Proposition 3.4.1 *Let $\beta > \beta(p,1)$. Then, there exist a process X and a sequence $(\widetilde{X}^k)_{k \geq 1}$ of processes defined on a probability space $(\Omega, \mathcal{F}, P)$ such that the following properties are satisfied:*

a) *The process X takes values in*

$$C([0,\infty) \to V_{2 \wedge p', -\beta}) \cap L_{\widetilde{p},\mathrm{loc}}([0,\infty) \to V_{\widetilde{p},1}). \tag{3.24}$$

b) *For some sequence $n(k) \nearrow \infty$, $\widetilde{X}^k$ has the same law as $X^{n(k)}$ and*

$$\lim_{k \to \infty} \widetilde{X}^k = X \text{ in the metric space (3.24), } P\text{-a.s.} \tag{3.25}$$

4 Proof of the Existence of Solutions

4.1 Proof of Theorem 2.1.3

Let X and $\widetilde{X}^k$ be as in Proposition 3.4.1. We will verify (2.1) (with $\beta = \beta(p,1)$), as well as (2.3)–(2.5), and (2.7) for X. (2.3) can easily be seen. In fact,

$$\widetilde{X}_0^k \to X_0 \quad \text{a.s. in } V_{2\wedge p', -\beta},$$

$$\widetilde{X}_0^k \overset{\text{law}}{=} X_0^{n(k)} = \mathcal{P}_{n(k)}\xi \to \xi \text{ in } V_{2,0}.$$

Thus the laws of X_0 and ξ are identical.

$$\widetilde{X}_0^k \overset{\text{law}}{=} X_0^{n(k)} = \mathcal{P}_{n(k)}\xi \to \xi \text{ in } V_{2,0}.$$

Note that the function:

$$v. \mapsto \sup_{t\le T} \|v_t\|_2^2 + \int_0^T \|v_t\|_{p,1}^p dt$$

is lower semi-continuous on the metric space (3.24). Thus, (2.7) follows from (3.10) and Proposition 3.4.1 via Fatou's lemma.

To show (2.4)–(2.5), we prepare the following:

Lemma 4.1.1 *Let $\varphi \in \mathcal{V}$ and $T > 0$. Then,*

$$\lim_{k\to\infty} \int_0^T |\langle \varphi, (\widetilde{X}_t^k \cdot \nabla)\widetilde{X}_t^k - (X_t \cdot \nabla)X_t \rangle| dt = 0 \textit{ in probability } (P), \quad (4.1)$$

$$\lim_{k\to\infty} \int_0^T |\langle e(\varphi), \tau(\widetilde{X}_t^k) - \tau(X_t) \rangle| dt = 0 \textit{ in } L_1(P), \quad (4.2)$$

$$\lim_{k\to\infty} \int_0^T \langle \varphi, \mathcal{P}_{n(k)} b(\widetilde{X}_t^k) - b(X_t) \rangle dt = 0 \textit{ in probability } (P). \quad (4.3)$$

Proof: We write $Z_t^k = \widetilde{X}_t^k - X_t$ to simplify the notation. We start by proving that:

$$\lim_{k\to\infty} E\left[\int_0^T \|Z_t^k\|_{p_1,1}^{p_1} dt\right] = 0, \text{ if } p_1 < p. \quad (4.4)$$

By Proposition 3.4.1,

$$I_k \overset{\text{def.}}{=} \int_0^T \|Z_t^k\|_{1,1} dt \overset{k\to\infty}{\longrightarrow} 0, \quad P\text{-a.s.}$$

Moreover, the the random variables $\{I_k\}_{k\ge1}$ are uniformly integrable, since

$$E\left[I_k^p\right] \overset{(3.10)}{\le} C_T < \infty.$$

Therefore,

2) $\lim_{k\to\infty} E\left[I_k\right] = 0.$

Let $k(m) \nearrow \infty$ be such that:

3) $\Phi_{m,t} \stackrel{\text{def}}{=} |Z_t^{k(m)}| + |\nabla Z_t^{k(m)}| \stackrel{m\to\infty}{\longrightarrow} 0, \ \ dt|_{[0,T]} \times dx \times P\text{-a.e.},$

where $dt|_{[0,T]} \times dx$ denotes the Lebesgue measure on $[0,T] \times \mathbb{T}^d$. Such a sequence $k(m)$ exists by 2). The sequence $\{\Phi_{m,\cdot}\}_{m\geq 1}$ are uniformly integrable with respect to $dt|_{[0,T]} \times dx \times P$. In fact,

$$E\left[\int_0^T \int_{\mathbb{T}^d} \Phi_{m,t}^p dt\right] \stackrel{(3.10)}{\leq} C_T < \infty.$$

Therefore, 3) together with this uniform integrability implies (4.4) along the subsequence $k(m)$. Finally, we get rid of the subsequence, since the subsequence as $k(m)$ above can be chosen from any subsequence of k given in advance.

We now prove (4.1): Since,

$$(\widetilde{X}_t^k \cdot \nabla)\widetilde{X}_t^k - (X_t \cdot \nabla)X_t = (Z_t^k \cdot \nabla)\widetilde{X}_t^k + (X_t \cdot \nabla)Z_t^k,$$

we have:

$$\int_0^T |\langle \varphi, (\widetilde{X}_t^k \cdot \nabla)\widetilde{X}_t^k - (X_t \cdot \nabla)X_t \rangle| dt \leq J_1 + J_2,$$

where

$$J_1 = \int_0^T |\langle \varphi, (Z_t^k \cdot \nabla)\widetilde{X}_t^k \rangle| dt, \ \text{ and } \ J_2 = \int_0^T |\langle \varphi, (X_t \cdot \nabla)Z_t^k \rangle| dt.$$

We may take p_1 in (4.4) is bigger than $\frac{3d}{d+2}$, so that there exists $0 < \alpha < 1$ such that $\frac{2d}{d+2\alpha} < p_1$. Then, by (1.24), we have that:

$$|\langle \varphi, (Z_t^k \cdot \nabla)\widetilde{X}_t^k \rangle| \leq C\|Z_t^k\|_{p_1,\alpha}\|\widetilde{X}_t^k\|_2\|\varphi\|_{p_1,\beta(p_1,\alpha)}$$

and hence that:

$$J_1 \leq C\|\varphi\|_{p_1,\beta(p_1,\alpha)} \sup_{t\leq T} \|\widetilde{X}_t^k\|_2 \int_0^T \|Z_t^k\|_{p_1,\alpha} dt.$$

By (3.10) and (4.4),

$$\sup_{k\geq 1} E[\sup_{t\leq T} \|\widetilde{X}_t^k\|_2^2] < \infty \ \text{ and } \ \lim_{k\to\infty} \int_0^T \|Z_t^k\|_{p_1,\alpha} dt = 0 \ \ P\text{-a.s.},$$

Thus, $\lim_{k\to\infty} J_1 = 0$ in probability. On the other hand, we have by (1.27) that:

$$|\langle \varphi, (X_t \cdot \nabla)Z_t^k \rangle| \leq C\|Z_t^k\|_{p_1,\alpha}\|X_t\|_2\|\varphi\|_{p_1,\beta(p_1,\alpha)}$$

and hence that:

$$J_2 \leq C\|\varphi\|_{p_1,\beta(p_1,\alpha)} \sup_{t\leq T} \|X_t\|_2 \int_0^T \|Z_t^k\|_{p_1,\alpha} dt.$$

By (2.7) and (4.4),

$$E[\sup_{t\leq T} \|X_t\|_2^2] < \infty \text{ and } \lim_{k\to\infty} \int_0^T \|Z_t^k\|_{p_1,\alpha} dt = 0 \ P\text{-a.s.},$$

Thus, $\lim_{k\to\infty} J_2 = 0$ in probability.
We now turn to (4.2): It is enough to prove that:

4) $$\lim_{k\to\infty} E\left[\int_0^T \|\tau(\widetilde{X}_t^k) - \tau(X_t)\|_1 dt\right] = 0 \ .$$

Again, let $k(m)$ be such that 3) holds. Then,

5) $$\lim_{m\to\infty} \tau(\widetilde{X}_t^{k(m)}) = \tau(X_t), \ \ dt|_{[0,T]} \times dx \times P\text{-a.e.}$$

On the other hand, we have for $p' = \frac{p}{p-1}$ that:

$$E\left[\int_0^T dt \int_{\mathbb{T}^d} |\tau(\widetilde{X}_t^k)|^{p'}\right] \leq CE\left[\int_0^T dt \int_{\mathbb{T}^d} \left(1 + |e(\widetilde{X}_t^k)|\right)^p\right] \overset{(3.10)}{\leq} C_T < \infty,$$

which implies that $\tau(\widetilde{X}_t^k)$, $k \in \mathbb{N}$ are uniformly integrable with respect to $dt|_{[0,T]} \times dx \times P$. Therefore, 5) together with this uniform integrability implies 4) along the subsequence $k(m)$. Finally, we get rid of the subsequence, since the subsequence as $k(m)$ above can be chosen from any subsequence of k given in advance.
(4.3) follows from (4.1) and (4.2). Since $\varphi \in \mathcal{V}$ is fixed and k is tending to ∞, we do not have to care about $\mathcal{P}_{n(k)}$ here. □

Remark: If $p = 2$, then Lemma 4.1.1 is valid for all d. This is for the following reason. By inspection of the proof above, we see immediately that (4.1) follows also from the modification of Proposition 3.4.1 mentioned at the end of section 3.4. Also, for $p = 2$, (4.2) is equivalent to:

$$\lim_{k\to\infty} \int_0^T \langle \Delta\varphi, \widetilde{X}_t^k - X_t \rangle dt = 0 \text{ in } L_1(P),$$

which also follows from the modification of Proposition 3.4.1 mentioned at the end of section 3.4.

Lemma 4.1.2 *Let:*

$$Y_t = Y_t(X) = X_t - X_0 - \int_0^t b(X_s)ds, \quad t \geq 0. \tag{4.5}$$

Then, $Y_\cdot$ is a $\mathrm{BM}(V_{2,0}, \Gamma)$*. Moreover, $Y_{t+\cdot} - Y_t$ and $\{\langle \varphi, X_s \rangle \; ; \; s \leq t, \varphi \in \mathcal{V}\}$ are independent for any $t \geq 0$.*

It is enough to prove that for each $\varphi \in \mathcal{V}$ and $0 \leq s < t$,

1) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle\right) | \mathcal{G}_s\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right), \quad \text{a.s.}$$

where $\mathcal{G}_s = \sigma(\langle \varphi, X_u \rangle \; ; \; u \leq s, \varphi \in \mathcal{V})$. We set

$$F(X) = f(\langle \varphi_1, X_{u_1} \rangle, ..., \langle \varphi_n, X_{u_n} \rangle),$$

where $f \in C_{\mathrm{b}}(\mathbb{R}^n)$, $0 \leq u_1 < .. < u_n \leq s$ and $\varphi_1, ..., \varphi_n \in \mathcal{V}$ are chosen arbitrary in advance. Then, 1) can be verified by showing that:

2) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle\right) F(X)\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right) E[F(X)].$$

Let:

$$Y_t^k = \widetilde{X}_t^k - \widetilde{X}_0^k - \int_0^t \mathcal{P}_{n(k)} b(\widetilde{X}_s^k)ds, \quad t \geq 0.$$

We then see from Theorem 3.1.1 that:

3) $$E\left[\exp\left(\mathbf{i}\langle \varphi, Y_t^k - Y_s^k \rangle\right) F(\widetilde{X}^k)\right] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\mathcal{P}_{n(k)}\varphi \rangle\right) E[F(\widetilde{X}^k)],$$

Moreover, we have

$$\lim_{k\to\infty} \langle \varphi, Y_t^k - Y_s^k \rangle \overset{(3.25),(4.3)}{=} \lim_{k\to\infty} \langle \varphi, Y_t - Y_s \rangle \quad \text{in probability,}$$

and hence

$$\lim_{k\to\infty} \text{LHS of 3)} = \text{LHS of 2)}.$$

On the other hand,

$$\lim_{k\to\infty} \text{RHS of 3)} \overset{(3.25)}{=} \text{RHS of 2)}.$$

These prove 2). □

Finally, we prove (2.1) with $\beta = \beta(p,1)$. It follows from (2.7) that:

$$X \in L_{p,\mathrm{loc}}([0,\infty) \to V_{p,1}) \cap L_{\infty,\mathrm{loc}}([0,\infty) \to V_{2,0}).$$

Thus, it remains to show that $X \in C([0,\infty) \to V_{2\wedge p', -\beta(p,1)})$. But this follows from Lemma 3.2.1 and that $Y \in C([0,\infty) \to V_{2,0})$. □

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