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Problems on Low-dimensional Topology, 2012

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 16–18, 2012.

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1 HOMFLY homology of knots

(Dylan Thurston)

Question 1.1 (D. Thurston). *Is there a good locally cancellable theory for HOMFLY homology or $SL(n)$ homology that allows one to do computations?*

Khovanov and Rozansky wrote down explicit chain complexes that compute a homology theory whose Euler characteristic is the HOMFLY polynomial and its specialization to the invariant associated to $SL(n)$. (This was later improved by Khovanov using Soergel bimodules.) But this theory is hard to make locally cancellable, in the sense that it is hard to extract a computable invariant for tangles. For $SL(2)$ homology (Khovanov homology), constructing a locally cancellable theory was the crucial step in making the invariant computable in practice. Can this be done for the more general case?

Question 1.2 (D. Thurston). *Is there a simpler theory of tangles for knot Heegaard Floer homology that is not locally cancellable?*

Knot Heegaard Floer homology is a homology theory for knots whose Euler characteristic is the Alexander polynomial (which can be thought of as the $n = 0$ specialization of the HOMFLY polynomial). There is a locally cancellable theory for it, namely bordered Floer homology. However, it is not very simple, and certainly not as easy as the Khovanov-Rozansky construction. Is there a simpler theory if we drop the requirement that it be locally cancellable?

Note: $SL(n)$ homology naively specialized to $n = 0$ does not work very well.

Question 1.3 (D. Thurston). *Is there a spectral sequence from HOMFLY homology to knot Heegaard Floer homology?*

For each $n > 0$, there is a spectral sequence starting from the triply-graded HOMFLY homology that converges to the doubly-graded $SL(n)$ homology. This was conjectured by Dunfield, Gukov, and Rasmussen, and proved by Rasmussen. There is strong evidence that there is also a spectral sequence from HOMFLY homology to knot Heegaard Floer homology, and it is a long-standing open problem to prove that. In order to do that, it would be very helpful to make the two theories more similar. A positive answer to either of the two questions above would likely answer this problem.

2 The additivity of the unknotting number of knots

(Tetsuya Abe)

The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes which convert K into the unknot. Let $K_1 \sharp K_2$ denote the connected sum of knots K_1 and K_2. The following conjecture is on the additivity of the unknotting number of knots.

Conjecture 2.1 ([27, Problem 1.69 (B)]). *For knots K_1 and K_2, $u(K_1 \sharp K_2) = u(K_1) + u(K_2)$.*

Scharlemann [43] showed that $u(K_1 \sharp K_2) \geq 2$ for non-trivial knots K_1 and K_2, which gives a partial answer to this conjecture.

There are many ways to estimate the unknotting number. One of them is $g_*(K) \leq u(K)$, where $g_*(K)$ denotes the 4-ball genus of K. Milnor [34] conjectured that this estimation determines the unknotting number of torus knots. By using gauge theory, Kronheimer and Mrowka [29] proved that $g_*(T_{p,q}) = u(T_{p,q}) = (|p|-1)(|q|-1)/2$ where $T_{p,q}$ denotes the torus knot of type (p,q). Other proofs were given in [39, 41]. On the other hand, little is known for the unknotting number of the connected sum of torus knots.

Question 2.2 (T. Abe). *Let p, q, p' and q' be non-zero integers such that $(p,q) = 1$ and $(p', q') = 1$. Does the equality $u(T_{p,q} \sharp T_{p',q'}) = u(T_{p,q}) + u(T_{p',q'})$ hold?*

When $g_*(T_{p,q} \sharp T_{p',q'}) = 0$, it might be difficult to show that $u(T_{p,q} \sharp T_{p',q'}) = u(T_{p,q}) + u(T_{p',q'})$. For example, it is not known whether $u(T_{2,5} \sharp T_{2,-5}) = 4$.

Question 2.3 ([2, Question 2]). *Let q' and q' be odd integers. Does the equality $u(T_{2,q} \# T_{2,q'}) = u(T_{2,q}) + u(T_{2,q'})$ hold?*

It seems that the following question is not solved yet.

Question 2.4 (T. Abe). *Let K be a given knot. Can we obtain a knot K' from K by a single crossing change such that $u(K') = u(K) + 1$?*

An obvious candidate for K' is $K \sharp T_{2,3}$. However no one succeeded to prove that $u(K \sharp T_{2,3}) = u(K) + 1$.

3 Invariants of symmetric links

(Yongju Bae)

A *symmetric link* L in $\mathbb{R}^3$ is a link with a diagram on which a finite group can act. The *periodic links* of order n are symmetric links whose acting group is the cyclic group $\mathbb{Z}_n$. One can construct symmetric links by using the covering graph construction. Indeed, one can consider a diagram D of a link as a 4-valent graph with the under/over information and a cyclic permutation at each crossing. By assigning an element of a finite group G to each edge of D, one can construct a covering graph $D \times_\phi G$. Since the local shape of $D \times_\phi G$ at a crossing is homeomorphic to that of D at the corresponding crossing, we can give the same under/over information and the same cyclic permutation to a crossing of $D \times_\phi G$ with those of D. If the embedding surface of $D \times_\phi G$ is the sphere, $D \times_\phi G$ is a symmetric link on which the finite group G can act. Otherwise, $D \times_\phi G$ can be considered as a symmetric virtual links.

The construction of a symmetric links on which the cyclic group $\mathbb{Z}_n$ can act; periodic links. At the moment, we construct symmetric links on which the Klein

4-group can act, and calculate the Alexander polynomial and the determinant of those Klein 4 symmetric links.

Problem 3.1. *Construct symmetric links such that the embedding surface is the sphere, and calculate link invariants of $D \times_\phi G$ by using those of D and the information of the acting group G.*

As a partial solution, we have a formula for the Alexander polynomial of $D \times_\phi G$ in the case that the acting group G is $\mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Problem 3.2. *In the case that the embedding surface is not the sphere, calculate link invariants of the virtual link $D \times_\phi G$ by using those of D and the information of the acting group G.*

For the study of Problem 3.1, we found the following specific matrices which are related with symmetric structure. The determinant formulae can be seen by using elementary linear algebra.

Theorem. Let $A, B, C,$ and D be $m \times m, m \times r, r \times m$ and $r \times r$ matrices, respectively. Let n denote the number of block A's. Then

$$\det \left(\begin{array}{cccc|c} A & 0 & \cdots & 0 & B \\ 0 & A & \cdots & 0 & B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A & B \\ \hline C & C & \cdots & C & nD \end{array}\right) = n^r (\det A)^{n-1} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$\det \left(\begin{array}{ccccc|cccc} A & 0 & 0 & \cdots & 0 & -B & -B & \cdots & -B \\ 0 & A & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & 0 & 0 & \cdots & B \\ \hline -C & C & 0 & \cdots & 0 & 2D & D & \cdots & D \\ -C & 0 & C & \cdots & 0 & D & 2D & \cdots & D \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -C & 0 & 0 & \cdots & C & D & D & \cdots & 2D \end{array}\right) = n^r \det A \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{n-1},$$

$$\det \left(\begin{array}{ccccc|cccc} A & 0 & 0 & \cdots & 0 & -B & -B & \cdots & -B \\ 0 & A & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & 0 & 0 & \cdots & B \\ \hline -C & C & 0 & \cdots & 0 & D+E & D & \cdots & D \\ -C & 0 & C & \cdots & 0 & D & D+E & \cdots & D \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -C & 0 & 0 & \cdots & C & D & D & \cdots & D+E \end{array}\right)$$

$$= \det A \cdot \sum_{k=1}^{n} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{n-k} \det \begin{pmatrix} A & B \\ C & E \end{pmatrix}^{k-1}.$$

Even though the proofs are not difficult, those formulae are not known yet according to linear algebraists who I consulted till now. We used those formulae for the Alexander polynomials of periodic links and Klein 4-symmetric links. We believe that those formulae can be used in other areas that are using the determinant of a matrix as a research tool.

4 Quantum invariants and Milnor invariants of links

(Sakie Suzuki)

We are interested in relationships between algebraic properties of the quantum invariants and topological properties of links and tangles. One method to understand the relationships is to study the quantum invariants in terms of classical invariants. In this note, we give several questions about the quantum invariants in terms of the Milnor invariants, which are generalizations of the linking numbers. More precisely, we aim to characterize the quantum invariants of links and tangles with all the Milnor invariants vanishing.

A bottom tangle is a tangle consisting of arc components each of whose endpoints are adjacent to each other on the bottom line of the cube. The universal $\mathfrak{sl}_2$ invariant of bottom tangles has a universality property for the colored Jones polynomial of links; see [15] for details.

The universal $\mathfrak{sl}_2$ invariant of n-component bottom tangles takes values in the completed n-fold tensor power $U_h(\mathfrak{sl}_2)^{\hat{\otimes} n}$ of the quantum enveloping algebra $U_h(\mathfrak{sl}_2)$. In [44, 45], we proved that the universal $\mathfrak{sl}_2$ invariant of n-component ribbon bottom tangles and n-component boundary bottom tangles are contained in a certain small subalgebra $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ of $U_h(\mathfrak{sl}_2)^{\hat{\otimes} n}$.

Since there is the one-to-one correspondence between the set of bottom tangles and the set of string links (see [15]), we can define the Milnor μ invariants [32, 33] of a bottom tangle as that of the corresponding string link. See [13] for the Milnor μ invariants of string links. In fact, all the Milnor μ invariants vanish both for ribbon bottom tangles and for boundary bottom tangles. It is natural to expect the following conjecture. In this note, we assume that links and bottom tangles are 0-framed.

Conjecture 4.1 (S. Suzuki [45, Conjecture 1.5]). *Let T be an n-component bottom tangle with all the Milnor μ invariants vanishing. Then $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$.*

The converse of this conjecture is also open.

Question 4.2 (S. Suzuki). *Let T be an n-component bottom tangle such that $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$. Then, is it true that all the Milnor μ invariants of T vanish?*

In [8], Eisermann proved that the Jones polynomial $V_L \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of an n-component ribbon link L is divisible by the Jones polynomial $(q^{1/2}+q^{-1/2})^n$ of the n-component unlink, *i.e.*, $V_L \in (q^{1/2}+q^{-1/2})^n\,\mathbb{Z}[q,q^{-1}]$. This result is generalized to links which are ribbon concordant to boundary links by Habiro [16]. Thus the following question arises naturally.

Question 4.3 (S. Suzuki). *Let L be an n-component link with all the Milnor $\bar{\mu}$ invariants vanishing. Then, is it true that $V_L \in (q^{1/2}+q^{-1/2})^n\,\mathbb{Z}[q,q^{-1}]$?*

The converse is also possible.

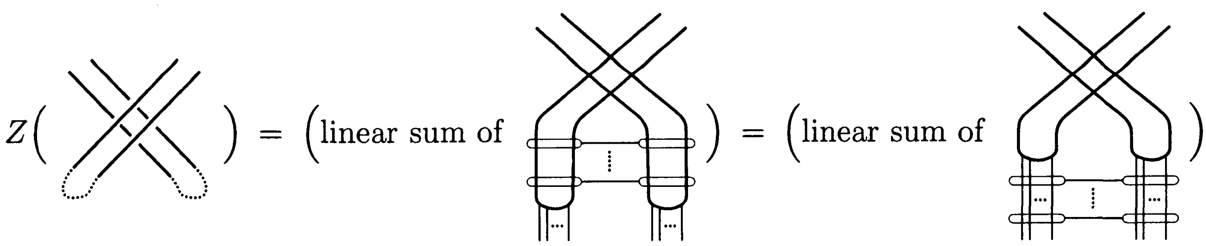
Question 4.4 (S. Suzuki). *Let L be an n-component link such that $V_L \in (q^{1/2}+q^{-1/2})^n\,\mathbb{Z}[q,q^{-1}]$. Then, is it true that all the Milnor $\bar{\mu}$ invariants of L vanish?*

In [46], we construct a subalgebra $\hat{Q}_0^{(n)}$ of $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ in which the universal $\mathfrak{sl}_2$ invariant of ribbon bottom tangles takes values. This result gives another proof of the result of Eisermann, *i.e.*, for a bottom tangle T and its closure link L, the fact $J_T \in \hat{Q}_0^{(n)}$ implies $V_L \in (q^{1/2}+q^{-1/2})^n\,\mathbb{Z}[q,q^{-1}]$. We do not know whether $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ implies $V_L \in (q^{1/2}+q^{-1/2})^n\,\mathbb{Z}[q,q^{-1}]$ or not. Anyway, we aim to solve the following problem ultimately.

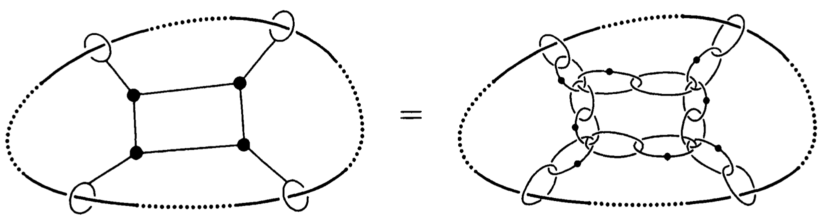
Problem 4.5 (S. Suzuki). *Characterize the universal $\mathfrak{sl}_2$ invariant of bottom tangles with all the Milnor μ invariants vanishing. Also, characterize the Jones polynomial of links with all the Milnor $\bar{\mu}$ invariants vanishing.*

Here, note that all the Milnor μ invariants of a bottom tangle T vanish if and only if all the Milnor $\bar{\mu}$ invariants of the closure link of T vanish.

Comment (T. Ohtsuki) Milnor invariants are coefficients of the tree part of the loop expansion of the Kontsevich invariant [14]. It is shown that all the Milnor μ invariants vanish for boundary links, since (roughly speaking) most parts of the Kontsevich invariant of the following tangle do not have tree diagrams,



Further, it is known (due to Habiro) that a clasper surgery along a graph clasper having a loop makes a concordant link,



where the left-hand side means a resulting link after a clasper surgery, and the right-hand side means a resulting link after a link surgery, which can be obtained by attaching 1-handles along marked components and 2-handles along the other components to the 4-ball, and hence, gives a concordant link. Therefore, from the viewpoint of the values of the Kontsevich invariant, links with vanishing Milnor invariants are similar to links which are concordant to the trivial link. It might be interesting to consider the above problems for links which are concordant to the trivial link, instead of links with vanishing Milnor invariants.

5 Intrinsically knotted graphs

(Ryo Nikkuni)

An embedding f of a finite graph G into the 3-sphere is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph.* We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. We call a subgraph γ of G homeomorphic to the circle a *cycle* of G, and a cycle of G containing exactly k edges a *k-cycle* of G. We denote the set of all cycles of G, the set of all k-cycles of G and the set of all pairs of two disjoint cycles of G by $\Gamma(G)$, $\Gamma_k(G)$ and $\Gamma^{(2)}(G)$, respectively. We say that a subset Γ of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$) is said to be *knotted* (resp. *linked*) if for any element f of $\mathrm{SE}(G)$, there exists an element γ of Γ such that $f(\gamma)$ is a nontrivial knot (resp. nonsplittable 2-component link). Note that a graph G is *intrinsically knotted* (resp. *linked*) if there exists a knotted (resp. linked) subset Γ of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$).

Now we say that a subset Γ of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$) is *minimally knotted* (resp. *linked*) if Γ is knotted (resp. linked) and each proper subset Γ' of Γ is not knotted (resp. linked). By definition, it is not hard to see that a knotted (resp. linked) subset of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$) includes a minimally knotted (resp. linked) subset of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$). Note that Γ is a minimally knotted (resp. linked) subset of $\Gamma(G)$ (resp. $\Gamma^{(2)}(G)$) if and only if for any element γ of Γ, there exist an element f of $\mathrm{SE}(G)$ such that $f(G)$ contains exactly one nontrivial knot (resp. nonsplittable 2-component link) $f(\gamma)$.

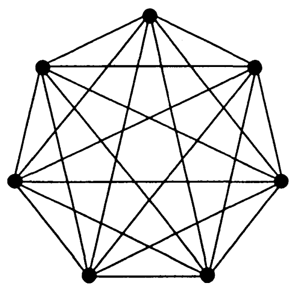
By a realization theorem of 2-component links in a spatial graph [47] and a characterization of intrinsically linked graphs [42], for an intrinsically linked graph G, we can find a minimally linked subset Γ of $\Gamma^{(2)}(G)$ explicitly. Thus, next we consider the case of intrinsically knotted graphs.

Problem 5.1 (R. Nikkuni). *For an intrinsically knotted graph G, find a minimally knotted subset Γ of $\Gamma(G)$.*

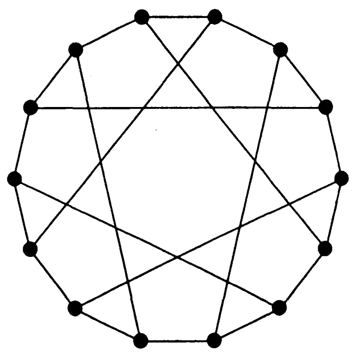
Example.
(1) Let K_n be the *complete graph* on n vertices. Conway-Gordon showed that $\Gamma_7(K_7)$ is knotted [6]. They also exhibited an element g of $\mathrm{SE}(K_7)$ whose image contains exactly one nontrivial knot as the image of a 7-cycle. Since every two k-cycles of K_7 can be transformed into each other by an automorphism of K_7, it follows that $\Gamma_7(K_7)$ is minimally knotted.

(2) Let C_{14} be the *Heawood graph*, which is an intrinsically knotted graph obtained from K_7 by seven times $\triangle Y$-exchanges. Nikkuni showed that $\Gamma_{14}(C_{14}) \cup \Gamma_{12}(C_{14})$ is knotted [36]. He also exhibited an element g of $\mathrm{SE}(C_{14})$ whose image contains exactly one nontrivial knot as the image of a 12-cycle. On the other hand, Kohara-Suzuki exhibited an element h of $\mathrm{SE}(C_{14})$ whose image contains exactly one nontrivial knot as the image of a 14-cycle [28]. Since every two k-cycles of C_{14} can be transformed

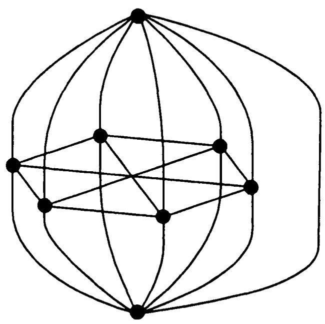
into each other by an automorphism of C_{14}, it follows that $\Gamma_{14}(C_{14}) \cup \Gamma_{12}(C_{14})$ is minimally knotted.



The complete graph K_7



The Heawood graph C_{14}



The complete four-partite graph $K_{3,3,1,1}$

It is known that the complete four-partite graph $K_{3,3,1,1}$ is also intrinsically knotted [11]. But a concrete example of minimally knotted subset Γ of $\Gamma(K_{3,3,1,1})$ has not been found yet.

Problem 5.2 (R. Nikkuni). *Find a minimally knotted subset* Γ *of* $\Gamma(K_{3,3,1,1})$.

For $n \geq 8$, it is known that every spatial complete graph on n vertices always contains nontrivial Hamiltonian knots more than two [3, 19]. This implies that K_n has a minimally knotted subset of $\Gamma(K_n)$ as a proper subset of $\Gamma_n(K_n)$.

Problem 5.3 (R. Nikkuni). *For* $n \geq 8$, *find a minimally knotted proper subset* Γ *of* $\Gamma_n(K_n)$.

6 The growth functions of groups

(Koji Fujiwara)

Let G be a group with a finite generating set S. For $g \in G$, let $|g|$ denote the word length with respect to S. Put

$$a_n = \#\{g \in G \mid |g| = n\}.$$

The *growth function* of (G, S) is defined by

$$\gamma_{G,S}(t) = \sum_{n=0}^{\infty} a_n t^n.$$

The (exponential) *growth rate* of (G, S) is defined by

$$r_{G,S} = \liminf_{n\to\infty} a_n^{1/n}.$$

The growth function is rational for certain classes of groups including hyperbolic groups, while (hyperbolic) knot groups are not hyperbolic groups (see [12]).

Question 6.1 (K. Fujiwara). *Let G be a knot group. Is $\gamma_{G,S}(t)$ rational for some/any S ?*

The growth functions of Coxeter groups G with respect to the standard (Coxeter) generators S are computable. The Coxeter group of type $(2,3,7)$

$$\langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^3, (zx)^7 \rangle$$

has the smallest volume among all 2-dimensional compact hyperbolic orbifolds. Also, it has the smallest growth rate among all 2-dimensional compact hyperbolic Coxeter groups (with respect to S) ([20]). The Coxeter group of type $(2,3,\infty)$ (obtained by erasing $(zx)^7$ from the above presentation) has the smallest volume/growth rate among finite volume ones in the same sense (Floyd). It might be interesting to characterize the standard generators S in terms of growth, for example,

Question 6.2 (K. Fujiwara). *Does S give the smallest growth rate in each case ?*

We can ask the same question for 3-dimensional hyperbolic Coxeter groups.

The figure-eight knot complement has the smallest hyperbolic volume among all hyperbolic knot complements (Cao-Meyerhoff).

Question 6.3 (K. Fujiwara). *Does the figure-eight knot group have the smallest growth rate among knots (w.r.t., say, "standard" generating set) ?*

It is known that the figure-eight knot complement is obtained by gluing two ideal tetrahedra. We obtain a polyhedron by gluing these tetrahedra along one face, and we obtain the figure-eight knot complement from this polyhedron by gluing its faces. From this construction of the figure-eight knot complement, we obtain the following presentation of the figure-eight knot group,

$$\langle x, y, z \mid x = z^{-1}yz,\ z = xy^{-1}x^{-1}y \rangle.$$

This presentation might give the "standard" generating set of the above question.

Let (G, S) be a hyperbolic group. It is known [7] that there exist $A, B, C > 0$ such that

$$A\, e^{Cn} \le a_n \le B\, e^{Cn} \tag{1}$$

for any $n \geq 0$.

Question 6.4 (K. Fujiwara). *Does* (1) *hold for (hyperbolic) knot groups ?*

Problem 6.5 (K. Fujiwara). *Let K be a knot, and let G be the knot group of K. Define*

$$r(K) = \inf_S r_{G,S}\,,$$

where the inf *runs over an "appropriate" class of generating sets of G. Calculate $r(K)$ for concrete knots K.*

For example, by considering ideal tetrahedral decompositions of the complement of K, we can consider a certain class of generating sets of G. It would be a problem to choose an "appropriate" class of generating sets so that we can calculate/estimate $r(K)$ concretely.

7 Quandles

A *quandle* is a set X equipped with the binary operation $*$ satisfying the following 3 axioms.

(1) $x * x = x$ for any $x \in X$.

(2) For any $y, z \in X$ there exists a unique $x \in X$ such that $z = x * y$.

(3) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

(Seiichi Kamada)

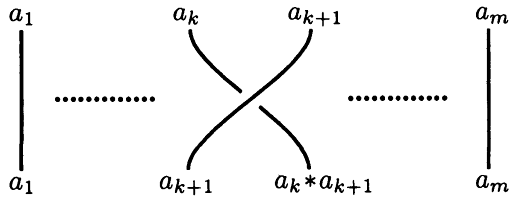
Problem 7.1 (S. Kamada). *Let $p : \widetilde{Q} \to Q$ be a surjective quandle homomorphism. Characterize a quandle homomorphism $f : P \to Q$ that has a lift $\widetilde{f} : P \to \widetilde{Q}$ with respect to p, i.e., $f = p \circ \widetilde{f}$.*

In S. Kamada's talk, it is treated the case where $\widetilde{Q}$ and Q are certain quandles in the braid group and the symmetry group. Precisely speaking, $\widetilde{Q}$ consists of all conjugates of standard generators of B_m and their inverses, and Q consists of all transpositions.

The following problem is a quandle version of the Hurwitz problem. Let Q be a quandle, and let Q^m be the m-fold product of Q. The *Hurwitz equivalence* among Q^m is generated by the moves

$$(a_1, \dots, a_k, a_{k+1}, \dots, a_m) \longmapsto (a_1, \dots, a_{k+1}, a_k * a_{k+1}, \dots, a_m) \quad (i = 1, \dots, m-1)$$

and their inverses. An equivalence class of this equivalence relation is an orbit of an action of B_m on Q^m.



Problem 7.2 (S. Kamada). *Let $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ be elements of the m-fold product of a quandle Q. Solve a Hurwitz word problem: Decide whether a and b are Hurwitz equivalent or not.*

The *HC equivalence* is generated by the moves

$$(a_1, \dots, a_k, a_{k+1}, \dots, a_m) \longmapsto (a_1, \dots, a_{k+1}, a_k * a_{k+1}, \dots, a_m) \quad (i = 1, \dots, m-1),$$
$$(a_1, \dots, a_m) \longmapsto (a_1 * g, \dots, a_m * g) \quad (g \in Q)$$

and their inverses.

Problem 7.3 (S. Kamada). *Let $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ be elements of the m-fold product of a quandle Q. Solve a Hurwitz conjugacy problem: Decide whether a and b are HC equivalent or not.*

(J. Scott Carter)

Problem 7.4 (J.S. Carter). *Develop algebraic-topological techniques for computing quandle and rack homology. This is particularly important for low-dimensional cocycles.*

The works of Nosaka [37] and Clauwens [4, 5] are relevant here.

Question 7.5 (J.S. Carter). *The quandles* $\mathbb{Z}_n[t,t^{-1}]/(t+1)$ *are interpreted as dihedral quandles — the quandle that consists of reflections of an* n*-gon. Are there similar geometric interpretations of the quandles* $\mathbb{Z}_n[t,t^{-1}]/(t-a)$ *?*

See in particular, Nelson's paper [35] and Chuichiro Hayashi, Miwa Hayashi, Kanako Oshiro's paper [18].

Problem 7.6 (J.S. Carter). *Consider the* G*-family of quandles* $\mathbb{Z}_q^n$ *with group* $\mathrm{GL}(n,\mathbb{Z}_q)$ *or subgroups thereof where* $q=p^k$*. Construct low dimensional non-trivial cocycles in the sense of Carter-Ishii.*

Problem 7.7 (J.S. Carter). *Use these cocycles to detect equivalences of knotted and embedded* n*-foams.*

Question 7.8 (J.S. Carter). *Is there an interpretation of quandle cocycles in these modular cases that is related to incidences of lines, planes,* etc. *in the Grassmanians of the vector space?*

Question 7.9 (J.S. Carter). *Any vector space over* $\mathbb{R}$ *(or* $\mathbb{C}$*) is also an* $\mathbb{R}$*-family of quandles with* $\vec{a}\triangleleft_t\vec{b}=t\vec{a}+(1-t)\vec{b}$ *where* $t\in\mathbb{R}$*. Is it possible to construct interesting cocycles in the infinite case?*

Problem 7.10 (J.S. Carter). *Develop a structure theorem for* G*-families of quandles analogous to that of Joyce* [23] *and Matveev* [30].

8 Surface-knots and 2-dimensional braids

A *surface-knot* is a smooth embedding of a closed surface into $\mathbb{R}^4$. In particular, a surface-knot of an embedding of the 2-sphere S^2 (resp. the projective plane P^2) is called a *2-knot* (resp. a *P^2-knot*). A *simple 2-dimensional braid* is a compact oriented surface embedded in a bidisk $D^2\times D^2$ satisfying a certain condition. It is described by an immersed graph on D^2, which is called a *chart*; for details, see [24, 25].

Let us recall the smooth unknotting conjecture for 2-knots.

Conjecture 8.1 (see [27, Problem 1.55 (A)]). *Any smooth 2-knot whose knot group is infinite cyclic is smoothly unknotted.*

See also [25, Conjecture 1.2.7] and [38, Conjecure 6.2], for comments to this conjecture. As an approach to this conjecture, we consider the following problem.

Section 8 was written by T. Ohtsuki, following Seiichi Kamada's explanations. He would like to thank Kamada for helpful comments.

Problem 8.2 (S. Kamada). *Characterize charts that represent unknotted 2-knots and charts that represent 2-knots with infinite cyclic knot groups.*

Question 8.3 (Kinoshita's question, see [25, Question 1.4.3], [38, Problem 7.5]). *Is every P^2-knot a connected sum of a 2-knot and a standard P^2-knot?*

All examples of P^2-knots known so far are such P^2-knots. Some supporting evidences to this question are given in Theorems 6.4.1, 6.4.2 and 6.4.3 of [25], where it is shown that P^2-knots obtained by certain constructions are such P^2-knots.

Problem 8.4 (S. Kamada). *Construct other examples of supporting evidences of the above question.*

It is known, see *e.g.* [25, Section 6.5], that for any P^2-knot F, the order of the meridian in $\pi_1(\partial N(F))$ is 4, and hence, the order of the meridian in $\pi_1(\mathbb{R}^4 - F)$ is 2 or 4.

Question 8.5 (Yoshikawa's question, see [25, Question 6.5.1]). *Is there a P^2-knot such that the order of the meridian is 4?*

Remark.
(1) If Question 8.3 is true, the order of the meridian is always 2. On the other hand, if there is a P^2-knot such that the order of the meridian is 4, it is a counter-example to Question 8.3.
(2) There is a counter-example for P^2-links; see [25, Proposition 6.5.2].

It is known, see *e.g.* [25, Section 7.1], that any ribbon knot is slice. The converse is an open problem.

Question 8.6 ([25, Question 7.1.3]). *Is every slice knot a ribbon knot?*

Question 8.7 (S. Kamada [25, Question 10.6.7]). *Let S and S' be simple 2-dimensional braids. If they are braid ambient isotopic, then are they equivalent?*

Question 8.8 (S. Kamada [25, Question 10.6.8]). *Let S and S' be simple 2-dimensional braids. If they are braid ambient isotopic, then are they equivalent after suitable application of conjugations and stabilizations?*

If either of Questions 8.7 and 8.8 is solved affirmatively, then the following conjecture holds.

Conjecture 8.9 (S. Kamada [25, Conjecture 10.6.9]). *Let S and S' be simple 2-dimensional braids, describing surface links F and F', respectively. Surface links F and F' are equivalent (i.e., ambient isotopic) if and only if S and S' are related by a sequence of equivalence moves, conjugations, stabilizations and destabilizations.*

9 Higher dimensional braids

(J. Scott Carter)

In Seiichi Kamada's talk at ILDT, he presented examples of 3-dimensional braids. These are constructed as embeddings into $S^2 \times D^2$ of 2- and 3-fold branched covers of S^3 branched over a knot or link. The covers have simple branch points. The 2-fold branched covers can always be embedded in $S^2 \times D^2$. Such embeddings and immersions are said to be *in braid form.*

Question 9.1 (J.S. Carter). *Which of these embeddings are knotted?*

Question 9.2 (J.S. Carter). *When can an embedded* 3*-manifold in* 5*-space can be put into braid form?*

Problem 9.3 (J.S. Carter). *Develop the theory of Kamada's braid chart into a theory of chart movies. Thus, chart moves represent scenes in movies for surface braids. We expect a movie-move theorem that imitates the Reidemeister-Roseman moves. Analogues of the Alexander and Markov theorems are also sought.*

A similar theory of embedded and immersed braided 4-manifolds in $S^4 \times D^2$ can be given. Thus branched covers of the 4-sphere branched along linked surfaces are embedded and immersed in $S^4 \times D^2$. In particular the 2-fold branched covers can be embedded. In analogue to Questions 9.1 and 9.2, we have the following:

Question 9.4 (J.S. Carter). *Which of these embeddings are knotted?*

Question 9.5 (J.S. Carter). *When can an embedded* 4*-manifold in* 5*-space can be put into braid form?*

10 Small dilatation mapping classes

(Eriko Hironaka)

Let $\phi : S \to S$ be a pseudo-Anosov mapping class on an oriented surface $S = S_{g,n}$ of genus g and n punctures. The *dilatation* $\lambda(\phi)$ is the expansion factor of ϕ along the stable transverse measured singular foliation associated to ϕ, and is a Perron algebraic unit greater than one. The set of dilatations for a fixed S is discrete [48]. Let $\mathcal{P}(S)$ be the set of all pseudo-Anosov mapping classes on S. Let $\delta(S)$ be the minimum dilatation for $\phi \in \mathcal{P}(S)$. Let $P_{g,n}$ be the set of pseudo-Anosov mapping classes on $S_{g,n}$ with dilatation equal to $\delta(S_{g,n})$.

The minimum dilatation problem (cf. [40, 31, 9]) can be stated as follows.

Problem 10.1 (Minimum dilatation problem I). *What is the behavior of $\delta(S_{g,n})$ as a function of g and n?*

The exact value of $\delta(S_{g,n})$ is not known except for very small cases (for example, for closed surfaces, the answer is only known for $g=2$ [17]). More is known about the *normalized dilatation* $L(\phi) = \lambda(\phi)^{|\chi(S)|}$. For $\ell > 1$, we say ϕ is ℓ-small if $L(\phi) \leq \ell$. Let $\mathcal{P}(\ell)$ be the set of ℓ-small pseudo-Anosov maps. The current smallest known accumulation point of the image of L (see [21, 1, 26]) is

$$\ell_0 = \Big(\frac{3+\sqrt{5}}{2}\Big)^2.$$

Question 10.2 (E. Hironaka). *Is there an accumulation point for $L(\phi)$ as ϕ ranges in $\mathcal{P} = \bigcup_S \mathcal{P}(S)$ that is smaller than ℓ_0?*

One can also formulate the minimum dilatation problem from a geometric rather than numerical standpoint.

Problem 10.3 (Minimum dilatation problem II). *What do small dilatation mapping classes look like?*

A mapping class $\phi \in \mathcal{P}(S)$ is *quasiperiodic with bound K* if $\phi = R \circ \eta$ for some $R, \eta : S \to S$ where R is periodic and η is the identity outside a subsurface $S_0 \subset S$ with $|\chi(S_0)| \leq K$. A mapping class $\phi \in \mathcal{P}(S)$ is *periodic rel. boundary*, if for some $k > 0$, $\phi^k = \partial$, where ∂ is a composition of Dehn twists centered at boundary parallel simple-closed curves.

Question 10.4 (E. Hironaka). *Can any ℓ-small pseudo-Anosov mapping class be constructed as the Murasugi sum of a mapping class that is periodic rel. boundary and a quasiperiodic mapping class with bound K_ℓ depending only on ℓ?*

Given a hyperbolic 3-manifold M (possibly with cusps), let $\Psi(M)$ be the set (possibly empty) of fibrations of M (with connected fibers) over the circle S^1. Let $\Phi(M)$ be the set of monodromies of elements of $\Psi(M)$. Let $\mathcal{P}^0(\ell) \subset \mathcal{P}(\ell)$ be the set of ℓ-small elements with no interior singularities. Farb-Leininger-Margalit [10] showed that given $\ell > 1$, there is a finite set of 3-manifolds $M_1, \ldots, M_r$ so that

$$\mathcal{P}^0(\ell) \subset \bigcup_{i=1}^{r} \Phi(M_i). \tag{2}$$

Since there exists an $\ell > 1$ so that the elements of $P_{g,0}$ are ℓ-small for large enough g [40] (cf. [31]), $\mathcal{P}^0(\ell)$ can be replace by $P^0_{g,0}$ in Equation (2), where $P^0_{g,0}$ is the set of mapping classes in $P_{g,0}$ with interior singularities removed; and similarly for $P^0_{0,n}$ [22] and $P^0_{1,n}$ [49]. Tsai showed in [49], however, that for fixed $g \geq 2$, the set $\bigcup_n P_{g,n}$ is not ℓ-small for any ℓ.

Question 10.5 (E. Hironaka). *For fixed $g \geq 2$, does there exist a finite set of M_i so that*

$$\bigcup_n P^0_{g,n} \subset \bigcup_{i=1}^{k} \Phi(M_i)\ ?$$

Let $S = S_{g,n}$, $\phi_{g,n} \in P_{g,n}$, and let M be the mapping torus. Then either $\overline{\phi}$ is not pseudo-Anosov, and hence the corresponding Dehn filling of M is not hyperbolic, or $\overline{\phi}$ is pseudo-Anosov and we have

$$\lambda(\phi) \geq \lambda(\overline{\phi}) \geq \lambda(\phi_{g,0}) > 1.$$

The latter can only happen for a finite number of n, since for fixed g,

$$\lim_{n \to \infty} \lambda(\phi_{g,n}) = 1$$

(see [49]).

Question 10.6 (E. Hironaka). *Let S be a fixed surface with boundary, and let $\phi \in \mathcal{P}(S)$ be an element of minimum dilatation. Is the Dehn filling of the mapping torus of (S, ϕ) corresponding to ϕ always non-hyperbolic?*

If the answer to Question 10.6 is negative, it implies that for some g, the sequence $\delta(S_{g,n})$ is not monotone decreasing, giving a negative answer to the following question (cf. [9]).

Question 10.7 (E. Hironaka). *Is $\delta(S_{g,n})$ monotone decreasing in g and n?*

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