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<th>A $G$-family of quandles and handlebody-knots (Intelligence of Low-dimensional Topology)</th>
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<tr>
<td>Author(s)</td>
<td>Iwakiri, Masahide</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2012), 1812: 98-110</td>
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<tr>
<td>Issue Date</td>
<td>2012-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194511">http://hdl.handle.net/2433/194511</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A $G$-family of quandles and handlebody-knots

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We introduce the notion of a $G$-family of quandles and use it to construct invariants for handlebody-knots. Our invariant can detect the chiralities of some handlebody-knots including unknown ones. This is a joint work with Atsushi Ishii, Yeonhee Jang and Kanako Oshiro ([8]).

1 Handlebody-links

A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere $S^3$. Two handlebody-links are equivalent if there is an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. A spatial graph is a finite graph embedded in $S^3$. Two spatial graphs are equivalent if there is an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. When a handlebody-link $H$ is a regular neighborhood of a spatial graph $K$, we say that $K$ represents $H$, or $H$ is represented by $K$. In this paper, a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A diagram of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link.

An IH-move is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a 3-ball embedded in $S^3$. Then we have the following theorem.

**Theorem 1.1** ([6]). For spatial trivalent graphs $K_1$ and $K_2$, the following are equivalent.

- $K_1$ and $K_2$ represent an equivalent handlebody-link.
- $K_1$ and $K_2$ are related by a finite sequence of IH-moves.
- Diagrams of $K_1$ and $K_2$ are related by a finite sequence of the moves depicted in Figure 2.
2 A $G$-family of quandles

A quandle \([12, 16]\) is a non-empty set \(X\) with a binary operation \(* : X \times X \to X\) satisfying the following axioms.

- For any \(x \in X\), \(x * x = x\).
- For any \(x \in X\), the map \(S_x : X \to X\) defined by \(S_x(y) = y * x\) is a bijection.
- For any \(x, y, z \in X\), \((x * y) * z = (x * z) * (y * z)\).

When we specify the binary operation \(*\) of a quandle \(X\), we denote the quandle by the pair \((X, *)\). An Alexander quandle \((M, \ast)\) is a \(\Lambda\)-module \(M\) with the binary operation defined by \(x \ast y = tx + (1 - t)y\), where \(\Lambda := \mathbb{Z}[t, t^{-1}]\). A conjugation quandle \((G, \ast)\) is a group \(G\) with the binary operation defined by \(x \ast y = y^{-1}xy\).

Let \(G\) be a group with identity element \(e\). A \(G\)-family of quandles is a non-empty set \(X\) with a family of binary operations \(\ast^g : X \times X \to X\) \((g \in G)\) satisfying the following axioms.

- For any \(x \in X\) and any \(g \in G\), \(x *^g x = x\).
- For any \(x, y \in X\) and any \(g, h \in G\),
  \[x *^{gh} y = (x *^g y) *^h y\text{ and }x *^e y = x.\]
- For any \(x, y, z \in X\) and any \(g, h \in G\),
  \[(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z).\]
When we specify the family of binary operations $*^g : X \times X \rightarrow X$ ($g \in G$) of a $G$-family of quandles, we denote the $G$-family of quandles by the pair $(X, \{*^g\}_{g \in G})$.

**Proposition 2.1.** Let $G$ be a group. Let $(X, \{*^g\}_{g \in G})$ be a $G$-family of quandles.

(1) For each $g \in G$, the pair $(X, *^g)$ is a quandle.

(2) We define a binary operation $\triangleright : (X \times G) \times (X \times G) \rightarrow X \times G$ by

$$(x, g) \triangleright (y, h) = (x *^h y, h^{-1}gh).$$

Then $(X \times G, \triangleright)$ is a quandle.

We call the quandle $(X \times G, *)$ in Proposition 2.1 the associated quandle of $X$.

**Example 2.2.** (1) Let $(X, *)$ be a quandle. Let $S_x : X \rightarrow X$ be the bijection defined by $S_x(y) = y * x$. Let $m$ be a positive integer such that $S_x^m = id_X$ for any $x \in X$ if such an integer exists. We define the binary operation $*^i : X \times X \rightarrow X$ by $x *^i y = S_y^i(x)$. Then $X$ is a $\mathbb{Z}$-family of quandles and a $\mathbb{Z}_m$-family of quandles, where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.

(2) Let $R$ be a ring, and $G$ a group with identity element $e$. Let $X$ be a right $R[G]$-module, where $R[G]$ is the group ring of $G$ over $R$. We define the binary operation $*^g : X \times X \rightarrow X$ by $x *^g y = xg + y(e - g)$. Then $X$ is a $G$-family of quandles.

## 3 Colorings

Let $D$ be a diagram of a handlebody-link $H$. We set an orientation for each edge in $D$. Then $D$ is a diagram of an oriented spatial trivalent graph $K$. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $A(D)$ the set of arcs of $D$, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. For an arc $\alpha$ incident to a vertex $\omega$, we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$\epsilon(\alpha; \omega) = \begin{cases} 
1 & \text{if the orientation of } \alpha \text{ points to } \omega, \\
-1 & \text{otherwise.}
\end{cases}$$

Let $X$ be a $G$-family of quandles, and $Q$ the associated quandle of $X$. Let $p_X$ (resp. $p_G$) be the projection from $Q$ to $X$ (resp. $G$). An $X$-coloring of $D$ is a map $C : A(D) \rightarrow Q$ satisfying the following conditions at each crossing $\chi$ and each vertex $\omega$ of $D$ (see Figure 3).

- Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$. 

such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Then

$$C(\chi_2) = C(\chi_1) \triangleright C(\chi_3).$$

- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex $\omega$ arranged clockwise around $\omega$. Then

$$(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),$$

$$(p_G \circ C)(\omega_1)^{s(\omega_1, \omega)}(p_G \circ C)(\omega_2)^{s(\omega_2, \omega)}(p_G \circ C)(\omega_3)^{s(\omega_3, \omega)} = e.$$

We denote by $\text{Col}_X(D)$ the set of $X$-colorings of $D$. For two diagrams $D$ and $E$ which locally differ, we denote by $\mathcal{A}(D, E)$ the set of arcs that $D$ and $E$ share.

**Lemma 3.1.** Let $X$ be a $G$-family of quandles. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the $R1$–$R6$ moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)$, there is a unique $X$-coloring $C_{D,E} \in \text{Col}_X(E)$ such that $C_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$.

**Remark 3.2.** Let $X$ be a $\mathbb{Z}$-family of quandles or a $\mathbb{Z}_m$-family of quandles defined as in Example 2.2 (2). Then an $X$-coloring be regarded as an $X$-coloring defined in [7].

Let $X$ be a $G$-family of quandles, and $Q$ the associated quandle of $X$. An $X$-set is a non-empty set $Y$ with a family of maps $*^g : Y \times X \rightarrow Y$ satisfying the following axioms, where we note that we use the same symbol $*^g$ as the binary operation of the $G$-family of quandles.

- For any $y \in Y$, $x \in X$, and any $g, h \in G$,

$$y *^{gh} x = (y *^g x) *^h x \text{ and } y *^e x = y.$$
For any $y \in Y$, $x_1, x_2 \in X$, and any $g, h \in G$,

$$(y \ast^g x_1) \ast^h x_2 = (y \ast^h x_2) \ast^{h^{-1}gh} (x_1 \ast^h x_2).$$

Put $y \triangleright (x, g) := y \ast^g x$ for $y \in Y$, $(x, g) \in Q$. Then the second axiom implies that $(y \triangleright q_1) \triangleright q_2 = (y \triangleright q_2) \triangleright (q_1 \triangleright q_2)$ for $q_1, q_2 \in Q$. Any $G$-family of quandles $(X, \{\ast^g\}_{g \in G})$ itself is an $X$-set with its binary operations. Any singleton set $\{y\}$ is also an $X$-set with the maps $\ast^g$ defined by $y \ast^g x = y$ for $x \in X$ and $g \in G$, which is a trivial $X$-set.

Let $D$ be a diagram of an oriented spatial trivalent graph. We denote by $\mathcal{R}(D)$ the set of complementary regions of $D$. Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $Q$ be the associated quandle of $X$. An $X_Y$-coloring of $D$ is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$ satisfying the following conditions.

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- The restriction $C|_{\mathcal{A}(D)}$ of $C$ on $\mathcal{A}(D)$ is an $X$-coloring of $D$.
- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$C(\alpha_1) \triangleright C(\alpha) = C(\alpha_2),$$

where $\alpha_1, \alpha_2$ are the regions facing the arc $\alpha$ so that the normal orientation of $\alpha$ points from $\alpha_1$ to $\alpha_2$ (see Figure 4).

We denote by $\text{Col}_X(D)_Y$ the set of $X_Y$-colorings of $D$.

For two diagrams $D$ and $E$ which locally differ, we denote by $\mathcal{R}(D, E)$ the set of regions that $D$ and $E$ share.

**Lemma 3.3.** Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the $R1$–$R6$ moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)_Y$, there is a unique $X_Y$-coloring $C_{D, E} \in \text{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D, E)} = C_{D, E}|_{\mathcal{A}(D, E)}$ and $C|_{\mathcal{R}(D, E)} = C_{D, E}|_{\mathcal{R}(D, E)}$.  

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node at (0,0) {$y_1$};
  \node at (1,0) {$y_1 \triangleright q$};
  \draw[->] (0,0) -- (1,0);
\end{tikzpicture}
\caption{Figure 4:}
\end{figure}
4 A homology

Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $(Q, \triangleright)$ be the associated quandle of $X$. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$(y, q_1, \ldots, q_i) \triangleright q, q_{i+1}, \ldots, q_n) := (y \triangleright q, q_1 \triangleright q, \ldots, q_{i} \triangleright q, q_{i+1}, \ldots, q_n)$$

for $y \in Y$ and $q, q_1, \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\partial_n(y, q_1, \ldots, q_n) = \sum_{i=1}^{n} (-1)^i (y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) - \sum_{i=1}^{n} (-1)^i ((y, q_1, \ldots, q_{i-1}) \triangleright q_i, q_{i+1}, \ldots, q_n)$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 4, 5]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+2}, \ldots, q_n) \middle| y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q \right\}$$

and

$$\bigcup_{i=1}^{n} \left\{ -(y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+1}, \ldots, q_n) \middle| y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q \right\}.$$

We remark that

$$(y, q_1, \ldots, q_{i-1}, (x, e), q_{i+1}, \ldots, q_n)$$

and

$$(y, q_1, \ldots, q_{i-1}, (x, g), q_{i+1}, \ldots, q_n) + ((y, q_1, \ldots, q_{i-1}) \triangleright (x, g), (x, g^{-1}), q_{i+1}, \ldots, q_n)$$

belong to $D_n(X)_Y$.

**Lemma 4.1.** For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group $A$, we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the $n$th homology group of $C_*(X)_Y$. 
5 Cocycle invariants

Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $D$ be a diagram of an oriented spatial trivalent graph. For an $X_Y$-coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing $\chi$ of $D$ as follows. Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Let $R_\chi$ be the region facing $\chi_1$ and $\chi_3$ such that the normal orientations $\chi_1$ and $\chi_3$ point from $R_\chi$ to the opposite regions with respect to $\chi_1$ and $\chi_3$, respectively. Then we define
\[
w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),\]
where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing $\chi$. We define a chain $W(D; C) \in C_2(X)_Y$ by
\[
W(D; C) = \sum_\chi w(\chi; C),
\]
where $\chi$ runs over all crossings of $D$.

**Lemma 5.1.** The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles $\theta, \theta'$ of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

**Lemma 5.2.** Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the $R1$–$R6$ moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D,E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by $G_H$ (resp. $G_K$) the fundamental group of the exterior of a handlebody-link $H$ (resp. a spatial graph $K$). When $H$ is represented by $K$, the groups $G_H$ and $G_K$ are identical. Let $D$ be a diagram of an oriented spatial trivalent graph $K$. By the definition

\[
\begin{align*}
\chi_4 & \\ 
\chi_1 & \quad \chi_2 \\
\chi_3 & \\
\epsilon(\chi) = 1 & \\
\chi_3 & \quad \chi_2 \\
\chi_4 & \\
\epsilon(\chi) = -1
\end{align*}
\]

Figure 5:
of an \(X_Y\)-coloring \(C\) of \(D\), the map \(p_G \circ C|_{A(D)}\) represents a homomorphism from \(G_K\) to \(G\), which we denote by \(\rho_C \in \text{Hom}(G_K, G)\). For \(\rho \in \text{Hom}(G_K, G)\), we define
\[
\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.
\]
For a 2-cocycle \(\theta\) of \(C^*(X; A)_Y\), we define
\[
\mathcal{H}(D) := \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\},
\]
\[
\Phi_\theta(D) := \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\},
\]
\[
\mathcal{H}(D; \rho) := \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\},
\]
\[
\Phi_\theta(D; \rho) := \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\}
\]
as multisets.

**Lemma 5.3.** Let \(D\) be a diagram of an oriented spatial trivalent graph \(K\). For \(\rho, \rho' \in \text{Hom}(G_K, G)\) such that \(\rho\) and \(\rho'\) are conjugate, we have \(\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho')\) and \(\Phi_\theta(D; \rho) = \Phi_\theta(D; \rho')\).

We denote by \(\text{Conj}(G_K, G)\) the set of conjugacy classes of homomorphisms from \(G_K\) to \(G\). By Lemma 5.3, \(\mathcal{H}(D; \rho)\) and \(\Phi_\theta(D; \rho)\) are well-defined for \(\rho \in \text{Conj}(G_K, G)\).

**Lemma 5.4.** Let \(D\) be a diagram of an oriented spatial trivalent graph \(K\). Let \(E\) be a diagram obtained from \(D\) by reversing the orientation of an edge \(e\). For \(\rho \in \text{Hom}(G_K, G)\), we have \(\mathcal{H}(D) = \mathcal{H}(E)\), \(\Phi_\theta(D) = \Phi_\theta(E)\), \(\mathcal{H}(D; \rho) = \mathcal{H}(E; \rho)\) and \(\Phi_\theta(D; \rho) = \Phi_\theta(E; \rho)\).

By Lemma 5.4, \(\mathcal{H}(D)\), \(\Phi_\theta(D)\), \(\mathcal{H}(D; \rho)\) and \(\Phi_\theta(D; \rho)\) are well-defined for a diagram \(D\) of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram \(D\) of a handlebody-link \(H\), we define
\[
\mathcal{H}^{\text{hom}}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\},
\]
\[
\Phi_\theta^{\text{hom}}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\},
\]
\[
\mathcal{H}^{\text{conj}}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\},
\]
\[
\Phi_\theta^{\text{conj}}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}
\]
as "multisets of multisets". We remark that, for \(X_Y\)-colorings \(C\) and \(C_{D,E}\) in Lemma 5.2, we have \(\rho_C = \rho_{C_{D,E}}\). Then, by Lemmas 5.1–5.4, we have the following theorem.
Theorem 5.5. Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $\theta$ be a 2-cocycle of $C^*(X; A)_Y$. Let $H$ be a handlebody-link represented by a diagram $D$. Then the following are invariants of a handlebody-link $H$.

$$
\mathcal{H}(D), \quad \Phi_\theta(D), \quad \mathcal{H}^{\text{hom}}(D), \quad \Phi_\theta^{\text{hom}}(D), \quad \mathcal{H}^{\text{conj}}(D), \quad \Phi_\theta^{\text{conj}}(D).
$$

We denote the invariants of $H$ given in Theorem 5.5 by

$$
\mathcal{H}(H), \quad \Phi_\theta(H), \quad \mathcal{H}^{\text{hom}}(H), \quad \Phi_\theta^{\text{hom}}(H), \quad \mathcal{H}^{\text{conj}}(H), \quad \Phi_\theta^{\text{conj}}(H),
$$

respectively.

We denote by $H^*$ the mirror image of a handlebody-link $H$. Then we have the following theorem.

Theorem 5.6. For a handlebody-link $H$, we have

$$
\mathcal{H}(H^*) = -\mathcal{H}(H), \quad \Phi_\theta(H^*) = -\Phi_\theta(H),
$$

$$
\mathcal{H}^{\text{hom}}(H^*) = -\mathcal{H}^{\text{hom}}(H), \quad \Phi_\theta^{\text{hom}}(H^*) = -\Phi_\theta^{\text{hom}}(H),
$$

$$
\mathcal{H}^{\text{conj}}(H^*) = -\mathcal{H}^{\text{conj}}(H), \quad \Phi_\theta^{\text{conj}}(H^*) = -\Phi_\theta^{\text{conj}}(H),
$$

where $-S = \{-a \mid a \in S\}$ for a multiset $S$.

6 Applications

In this section, we calculate cocycle invariants defined in the previous section for the handlebody-knots $0_1, \ldots, 6_{16}$ in the table given in [9], by using a 2-cocycle given by Nosaka [18]. This calculation enables us to distinguish some of handlebody-knots from their mirror images, and a pair of handlebody-knots whose complements have isomorphic fundamental groups.

Let $G = SL(2; \mathbb{Z}_3)$ and $X = (\mathbb{Z}_3)^2$. Then $X$ is a $G$-family of quandles with the proper binary operation as given in Proposition 2.2 (2). Let $Y$ be the trivial $X$-set $\{y\}$. We define a map $\theta : Y \times (X \times G)^2 \to \mathbb{Z}_3$ by

$$
\theta(y, (x_1, g_1), (x_2, g_2)) := \lambda(g_1) \det(x_1 - x_2, x_2(1 - g_2^{-1})),
$$

where the abelianization $\lambda : G \to \mathbb{Z}_3$ is given by

$$
\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d)(b - c)(1 - bc).
$$
By [18], the map $\theta$ is a 2-cocycle of $C^*(X; \mathbb{Z}_3)$. Table 1 lists the invariant $\Phi_{\theta}^{\text{conj}}(H)$ for the handlebody-knots $0_1, \ldots, 6_{16}$. We represent the multiplicity of elements of a multiset by using subscripts. For example, $\{(0_2, 1_3)\}$ represents the multiset $\{0, 0, 1, 1, 1\}$.

Table 1:

<table>
<thead>
<tr>
<th>$\Phi_{\theta}(H)$</th>
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<tbody>
<tr>
<td>0_1: {(0_9)}</td>
</tr>
<tr>
<td>4_1: {(0_9)<em>{33}, (0</em>{27})<em>{22}, (0</em>{27})_{11}}</td>
</tr>
<tr>
<td>5_1: {(0_9)}</td>
</tr>
<tr>
<td>5_2: {(0_9)<em>{35}, (0</em>{27})<em>6, (0</em>{27})<em>{11}, 1, (0_9, 1</em>{18})<em>4, (0</em>{27}, 1_{54})_2}</td>
</tr>
<tr>
<td>5_3: {(0_9)<em>{102}, (0</em>{27})<em>4, (0</em>{27}, 2_{54})_2}</td>
</tr>
<tr>
<td>5_4: {(0_9)<em>{74}, (0</em>{27})_2}</td>
</tr>
<tr>
<td>6_1: {(0_9)<em>{31}, (0</em>{27})<em>{16}, (0</em>{27})_4}</td>
</tr>
<tr>
<td>6_2: {(0_9)<em>{106}, (0</em>{45}, 1_{18})_2}</td>
</tr>
<tr>
<td>6_3: {(0_9)<em>{74}, (0</em>{27})_2}</td>
</tr>
<tr>
<td>6_4: {(0_9)}</td>
</tr>
<tr>
<td>6_5: {(0_9)<em>{74}, (0_9, 1</em>{18})_2}</td>
</tr>
<tr>
<td>6_6: {(0_9)<em>{72}, (0</em>{27})_4}</td>
</tr>
<tr>
<td>6_7: {(0_9)<em>{35}, (0</em>{27})<em>{16}, (0</em>{27})<em>{11}, (0_9, 1</em>{18})<em>4, (0</em>{27}, 1_{54})_2}</td>
</tr>
<tr>
<td>6_8: {(0_9)}</td>
</tr>
<tr>
<td>6_9: {(0_9)<em>{31}, (0</em>{27})<em>6, (0</em>{27})<em>{11}, (0_9, 1</em>{18})<em>6, (0</em>{27}, 1_{54})<em>2, (0</em>{27}, 2_{54})_2}</td>
</tr>
<tr>
<td>6_{10}: {(0_9)}</td>
</tr>
<tr>
<td>6_{11}: {(0_9)<em>{70}, (0_9, 1</em>{18})_6}</td>
</tr>
<tr>
<td>6_{12}: {(0_9)<em>{37}, (0</em>{27})<em>{11}, (0_9, 1</em>{18})<em>8, (0_9, 1</em>{36})<em>2, (0</em>{27})_2}</td>
</tr>
<tr>
<td>6_{13}: {(0_9)<em>{35}, (0</em>{27})<em>6, (0</em>{27})<em>{11}, (0_9, 2</em>{18})<em>4, (0</em>{27}, 2_{54})_2}</td>
</tr>
<tr>
<td>6_{14}: {(0_9)<em>{19}, (0</em>{27})<em>6, (0</em>{27})<em>{11}, (0_9, 1</em>{18})<em>{12}, (0</em>{27}, 1_{54})_4}</td>
</tr>
<tr>
<td>6_{15}: {(0_9)<em>{19}, (0</em>{27})<em>6, (0</em>{27})<em>{11}, (0_9, 2</em>{18})<em>{12}, (0</em>{27}, 1_{54})_4}</td>
</tr>
<tr>
<td>6_{16}: {(0_9)<em>{44}, (0</em>{27})_{32}}</td>
</tr>
</tbody>
</table>

From Table 1, we see that our invariant can distinguish the handlebody-knots $6_{14}, 6_{15}$, whose complements have the isomorphic fundamental groups. Together with Theorem 5.6, we also see that handlebody-knots $5_2, 5_3, 5_6, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15}$ are not equivalent to their mirror images. In particular, the chiralities of $5_3, 6_5, 6_{11}$ and $6_{12}$ were not known. Table 2 shows us known facts on the chirality of handlebody-knots in [9] so far. In the column of “chirality”, the symbols $\bigcirc$ and $\times$ mean that the handlebody-knot is amphichiral and chiral, respectively, and the symbol $?_\bigcirc$ means that it is not known whether the handlebody-knot is amphichiral or chiral. The symbols $\bigcirc$ in the right five columns mean that the handlebody-knots can be proved chiral by using the method introduced
in the papers corresponding to the columns. Here, M, II, LL, IKO and IIJO denote the papers [17], [7], [15], [10] and this paper, respectively.

References


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