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On the Alexander polynomials of links with symmetry

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1. Introduction

A *symmetric link* L in $\mathbb{R}^3$ is a link with a diagram on which a finite group can act. Figure 1 shows a link diagram, on which the Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ can act. The periodic links of order n are symmetric links whose acting group is the cyclic group $\mathbb{Z}_n$.

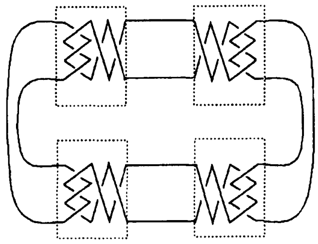


FIGURE 1. Symmetric link

It is well-known in topological graph theory that the covering graph is constructed by a voltage assignment on the set of edges of the base graph, and that there is one-to-one correspondence between the set of all embeddings of a graph and the set of all rotation schemes. Also, by lifting the rotation scheme of a base graph to that of the covering graph, one can get the embedding of the covering graph, see [16][6].

In this paper, we will introduce a method to construct symmetric links by adapting the graph theoretical settings to link diagrams, which are 4-valent graphs embedded in S2 with under over information, and try to calculate the Seifert matrix of such a resulting symmetric link from the information of the base link and the information of the acting group. Also we will find a Seifert matrix of link admitting Klein 4-group ($\mathbb{Z}_2 \oplus \mathbb{Z}_2$) action. And by using the Seifert matrix, we will calculate the Alexander polynomial and the determinant of links admitting Klein 4-group action. To calculate our invariants, we shall use a new notation.

Definition 1.1. [4] *A $(k,k)-$tangle in I^2 is defined by a diagram $D \cap I^2$ in $I^2 = \{(x,y)|0 \leq x,y \leq 1\}$, where D is an unoriented link diagram in $\mathbb{R}^2$ such that $D \cap \partial I^2$ is the set $\{(\frac{i}{k+1},1),(\frac{i}{k+1},0)|i = 1,2,\cdots,k\}$ and the set is disjoint from the vertices of D. A typical example is the braid group. By a $2k-$tangle, we mean a $(k,k)-$tangle.*

For a $2k-$tangle T, $D(T)$, the *denominator*, of T is defined from T by applying the closure operation as the braid group theory. see Figure 2. In particular, if T is a 4-tangle, the *numerator* $N(T)$ of T is defined as the last picture in Figure 2. If an orientation on $D(T)$ is given, it induces the natural orientation of a resulting symmetric link from the the orientation of $D(T)$.

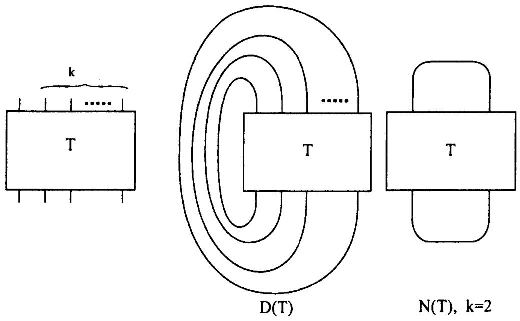


FIGURE 2.

Let T be a 4-tangle. Suppose the denominator $D(T)$ of T is oriented. The orientations of outer arcs of T are either parallel or opposite as in Figure 3. An oriented 4-tangle T is said to be *co-oriented* if the orientations of outer arcs of T are parallel, while T is said to be *contra-oriented* if the orientations of outer arcs of T are opposite, see Figure 3.

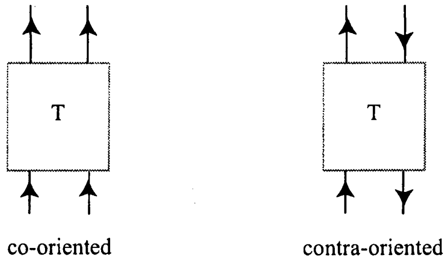


FIGURE 3

The followings are the main results of the paper.

Theorem 1.2. *Let $D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the symmetric link constructed by the base link $D(T)$ and Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action. Suppose that $D(T)$ is oriented 4−tangle with $\phi(e) \neq\neq 1_G$ and $\phi(f) \neq 1_G$. Suppose that $\widetilde{L} = D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and $L = D(T)$ for some 4-tangle T.*

(1) *If T is co-oriented, then*

$$\Delta_{\widetilde{L}}(t) = 2(\Delta_L)^3(t)\{\Delta_{D(T_+)}(t) + \Delta_{D(T_-)}(t)\}.$$

(2) *If T is contra-oriented, then*

$$\Delta_{\widetilde{L}}(t) = 4(\Delta_L)^3(t)\Delta_{N(T)}(t).$$

In (1), T_+ and T_- are the 4-tangles defined in Figure 4.

2. PRELIMINARY

In this section, we recall the known results which are related with the calculation of a Seifert matrix of link and the constuction of a symmetric link, see [1][2][16][4][5][6].

A *Seifert surface* for an oriented link L in S^3 is a connected compact oriented surface contained in S^3 which has L as its boundary. We will give a brief sketch about the Seifert algorithm.

For

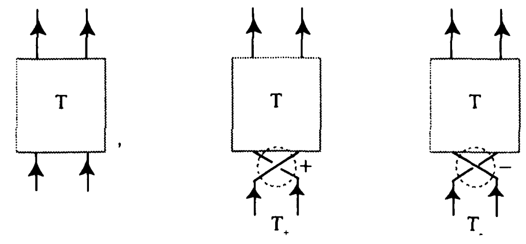


FIGURE 4.

Let D be a diagram of an oriented link L. In a small neighborhood of each crossing, make the following local change to the diagram;

> Delete the crossing and reconnect the loose ends in the only way compatible with the orientation.

When this has been done at every crossing, the diagram becomes a set of disjoint simple loops in the plane. It is a diagram with no crossings. These loops are called *Seifert circles.* By attaching a disc to each Seifert circle and by connecting a half-twisted band at the place of each crossing of D according to the crossing sign, we get a Seifert surface for L. The *Seifert graph* Γ of D is constructed as follows;

> Associate a vertex with each Seifert circle and connect two vertices with an edge if their Seifert circles are connected by a twisted band.

Note that a Seifert graph Γ is planar, and that if D is connected, so do Γ. Since Γ is a deformation retract of Seifert surface F, their homology groups are isomorphic: $H_1(F) \cong H_1(\Gamma)$. Let T be a spanning tree for Γ. For each edge $e \in E(\Gamma) \setminus E(T)$, $T \cup e$ contains a unique simple closed circuit T_e, which represents an $1-$ *cycle* in F. The set $\{T_e \mid e \in E(\Gamma) \setminus E(T)\}$ of these $1-$cycles is a homology basis for F. For such a circuit T_e, let T_e^+ denote the circuit in S^3 obtained by lifting slightly along the positive normal direction of F. A *Seifert matrix* of L associated to F is the $n \times n$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = lk(T_{e_i}, T_{e_j}^+),$$

where $E(\Gamma) \setminus E(T) = \{e_1, \cdots, e_n\}$. The Seifert matrix of L depends on the Seifert surface F and the choice of generators of $H_1(F)$.

Let M be any Seifert matrix for an oriented link L. The *Alexander polynomial* $\Delta_L(x) \in \mathbb{Z}[x^{\pm}]$, the *determinant* $\det(L)$ and the *signature* $\sigma(L)$ of L are defined by

$$\begin{aligned} \Delta_L(x) &= det(xM - x^{-1}M^T) \\ \det(L) &= |\det(M + M^T)| \end{aligned}$$

If $T_e \cap T_f$ is not an empty set, let v_0 and v_1 denote two ends of $T_e \cap T_f$. Without loss of generality, we may assume that the neighborhood of v_0 looks like Figure 5. In other words, the cyclic order of edges incident to v_0 is given by $T_e \cap T_f, T_e, T_f$ with respect to the positive normal direction of the Seifert surface. Also we may assume that the directions of T_e and T_f are given so that v_0 is the starting point of $T_e \cap T_f$. For, if the direction is reversed, one can change the direction to adapt to our setting so that the resulting linking number changes its sign.

Proposition 2.1. [1] *For $e, f \in E(\Gamma) \setminus E(T)$, let p and q denote the numbers of edges in $T_e \cap T_f$ corresponding to positive crossings and negative crossings, respectively. Suppose that the local*

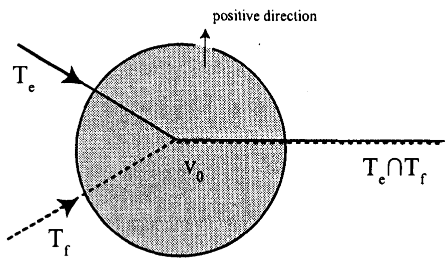


FIGURE 5

shape of $T_e \cap T_f$ in F looks like Figure 5. Then,

$$lk(T_e, T_f^+) = \begin{cases} -\frac{1}{2}(p-q), & \textit{if } p+q \textit{ is even;} \\ -\frac{1}{2}(p-q+1), & \textit{if } p+q \textit{ is odd, and} \end{cases}$$

$$lk(T_f, T_e^+) = \begin{cases} -\frac{1}{2}(p-q), & \textit{if } p+q \textit{ is even;} \\ -\frac{1}{2}(p-q-1), & \textit{if } p+q \textit{ is odd.} \end{cases}$$

A *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ consists of a finite set $V(\Gamma)$ of vertices and a finite set $E(\Gamma)$ of edges. An *embedding* of Γ into a surface F is a continuous injection $i : \Gamma \to F$. An embedding of Γ into a surface F is called a 2-*cell embedding* if each component of $F \setminus i(\Gamma)$, called a *region* of the embedding, is homeomorphic to the standard disc. For a vertex $v_i \in V(\Gamma)$, let $V(v_i)$ be the set of all vertices incident to v_i, and let $P_{v_i} : V(v_i) \to V(v_i)$ be a cyclic permutation on $V(v_i)$. We call $(P_{v_1} P_{v_2} \cdots, P_{v_n})$ *Edmond's rotation scheme.*

Proposition 2.2. [6] *A rotation scheme $(P_{v_1}, P_{v_2}, \cdots, P_{v_n})$ determine a 2-cell embedding $\Gamma(M)$ of Γ in a surface F, such that there is an orientation on F which a cyclic ordering of the edge $[v_i, v_k]$ at i in which the immediate successor to $[v_i, v_k]$ is $[v_j, P_{v_i}(v_k)]$. Conversely, for a given 2-cell embedding $i : \Gamma \to F$ in a surface F with a given orientation, there is a corresponding rotation scheme $(P_{v_1}, P_{v_2}, \cdots, P_{v_n})$ determining that embedding.*

Let Γ be a graph with and G a finite group. Let $D(\Gamma)$ denote the set of all directed edges of Γ. and let $\phi : D(\Gamma) \to G$ be a function, called a *voltage assignment*, satisfying $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in D(\Gamma)$. We call a triple (Γ, G, ϕ) a *voltage graph.* The *covering graph* $\Gamma \times_\phi G$ for (Γ, G, ϕ) has the vertex set $V(\Gamma) \times G$ and each edge $e = uv$ of Γ determines the edges $(u, g)(v, g\phi(e))$ of $\Gamma \times_\phi G$, for all $g \in G$. Notice that $\Gamma \times_\phi G$ is a $|G|$-fold regular covering space of Γ; in fact, every regular covering space of Γ can be obtained in this manner.

Example. Let Γ be a graph as the left of Figure 6 and let $D(\Gamma)$ be the set of directed edges of Γ. Let $\phi : D(\Gamma) \to \mathbb{Z}_5$ be a voltage assignment defined by $(u, u) \to 1$, $(u, v) \to 0$, $(v, v) \to 2$, where $(u, u), (u, v), (v, v) \in D(\Gamma)$. Then, $\Gamma \times_\phi G$ is the graph in the right of Figure 6.

The number of components of $D \times_\phi G$ can be calculated by the following proposition.

Proposition 2.3. [6] *Let D be a connected link diagram. Let $\phi : E(D) \to G$ be a voltage assignment. Let H be the subgroup of G generated by* $\mathrm{Im}(\phi)$. *Then the number of components of $D \times_\phi G$ is*

$$[G : H] = |G|/|H|.$$

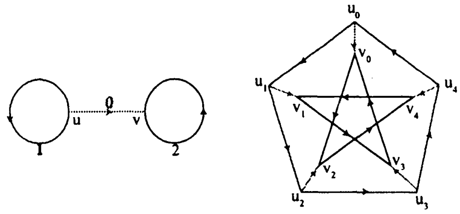


FIGURE 6.

Now consider a voltage graph (Γ, G, ϕ) which is 2-cell embedded in an orientable surface S, as described algebraically by the rotation scheme $P = (P_1, P_2, \ldots, P_n)$. We define the *lift* $\tilde{P}$ of P to $\Gamma \times_\phi G$ as follows: if $P_v(v, u) = (v, w)$, then

$$\tilde{P}_{(v,g)}((v, g), (u, g\phi(v, u))) = ((v, g), (w, g\phi(v, w))),$$

for each $g \in G$. Since $\tilde{P} = \{\tilde{P}_{(v,g)} | (v, g) \in V(\Gamma \times_\phi G)\}$ is a rotation scheme of $\Gamma \times_\phi G$, it determines the natural embedding of $\Gamma \times_\phi G$ into a surface $\tilde{S}$.

For a region R of the embedding of Γ on S induced by P, let $|R|_\phi$ be the order of $\phi(\partial R) = \phi(e_1)\phi(e_2)\cdots\phi(e_n)$ in G, where $\partial R = e_1, e_2, \ldots, e_n$ is the ordered boundary of R. Since $\phi(\partial R)$ is unique up to inverses and conjugacy, $|R|_\phi$ is independent of the orientation of R and of the initial vertex of w. The following is well-known in topological graph theory, from which one can calculate the genus of the embedding surface $\tilde{S}$.

Proposition 2.4. [6] *Let (Γ, G, ϕ) be a voltage graph with rotation scheme P and $\widetilde{P}$ which determine a 2-cell embedding of Γ and $\Gamma \times_\phi G$ on the orientable surfaces S and $\widetilde{S}$, respectively. Then, there exists a branched covering $\rho : \widetilde{S} \to S$ such that*

(1) $\rho^{-1}(\Gamma) = \Gamma \times_\phi G$.
(2) $\rho| : \Gamma \times_\phi G \to \Gamma$ *is the graph covering map.*
(3) *If b is a branch point of multiplicity m, then there exists a face R in Γ embedded in S such that $b \in Int(R)$ and $|R|_\phi = m$.*
(4) *If R is a k-gon in Γ embedded in S, then $\rho^{-1}(R)$ has $\frac{|G|}{|R|_\phi}$ components, each of which is a $k|R|_\phi$-gon region in $\Gamma \times_\phi G \to \widetilde{S}$.*

Let D be a diagram of L embedded in $\mathbb{R}^2 \subset S^2$, which can be seen as a 4-valent graph with under/over information at each vertex. Let V, E and F denote the numbers of vertices, edges and faces of the embedded 4-valent graph D, respectively. Let G be a finite group of order n and $\phi : \vec{E}(D) \to G$ be a voltage assignment on a link diagram D, where $\vec{E}(D)$ is the set of directed edges of D. Let $D \times_\phi G \hookrightarrow \widetilde{S}$ denote the embedding of $D \times_\phi G$ determined by the lifted rotation scheme with $\widetilde{V}$ vertices, $\widetilde{E}$ edges and $\widetilde{F}$ faces. If the embedding surface $\widetilde{S}$ is the sphere S^2, one can obtain a *symmetric link* $D \times_\phi G$ by recovering the under/over information at each vertex according to the under/over information of the corresponding vertex of D. If $\widetilde{S}$ is *not* the sphere S^2, one may see the embedding $D \times_\phi G \hookrightarrow \widetilde{S}$ as a kind of *virtual symmetric link.*

We can calculate the genus $\widetilde{g}$ of $\widetilde{S}$, by using the following propositions.

Proposition 2.5. [1] *Let D be a connected link diagram. Let $\phi : \vec{E}(D) \to G$ be a voltage assignment. Let H be the subgroup of G generated by* $\mathrm{Im}(\phi)$. *The genus $\widetilde{g}$ of $\widetilde{S}$ is given by*

$$\widetilde{g} = |G|\,(\frac{1}{|H|} + \frac{1}{2}\sum_R (1 - \frac{1}{|R|_\phi}) - 1).$$

In particular, if $\widetilde{S}$ is connected, then

$$\widetilde{g} = 1 - |G| + \frac{|G|}{2}\sum_R (1 - \frac{1}{|R|_\phi}).$$

Proposition 2.6. [1] *Suppose that $\phi : \vec{E}(D) \to G$ be a voltage assignment such that there are exactly two edges e and f with non-trivial voltages $\phi(e) = a$ and $\phi(f) = b$, and that e and f are on the boundary of the same region, see Figure 7. Then*

(1) *if $b = a^{-1} \in G$, then $\widetilde{S}$ is the disjoint union of $\frac{|G|}{|a|}$ copies of S^2.*
(2) *if $G = \mathbb{Z}_p$ (p is prime) and $b \neq a^{-1}$, then $\widetilde{S}$ is the surface of genus $\frac{p-1}{2}$.*
(3) *if every element of G is of order 2, then $\widetilde{S}$ is a disjoint union of suitable copies of S^2.*

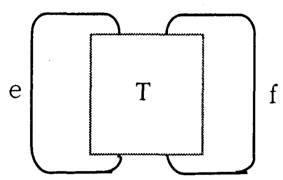


FIGURE 7. redraw

In this paper, we will find a Seifert matrix of link admitting Klein 4-group ($\mathbb{Z}_2 \oplus \mathbb{Z}_2$) action. Since every element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is order 2, $\widetilde{S}$ is a disjoint union of suitable copies of S^2 by Theorem 2.6. If $\phi(e)$ or $\phi(f)$ is 1_G in Figure 7, then one can calculate the Alexander polynomial of $D \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ by using Proposition 2.3 and Proposition 2.6 in Figure ??.

From now on, we assume that there exist two edges e and f in D such that $\phi(e) \neq 1_G$, $\phi(f) \neq 1_G$ and $\phi(g) = 1_G$ for every edge $g \neq e, f$ and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If $\phi(e) = \phi(f) \neq 1_G$, then $D \times_\phi G$ has 2-components with $C_1 = C_2 \cong D \times_\phi \mathbb{Z}_2$. Then from Proposition 2.3 and Proposition 2.5, one can know that$\Delta_{\widetilde{L}} = 0$ and $\det(\widetilde{L}) = 0$.

3. Proof of the main theorem

Now we shall introduce formula for the determinant of matrix. This formula is the key tool for the calculation of our main theorem in the paper.

Lemma 3.1. *Let A, B, C, D and E be $m \times m, m \times 1, 1 \times m, 1 \times 1$ and 1×1 matrices, respectively. Then*

$$\det\begin{pmatrix} A & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & B \\ 0 & 0 & A & 0 & B \\ 0 & 0 & 0 & A & B \\ C & C & C & C & 2(D+E) \end{pmatrix} = 2(\det A)^3\{\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \det\begin{pmatrix} A & B \\ C & E \end{pmatrix}\},$$

where 0 *denotes the zero-matrix.*

Proof.

$$
\begin{aligned}
\det\begin{pmatrix} A & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & B \\ 0 & 0 & A & 0 & B \\ 0 & 0 & 0 & A & B \\ C & C & C & C & 2(D+E) \end{pmatrix}
&\underset{(1)}{=} \det\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & B \\ \hline 0 & A & 0 & 0 & 2B \\ 0 & 0 & A & 0 & B \\ 0 & 0 & 0 & A & B \\ 0 & C & C & C & 2(D+E) \end{array}\right) \\
&= \det(A)\det\left(\begin{array}{ccc|c} A & 0 & 0 & 2B \\ 0 & A & 0 & B \\ 0 & 0 & A & B \\ \hline C & C & C & 2(D+E) \end{array}\right) \\
&\underset{(2)}{=} \det(A)^3\det\left(\begin{array}{c|c} A & 4B \\ \hline C & 2(D+E) \end{array}\right) \\
&= 2\det(A)^3\det\left(\begin{array}{c|c} A & 2B \\ \hline C & (D+E) \end{array}\right) \\
&= 2\det(A)^3\{\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \det\begin{pmatrix} A & B \\ C & E \end{pmatrix}\}
\end{aligned}
$$

The reason for the identities (1)(2) are;

(1) Add (-1)(the second column) to the first column and then, add the first row to the second row.

(2) Apply the method (1) to the matrix repeatedly.

□

From now on, we assume that T is a 4−tangle whose denominator $D(T)$ is oriented. If T is co-oriented, the two outer arcs of $D(T)$ have to be contained in different Seifert circles (CASE I), see the first picture in Figure 8, while if T is conta-oriented, the two outer arcs of $D(T)$ can be contained in either the same Seifert circle (CASE II) or different Seifert circles (CASE III), see Figure 8.

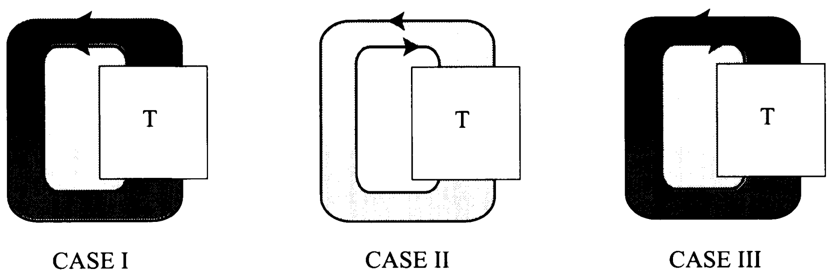


FIGURE 8

CASE I. T is co-oriented.

For the convenience of readers, we may apply the Reidemeister move II between the the left two outer strands and the right two right strands of T to get a new tangle T' so that Seifert circles at the left and the right are 2-gons, and hence the corresponding vertices of the Seifert graph have degree 2, respectively, see Figure 9. Note that $D(T')$, $D(T'_+)$, $D(T'_-)$ and $D(\widetilde{L}')$ are ambient isotopic to $D(T)$, $D(T_+)$, $D(T_-)$ and $D(\widetilde{L})$, respectively.

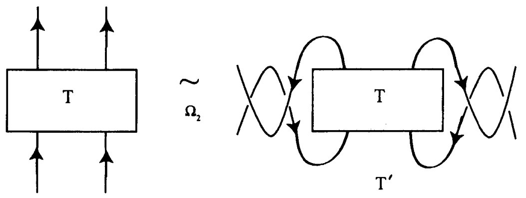


FIGURE 9

The Seifert graphs $\Gamma_{D(T)}$, $\Gamma_{D(T_+)}$ and $\Gamma_{D(T_-)}$ of $D(T)$ $D(T_+)$ and $D(T_-)$ are of the form in Figure 10, in which spanning trees $\tau_{D(T)}$, $\tau_{D(T_+)}$ and $\tau_{D(T_-)}$ of $\Gamma_{D(T)}$ $\Gamma_{D(T_+)}$ and $\Gamma_{D(T_-)}$ are given by dotted edges, respectively. Notice that $\Gamma_{D(T_+)}$ and $\Gamma_{D(T_-)}$ are obtained from $\Gamma_{D(T)}$ by connecting the left vertex to the right. If $E(\Gamma_{D(T')}) \setminus E(\tau_{D(T')}) = \{e_1, \cdots, e_k\}$, then $E(\Gamma_{D(T'_+)}) \setminus E(\tau_{D(T'_+)}) = \{e_1, \cdots, e_k\} \cup \{d\}$ and $E(\Gamma_{D(T'_-)}) \setminus E(\tau_{D(T'_-)}) = \{e_1, \cdots, e_k\} \cup \{d_*\}$, where d and d_* are the long (newly appeared) edge of $\Gamma_{D(T)}$ given in Figure 10.

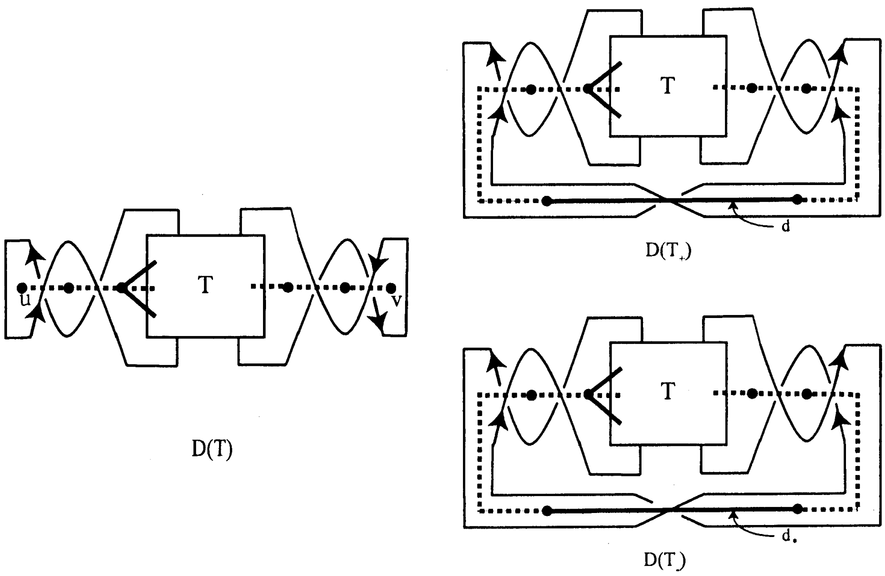


FIGURE 10

The corresponding Seifert matrix $M_{D(T)} = [m_{ij}(D(T))]$ is a $k \times k$ matrix, while the Seifert matrix $M_{D(T_+)} = [m_{ij}(D(T))]$ and $M_{D(T_-)} = [m_{ij}(D(T))]$ are $(k+1) \times (k+1)$ matrices. Furthermore, the linking number between T_{e_i} and $T^+_{e_j}$ in $D(T)$ is equal to the linking number between T_{e_i} and $T^+_{e_j}$ in $D(T_+)$ and $D(T_-)$, by Proposition 2.1. Indeed, $m_{ij}(D(T)) = m_{ij}(D(T_+)) = m_{ij}(D(T_-))$ for all $i, j = 1, 2, \cdots, k$. Furthermore, $lk(T_{e_i}, T^+_d)$ and $lk(T_d, T^+_{e_k}))$ is equal to $lk(T_{e_i}, T^+_{d_*})$ and $lk(T_{d_*}, T^+_{e_k})$ for $i = 1, 2, \cdots, k$, respectively. To calculate self linking numbers of d and d_*, let p and q denote the number of edges between the vertex u and v in

Figure 10 corresponding positive crossings and negative crossings, respectively. Since the two outer arcs of $D(T)$ have the same orientation, $p+q$ is odd. Hence T_d consists of $p+1$ positive crossings and q negative crossings, while T_{d_*} consists of p positive crossings and $q+1$ negative crossings by the definition of $D(T_+)$ and $(D(T_-)$, respectively. Hence by Proposition 2.1, we obtain $lk(T_d, T_d^+) = -\frac{1}{2}(p+1-q) \equiv D$ and $lk(T_{d_*}, T_{d_*}^+) = -\frac{1}{2}(p-q-1) \equiv E$. Hence the Seifert matrix of $D(T)$ is given by

$$M_{D(T_+)} = \begin{pmatrix} M_{D(T)} & B \\ C & D \end{pmatrix} \text{ and } M_{D(T_-)} = \begin{pmatrix} M_{D(T)} & B \\ C & E \end{pmatrix},$$

where $B = (lk(T_{e_1}, T_d^+), \cdots, lk(T_{e_k}, T_d^+))^T, C = (lk(T_d, T_{e_1}^+), \cdots, lk(T_d, T_{e_k}^+))$, $D = lk(T_d, T_d^+)$ and $E = lk(T_{d_*}, T_{d_*}^+)$.

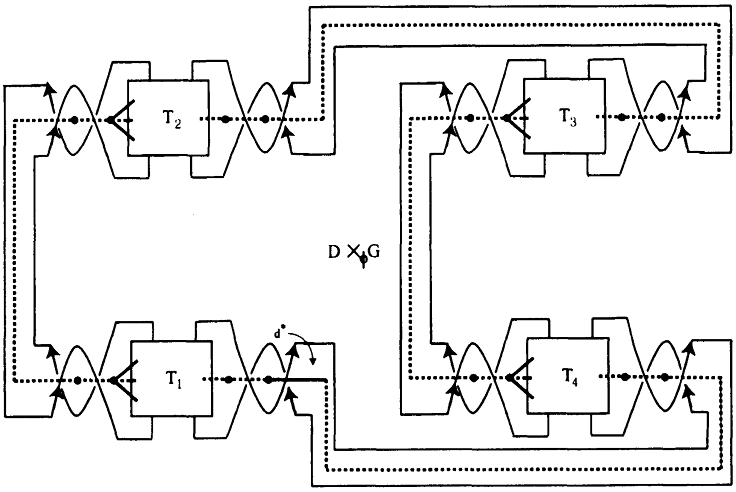


FIGURE 11. Seifert graph of $D(\widetilde{L})$

From now on, we will try to find a Seifert matrix $M_{D(\widetilde{L})}$. Notice that since $\Gamma_{D(T_+)} = \Gamma_{D(T_-)}$, $\tau_{D(T_+)} = \tau_{D(T_-)}$, and then $\tau_{D(T_+)} \cup \{d\} = \tau_{D(T_-)} \cup \{d_*\}$.

The Seifert graph $\Gamma_{D(\widetilde{L})}$ of $D(\widetilde{L})$ consists of 4 copies of $\Gamma_{D(T)}$ whose left edge and right edge are used to connect the copies of $\Gamma_{D(T)}$, as in Figure 11. Since T is connected, the union of 4 copies of $\tau_{D(T_+)} \cup \{d\}$ and $\tau_{D(T_-)} \cup \{d_*\}$ have 3−closed circuits which are the same, as in Figure 11. Hence we get the following spanning tree $\tau_{D(\widetilde{L})}$ of $D(\widetilde{L})$.

$$E(\Gamma_{D(\widetilde{L})}) \setminus E(\tau(D(\widetilde{L}))) = \cup_{p=1}^{4} \{e_1^p, \cdots, e_k^p\} \cup \{d_2, d_3, d_4\},$$

where $\{e_1^p, \cdots, e_k^p\}$ is the corresponding p-th copy of $\{e_1, \cdots, e_k\}$. Since the linking number between $T_{e_i^p}$ and $T_{e_j^p}^+$ is equal to the linking number between T_{e_i} and $T_{e_j}^+$ in $D(T_+)$, we have, for all $i, j = 1, 2, \cdots, k$, and $p = 1, 2, 3, 4$, $m_{i^p j^p}(D(\widetilde{L})) = m_{ij}(D(T))$, where $m_{i^p j^p}(D(\widetilde{L})) = lk(T_{e_i^p}, T_{e_j^q}^+)$. If $p \neq q$, since $T_{e_i^p}$ and $T_{e_j^q}$ do not intersect, by Proposition 2.1, $m_{i^p j^q}(D(\widetilde{L})) = 0$ for all $i, j = 1, 2, \cdots, k$. Hence,

$$m_{i^p j^q}(D(\widetilde{L})) = \begin{cases} m_{ij}(D(T)), & \text{if } p = q; \\ 0, & \text{if } p \neq q. \end{cases}$$

Notice that by the construction of $D(\widetilde{L})$, d and d_* correspond to the same edge in $D(\widetilde{L})$. Let d_p be the corresponding $p-$th copy of d for all $p = 1,2,3,4$ in $D(\widetilde{L})$.

Since $T_{e_i^p} \cap T_{d^*}$ lies in the just p-th copy of $T_{e_i} \cap T_d$, we have $lk(T_{e_i^p}, T_{d^*}^+) = lk(T_{e_i}, T_d^+) = B$ and $lk(T_{d^*}, T_{e_i^p}^+) = lk(T_d, T_{e_i}^+) = C$ for all $p = 1,2,3,4$. Finally, since T is connected, the generator T_{d^*} runs through 2 copies, T_{d_p} consists of $2p-$positive crossings and $2q-$negative crossings. Hence by Theorem ??, $lk(T_{d^*}, T_{d^*}^+) = D + E$.

Hence we get the following lemma.

Lemma 3.2. *Let T be a 4-tangle whose denominator $D(T)$ is oriented. If T is co-oriented, then there exist Seifert matrices $M_{D(T)}, M_{D(T_+)}, M_{D(T_-)}$ and $M_{\widetilde{L}}$ of $D(T), D(T_+), D(T_-)$ and $D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, respectively, such that*

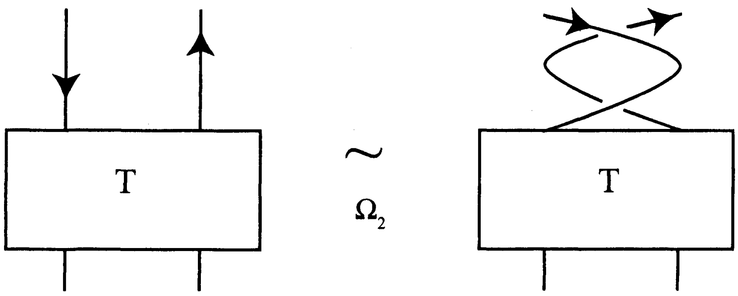
$$
\begin{aligned}
M_{D(T_+)} &= \begin{pmatrix} M_{D(T)} & B \\ C & D \end{pmatrix}, \\
M_{D(T_-)} &= \begin{pmatrix} M_{D(T)} & B \\ C & E \end{pmatrix} \quad \text{and}
\end{aligned}
$$

$$
M_{\widetilde{L}} = \left(\begin{array}{cccc|c}
M_{D(T)} & 0 & 0 & 0 & B \\
0 & M_{D(T)} & 0 & 0 & B \\
0 & 0 & M_{D(T)} & 0 & B \\
0 & 0 & 0 & M_{D(T)} & B \\
\hline
C & C & C & C & 2(D+E)
\end{array}\right),
$$

where B is a row vector, C is a column vector, $E = D + 1$.

CASE II. T is contra-oriented and the two outer arcs of $D(T)$ are contained in the same Seifert circle.

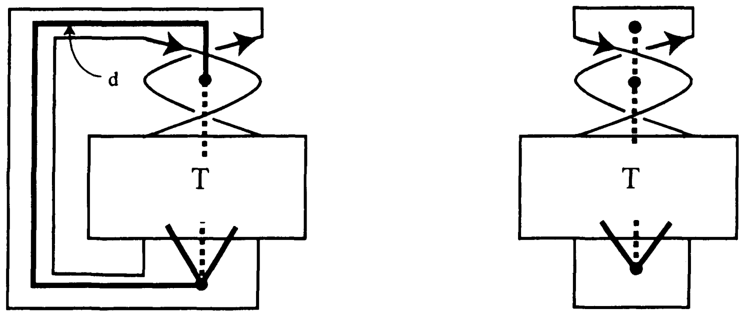
By applying the Reidemeister move II between the top strands to get a new tangle T' so that the Seifert circle at the top is a 2-gon, and hence the corresponding vertex of the Seifert graph has degree 2. See Figure 3. Note that $D(T')$, $N(T')$ and $D(\widetilde{L}')$ are ambient isotopic to $D(T)$, $N(T)$ and $D(\widetilde{L})$, respectively.



The Seifert graphs $\Gamma_{D(T)}$ and $\Gamma_{N(T)}$ of $D(T)$ and $N(T)$ are of the form in Figure 3, in which spanning trees $\tau_{D(T)}$ and $\tau_{N(T)}$ of $\Gamma_{D(T)}$ and $\Gamma_{N(T)}$ are given by dotted edges, respectively. Notice that $\Gamma_{D(T)}$ is obtained from $\Gamma_{N(T)}$ by connecting the top vertex to the bottom vertex.

If $E(\Gamma_{N(T')}) \backslash E(\tau_{N(T')}) = \{e_1, \cdots, e_k\}$, then $E(\Gamma_{D(T')}) \backslash E(\tau_{D(T')}) = \{e_1, \cdots, e_k\} \cup \{d\}$, where d is the long (newly appeared) edge of $\Gamma_{N(T)}$ given in Figure 3.

The corresponding Seifert matrix $M_{N(T)} = [m_{ij}(N(T))]$ is a $k \times k$ matrix, while the Seifert matrix $M_{D(T)} = [m_{ij}(D(T))]$ is a $(k+1) \times (k+1)$ matrix. Furthermore, the linking number



D(T) N(T)

between T_{e_i} and $T_{e_j}^+$ in $N(T)$ is equal to the linking number between T_{e_i} and $T_{e_j}^+$ in $D(T)$, by Theorem 2.1. Indeed, $m_{ij}(D(T)) = m_{ij}(N(T))$ for all $i, j = 1, 2, \cdots, k$. Hence the Seifert matrix of $D(T)$ is given by

$$M_{D(T)} = \begin{pmatrix} M_{N(T)} & B \\ C & D \end{pmatrix},$$

where $B = (lk(T_{e_1}, T_d^+), \cdots, lk(T_{e_k}, T_d^+))^T$, $C = (lk(T_d, T_{e_1}^+), \cdots, lk(T_d, T_{e_k}^+))$ and $D = lk(T_d, T_d^+)$.

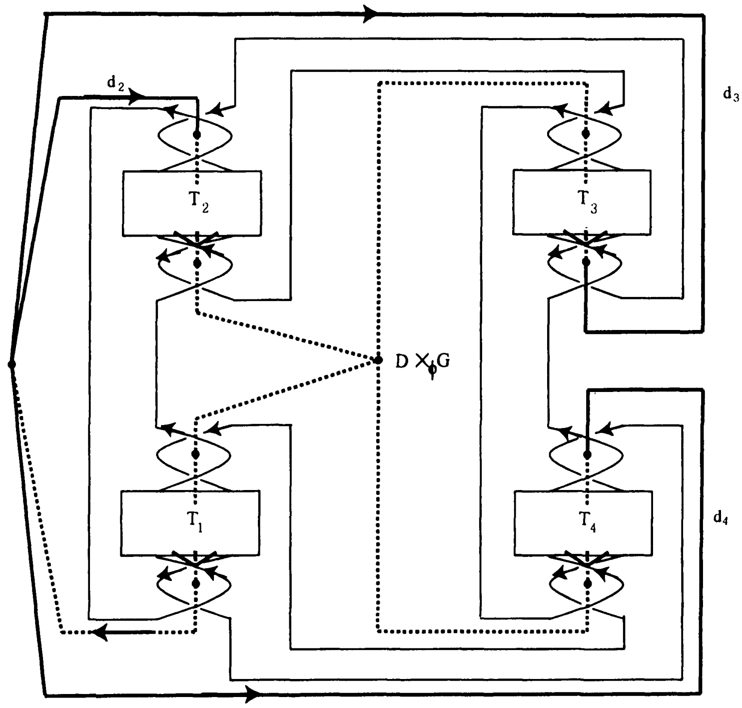


FIGURE 12. Seifert graph of $D(\widetilde{L})$

From now on, we will try to find a Seifert matrix $M_{\widetilde{L}}$. The Seifert graph $\Gamma_{D(\widetilde{L})}$ of $D(\widetilde{L})$ consists of 4 copies of $\Gamma_{N(T)}$whose top edge is used to connect the copies of $\Gamma_{N(T)}$, as in Figure 12.

Since T is connected, the union of n copies of $\tau_{N(T)} \cup \{d\}$ has 4−closed circuits which are the same, as in Figure12. Hence we get a spanning tree $\tau_{D(\widetilde{L})}$ of $D(\widetilde{L})$. Indeed,

$$E(\Gamma_{D(\widetilde{L})}) \setminus E(\tau(D(\widetilde{L}))) = \cup_{p=1}^{4}\{e_1^p, \cdots, e_k^p\} \cup \{d_2, d_3, d_4\},$$

where $\{e_1^p, \cdots, e_k^p\}$ is the corresponding p-th copy of $\{e_1, \cdots, e_k\}$.

Since the linking number between $T_{e_i^p}$ and $T_{e_j^p}^+$ is equal to the linking number between T_{e_i} and $T_{e_j}^+$ in $N(T)$, we have, for all $i, j = 1, 2, \cdots, k$, and $p = 1 \cdots, 4$, $m_{i^p j^p}(D(\widetilde{L})) = m_{ij}(D(T))$, where $m_{i^p j^p}(D(\widetilde{L})) = lk(T_{e_i^p}, T_{e_j^q}^+)$. If $p \neq q$, since $T_{e_i^p}$ and $T_{e_j^q}$ do not intersect, by Theorem 2.1, $m_{i^p j^q}(D(\widetilde{L})) = 0$ for all $i, j = 1, 2, \cdots, k$. Hence,

$$m_{i^p j^q}(D(\widetilde{L})) = \begin{cases} m_{ij}(N(T)), & \text{if } p = q; \\ 0, & \text{if } p \neq q. \end{cases}$$

On the other hand, since d_p lies in the 1st copy and pth copy of $T_{e_i} \cap T_{d_p}$, we have $lk(T_{e_i^1}, T_{d_2}^+) = lk(T_{e_i}, T_d^+) = B$, $lk(T_{d_2}, T_{e_i^1}^+) = lk(T_d, T_{e_i}^+) = \pm C$, $lk(T_{e_i^1}, T_{d_4}^+) = lk(T_{e_i}, T_d^+) = B$, $lk(T_{d_4}, T_{e_i^1}^+) = lk(T_d, T_{e_i}^+) = \pm C$, and $lk(T_{e_i^1}, T_{d_3}^+) = -lk(T_{e_i}, T_d^+) = -B$ and $lk(T_{d_3}, T_{e_i^1}^+) = -lk(T_d, T_{e_i}^+) = -C$. Finally, since T is connected, the generator T_{d_p} runs through 2 copies, in each of which self linking number is equal to $D = lk(T_d, T_d^+)$ for all $p = 2, 3, 4$. Hence, $lk(T_{d_p}, T_{d_p}^+) = 2 \cdot lk(T_{d_p}, T_{d_p}^+) = 2D$ for all $p = 2, 3, \cdots, n$. Furthermore, since generators T_{d_p} and T_{d_q} meet in the just 1st copy and its linking number is equal to $D = lk(T_d, T_d^+)$.

Hence we get the following lemma.

Lemma 3.3. *Let T be a 4-tangle whose denominator $D(T)$ is oriented. If T is contra-oriented and the two outer arcs of $D(T)$ is contained in the same Seifert circle, then there exist Seifert matrices $M_{D(T)}, M_{N(T)}$ and $M_{\widetilde{L}}$ of $D(T)$, $N(T)$ and $D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, respectively, such that*

$$M_{D(T)} = \begin{pmatrix} M_{N(T)} & B \\ C & D \end{pmatrix} \quad \textit{and}$$

$$M_{\widetilde{L}} = \left(\begin{array}{cccc|ccc} M_{N(T)} & 0 & 0 & 0 & B & B & B \\ 0 & M_{N(T)} & 0 & 0 & B & 0 & 0 \\ 0 & 0 & M_{N(T)} & 0 & 0 & B & 0 \\ 0 & 0 & 0 & M_{N(T)} & 0 & 0 & -B \\ \hline C & C & 0 & 0 & 2D & D & D \\ C & 0 & C & 0 & D & 2D & D \\ C & 0 & 0 & -C & D & D & 2D \end{array}\right)$$

where B is a row vector, C is a column vector.

CASE III. T is contra-oriented and the two outer arcs of $D(T)$ are contained in different Seifert circles.

As the CASE I, by applying the Reidemeister move II between the the left two outer strands and the right outer strands of T to get a new tangle T' so that Seifert circles at the left and the right are 2-gons, and hence the corresponding vertices of the Seifert graph have degree 2,

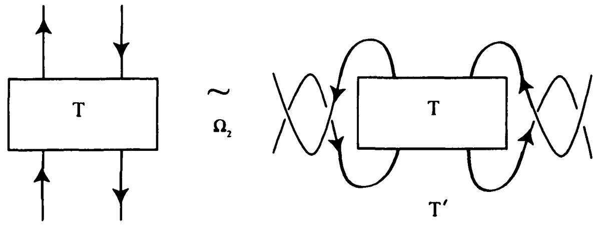
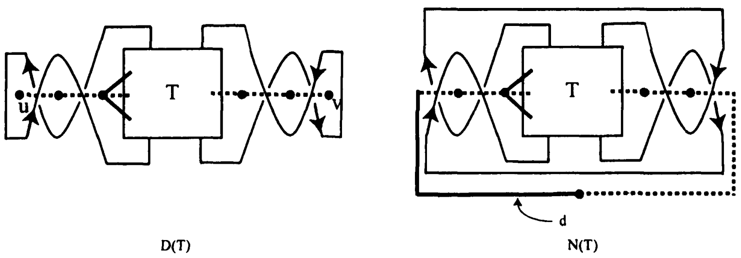


FIGURE 13

respectively, see Figure 13. Note that $D(T')$, $N(T')$ and $D(\widetilde{L}')$ are ambient isotopic to $D(T)$, $N(T)$ and $D(\widetilde{L})$, respectively.

The Seifert graphs $\Gamma_{D(T)}$ and $\Gamma_{N(T)}$ of $D(T)$ and $N(T)$ are of the form in Figure 3, in which spanning trees $\tau_{D(T)}$ and $\tau_{N(T)}$ of $\Gamma_{D(T)}$ and $\Gamma_{N(T)}$ are given by dotted edges, respectively. Notice that $\Gamma_{N(T)}$ is obtained from $\Gamma_{D(T)}$ by connecting the left vertex to the right vertex.



If $E(\Gamma_{D(T')})\setminus E(\tau_{D(T')}) = \{e_1,\cdots,e_k\}$, then $E(\Gamma_{N(T')})\setminus E(\tau_{N(T')})= \{e_1,\cdots,e_k\}\cup\{d\}$, where d is the long (newly appeared) edge of $\Gamma_{D(T)}$ given in Figure 3.

The corresponding Seifert matrix $M_{D(T)} = [m_{ij}(D(T))]$ is a $k \times k$ matrix, while the Seifert matrix $M_{N(T)} = [m_{ij}(D(T))]$ is a $(k+1)\times(k+1)$ matrix. Furthermore, the linking number between T_{e_i} and $T^+_{e_j}$ in $D(T)$ is the same with the linking number between T_{e_i} and $T^+_{e_j}$ in $N(T)$, by Proposition 2.1. Indeed, $m_{ij}(D(T)) = m_{ij}(N(T))$ for all $i,j = 1,2,\cdots,k$. Hence the Seifert matrix of $D(T)$ is given by

$$M_{N(T)} = \begin{pmatrix} M_{D(T)} & B \\ C & D \end{pmatrix},$$

where $B = (lk(T_{e_1},T^+_d),\cdots,lk(T_{e_k},T^+_d))^T, C = (lk(T_d,T^+_{e_1}),\cdots,lk(T_d,T^+_{e_k}))$ and $D = lk(T_d,T^+_d)$.

From now on, we will try to find a Seifert matrix $M_{\widetilde{L}}$. The Seifert graph $\Gamma_{D(\widetilde{L})}$ of $D(\widetilde{L})$ consists of 4 copies of $\Gamma_{D(T)}$ whose left edge and right edge are used to connect the copies of $\Gamma_{D(T)}$, as in Figure 14.

Since T is connected, the union of 4 copies of $\tau_{D(T)}\cup\{d\}$ has a closed circuit which is the longest circle in Figure 14. By removing one of the 3 copies of the edge d, e.g. d^* in Figure 14, we get a spanning tree $\tau_{D(\widetilde{L})}$ of $D(\widetilde{L})$. Indeed,

$$E(\Gamma_{\widetilde{L}})\setminus E(\tau(D(\widetilde{L}))) = \cup_{p=1}^{4}\{e_1^p,\cdots,e_k^p\}\cup\{d^*\},$$

where $\{e_1^p,\cdots,e_k^p\}$ is the corresponding p-th copy of $\{e_1,\cdots,e_k\}$.

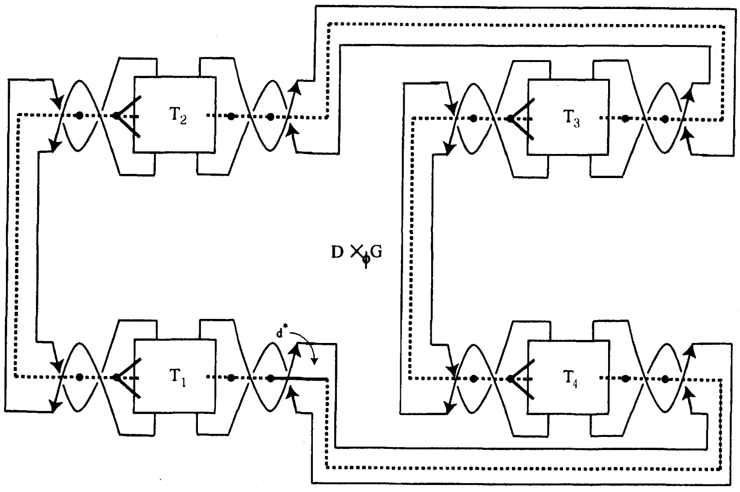


FIGURE 14. Seifert graph of $D(\widetilde{L})$

Since the linking number between $T_{e_i^p}$ and $T^+_{e_j^p}$ is equal to the linking number between T_{e_i} and $T^+_{e_j}$ in $D(T)$, we have, for all $i,j = 1,2,\cdots,k$, and $p = 1,2,3,4$, $m_{i^pj^p}(D(\widetilde{L})) = m_{ij}(D(T))$, where $m_{i^pj^p}(D(\widetilde{L})) = lk(T_{e_i^p}, T^+_{e_j^q})$. If $p \neq q$, since $T_{e_i^p}$ and $T_{e_j^q}$ do not intersect, by Theorem 2.1, $m_{i^pj^q}(D(\widetilde{L})) = 0$ for all $i,j = 1,2,\cdots,k$. Hence,

$$m_{i^pj^q}(D(\widetilde{L})) = \begin{cases} m_{ij}(D(T)), & \text{if } p = q; \\ 0, & \text{if } p \neq q. \end{cases}$$

On the other hand, since $T_{e_i^p} \cap T_{d^*}$ lies in the just p-th copy of $T_{e_i} \cap T_d$, we have $lk(T_{e_i^1}, T^+_{d^*}) = lk(T_{e_i}, T^+_d) = B$, $lk(T_{d^*}, T^+_{e_i^1}) = lk(T_d, T^+_{e_i}) = C$, $lk(T_{e_i^2}, T^+_{d^*}) = -lk(T_{e_i}, T^+_d) = -B$, $lk(T_{d^*}, T^+_{e_i^2}) = -lk(T_d, T^+_{e_i}) = -C$, $lk(T_{e_i^3}, T^+_{d^*}) = -lk(T_{e_i}, T^+_d) = -B$, $lk(T_{d^*}, T^+_{e_i^3}) = -lk(T_d, T^+_{e_i}) = -C$, and $lk(T_{e_i^4}, T^+_{d^*}) = lk(T_{e_i}, T^+_d) = B$, $lk(T_{d^*}, T^+_{e_i^4}) = lk(T_d, T^+_{e_i}) = C$. We assume that the orientation of d^* is depicted as in Figure 14. Then we know the signs of $lk(T_{e_i}, T^+_d)$ and $lk(T_d, T^+_{e_i})$, respectively. Finally, since T is connected, the generator T_{d^*} runs through all copies, in each of which its self linking number is equal to $D = lk(T_d, T^+_d)$. Hence, $lk(T_{d^*}, T^+_{d^*}) = 4 \cdot lk(T_d, T^+_d) = 4D$.

Hence we get the following lemma.

Lemma 3.4. *Let T be a 4-tangle whose denominator $D(T)$ is oriented. If T is contra-oriented and the two outer arcs of $D(T)$ are contained in different Seifert circles, then there exist Seifert matrices $M_{D(T)}, M_{N(T)}$ and $M_{\widetilde{L}}$ of $D(T)$, $N(T)$ and $D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, respectively, such that*

$$M_{N(T)} = \begin{pmatrix} M_{D(T)} & B \\ C & D \end{pmatrix} \quad \textit{and}$$

$$M_{D(\widetilde{L})} = \left(\begin{array}{cccc|c} M_{D(T)} & 0 & 0 & 0 & B \\ 0 & M_{D(T)} & 0 & 0 & -B \\ 0 & 0 & M_{D(T)} & 0 & -B \\ 0 & 0 & 0 & M_{D(T)} & B \\ \hline C & -C & -C & C & 4D \end{array}\right)$$

where B is a row vector, C is a column vector.

Theorem 3.5. *Let $D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the symmetric link constructed by the base link $D(T)$ and Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action. Suppose that $D(T)$ is oriented 4−tangle with $\phi(e) \neq\neq 1_G$ and $\phi(f) \neq 1_G$. Suppose that $\widetilde{L} = D(T) \times_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and $L = D(T)$ for some 4-tangle T.*

(1) *If T is co-oriented, then*

$$\Delta_{\widetilde{L}}(t) = 2(\Delta_L)^3(t)\{\Delta_{D(T_+)}(t) + \Delta_{D(T_-)}(t)\}.$$

(2) *If T is contra-oriented, then*

$$\Delta_{\widetilde{L}}(t) = 4(\Delta_L)^3(t)\Delta_{N(T)}(t).$$

In (1), T_+ and T_- are the 4-tangles defined in Fiure 4.

Proof. If T is co-oriented (CASE I), we have, by Lemma 3.2 and Lemma 3.1,

$$\begin{aligned}
\Delta_{D(T_+)} &= \det\begin{pmatrix} xM_{D(T)} - x^{-1}M_{D(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})D \end{pmatrix}, \\
\Delta_{D(T_-)} &= \det\begin{pmatrix} xM_{D(T)} - x^{-1}M_{D(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})E \end{pmatrix}, \text{and} \\
\Delta_{\widetilde{L}} &= \det\begin{pmatrix} xM_{D(T)} - x^{-1}M_{D(T)}{}^T & \cdots & 0 & xB - x^{-1}C^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & xM_{D(T)} - x^{-1}M_{D(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & \cdots & xC - x^{-1}B^T & 2x(D+E) - 2x^{-1}(D+E) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= 2\det(xM_{D(T)} - x^{-1}M_{D(T)}{}^T)^3 \times \{\det\begin{pmatrix} xM_{D(T)} - x^{-1}M_{D(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})D \end{pmatrix} \\
&\quad + \det\begin{pmatrix} xM_{D(T)} - x^{-1}M_{D(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})E \end{pmatrix}\} \\
&= 2(\Delta_{D(T)})^3\{\Delta_{D(T_+)} + \Delta_{D(T_-)}\}.
\end{aligned}$$

If T is contra-oriented and the two outer arcs of $D(T)$ are contained in the same Seifert circle (CASE II), by Lemma 3.3 and Lemma **??**,by lemma 3.3 in [3],
$\Delta_{D(T)} = \det\begin{pmatrix} xM_{N(T)} - x^{-1}M_{N(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})D \end{pmatrix}$, and

$$\begin{aligned}
&\Delta_{\widetilde{L}} \\
&= \det\begin{pmatrix} xM_{N(T)} - x^{-1}M_{N(T)}{}^T & \cdots & 0 & xB - x^{-1}C^T & \cdots & xB - x^{-1}C^T \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & xM_{N(T)} - x^{-1}M_{N(T)}{}^T & 0 & \cdots & -(xB - x^{-1}C^T) \\ xC - x^{-1}B^T & \cdots & 0 & 2(x - x^{-1})D & \cdots & (x - x^{-1})D \\ C - x^{-1}B^T & \cdots & 0 & (x - x^{-1})D & \cdots & (x - x^{-1})D \\ xC - x^{-1}B^T & \cdots & -(xC + -^{-1}B^T) & (x - x^{-1})D & \cdots & 2(x - x^{-1})D \end{pmatrix} \\
&= 4\det(xM_{N(T)} - x^{-1}M_{N(T)}{}^T)\det\begin{pmatrix} xM_{N(T)} - x^{-1}M_{N(T)}{}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x - x^{-1})D \end{pmatrix}^3 \\
&= 4\Delta_{N(T)}(\Delta_{D(T)})^3.
\end{aligned}$$

If T is contra-oriented and the two outer arcs of $D(T)$ are contained in different Seifert circles (CASE III), we have, by Lemma 3.4 and Lemma 3.5 [2],

$$\begin{aligned}
\Delta_{N(T)} &= \det\begin{pmatrix} xM_{D(T)} - x^{-1}{M_{D(T)}}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x-x^{-1})D \end{pmatrix}, \text{ and} \\
\Delta_{\widetilde{L}} &= \det\begin{pmatrix} xM_{D(T)} - x^{-1}{M_{D(T)}}^T & \cdots & 0 & xB - x^{-1}C^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & xM_{D(T)} - x^{-1}{M_{D(T)}}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & \cdots & xC - x^{-1}B^T & 4(x-x^{-1})D \end{pmatrix} \\
&= 4\det(xM_{D(T)} - x^{-1}{M_{D(T)}}^T)^3 \det\begin{pmatrix} xM_{D(T)} - x^{-1}{M_{D(T)}}^T & xB - x^{-1}C^T \\ xC - x^{-1}B^T & (x-x^{-1})D \end{pmatrix} \\
&= 4(\Delta_{D(T)})^3\Delta_{N(T)}.
\end{aligned}$$

$\square$

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