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Braids and branched coverings of dimension three

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1 Introduction

This is on a part of our work in progress, which was introduced at the conference "Intelligence of Low-dimensional Topology" held in RIMS in May, 2012. The purpose of our research is to understand branched coverings and m-dimensional braids which are generalizations of classical braids. Here we discuss chart descriptions of branched coverings and braids in dimension $m=2$ first, and then those for which $m=3$.

We work in the PL category ([9, 20]). Let S^{m} denote the m-sphere, and let M^{m} denote a closed oriented m-manifold.

2 Preliminaries

We start by giving some definitions and theorems on branched coverings.

Definition 2.1 A PL map $f : M^{m}\rightarrow S^{m}$ is a *branched covering* (*map*) if there exists an $(m-2)$-subcomplex L of S^{m} such that the restriction $\underline{f} : M^{m}\setminus f^{-1}(L)\rightarrow S^{m}\setminus L$ is a covering map.

We denote the covering degree by d. We call f a *d-fold* branched covering.

We assume that L is minimum, i.e., $\forall y\in L$, $\#(f^{-1}(y))<d$. Then we call L the *branch set* of f.

Definition 2.2 A d-fold branched covering f is *simple* if $\forall y\in L$, $\#(f^{-1}(y))=d-1$.

Remark 2.3 (1) A branched covering is defined in general as follows (cf. [2, 3]): A PL map between manifolds is called *proper* if the inverse image of the boundary is the boundary. A proper PL map between manifolds f : $M^{m}\rightarrow N^{m}$ is called a branched covering if it is finite-to-one and open.

(2) A branched covering $f : M\rightarrow N$ is *primitive* if $f_{*} : \pi_{1}(M)\rightarrow\pi_{1}(N)$ is surjective. It is often assumed that a branched covering is primitive.

Note that M^{m} is closed, oriented and <u>connected</u> in what follows in this section.

Theorem 2.4 (J.W. Alexander [1]) *For any closed oriented and connected m-manifold M^m, there exists a simple branched covering $f : M^m \to S^m$ for some degree d.*

Remark 2.5 (1) A closed oriented and connected 1-manifold M^1 is homeomorphic to S^1. Thus there exists a 1-fold covering $f : M^1 \to S^1$.

(2) For any closed oriented and connected 2-manifold M^2, there exists a 2-fold simple branched covering $f : M^2 \to S^2$.

Theorem 2.6 (H. M. Hilden [8], J. M. Montesinos [17]) *For any closed oriented and connected 3-manifold M^3, there exists a 3-fold simple branched covering $f : M^3 \to S^3$ such that the branch set L is a link (or a knot).*

The following is a conjecture due to Montesinos.

Conjecture 2.7 *For any closed oriented and connected 4-manifold M^4, there exists a 4-fold simple branched covering $f : M^4 \to S^4$ such that L is an embedded surface in S^4.*

Some partial answers to this conjecture are known as follows.

Theorem 2.8 (R. Piergallini [19]) *For any closed oriented and connected 4-manifold M^4, there exists a 4-fold simple branched covering $f : M^4 \to S^4$ such that L is an immersed surface in S^4.*

Theorem 2.9 (M. Iori and R. Piergallini [11]) *For any closed oriented and connected 4-manifold M^4, there exists a 5-fold simple branched covering $f : M^4 \to S^4$ such that L is an embedded surface in S^4.*

3 Two dimensional case ($m = 2$)

Let $f : M^2 \to S^2$ be a d-fold simple branched covering with branch set L, and let $\underline{f} : M^2 \setminus f^{-1}(L) \to S^2 \setminus L$ be the associated covering map.

Take a base point $*$ of $S^2 \setminus L$ to consider the fundamental group $\pi_1(S^2 \setminus L, *)$. The preimage $f^{-1}(*)$ of the base point $*$ consists of d points of M^2. Then we have a *monodromy* $\rho : \pi_1(S^2 \setminus L, *) \to S_d$, where the symmetric group S_d on letters $\{1, 2, \dots, d\}$ is identified with the symmetric group on $f^{-1}(*)$. (A monodromy ρ depends on the identification between $\{1, 2, \dots, d\}$ and $f^{-1}(*)$.) The covering $\underline{f}$ is determined by the monodromy.

By the Riemann-Hurwitz formula, L consists of an even number of points.

In Figure 1, a branch set, a monodromy, and a chart are depicted. (A chart description is explained later.)

When a monodromy is described by a chart, it is easy to construct M^2. We explain it by using an example. Let Γ be the chart depicted on the right of Figure 1. Consider three copies of S^2 labeled by 1, 2, and 3, say S_1^2, S_2^2 and S_3^2, respectively. On the copy S_1^2, draw the edges with label (12) of Γ, on the copy S_2^2, draw the edges with label (12) of Γ and those with label (23), and on the copy S_3^2, draw the edges with label (23). Cut the three 2-spheres along these edges, and we obtain three compact surfaces, say M_1, M_2 and M_3, as in the bottom of Figure 2. The surface M^2 is obtained from the union $M_1 \cup M_2 \cup M_3$

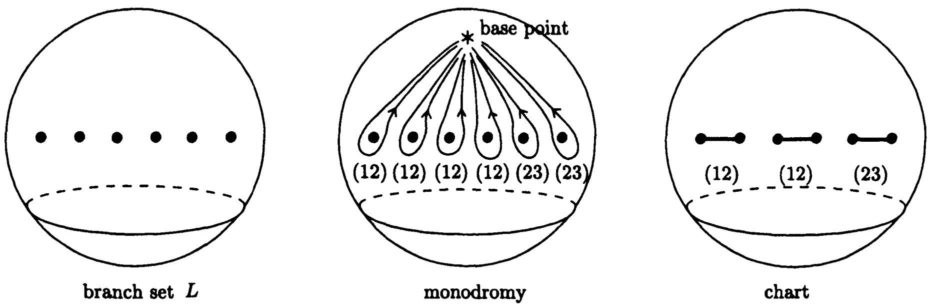

Figure 1: A branch set, a monodromy and a chart

by identifying the boundary as follows: Let e be an edge with label (12) on S_1^2, and let e_+ and e_- be the copies of e in ∂M_1. Let e' be the corresponding edge on S_2^2, and let e'_+ and e'_- be the corresponding copies in ∂M_2. Then we identify e_+ with e'_-, and identify e_- with e'_+, respectively. All boundary edges of $M_1 \cup M_2 \cup M_3$ are identified in this fashion, and we have a closed surface. This is the desired M^2.

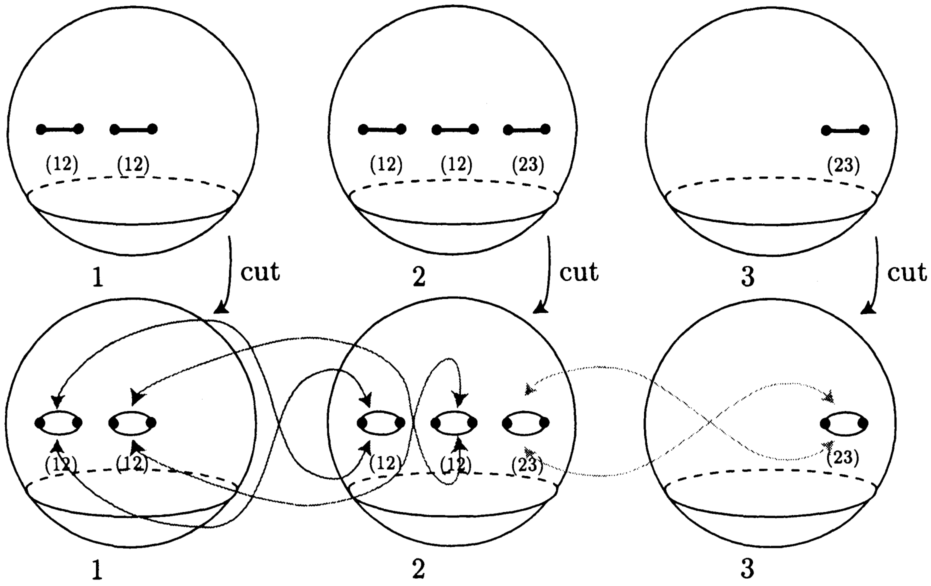

Figure 2: How to construct M^2

The classification of simple branched coverings was studied by J. Lüroth [15], A. Clebsch [6], A. Hurwitz [10], and others. The classification theorem is stated as follows.

Theorem 3.1 *Let $f : M^2 \to S^2$ and $f' : M^{2'} \to S^2$ be d-fold simple branched coverings with branch sets L and L', respectively. We assume that M^2 and $M^{2'}$ are connected. Then f and f' are equivalent if and only if $\#L = \#L'$.*

Hurwitz [10] studied branched coverings by using of a system of monodromies of meridian elements of the branch set, called a *Hurwitz system*, and studied when two systems present the same (up to equivalence) branched coverings.

A Hurwitz system depends on a system of generating set of $\pi_1(S^2 \setminus L, *)$. For a generating system depicted in the middle of Figure 1, the Hurwitz system is

$$\alpha = ((12), (12), (12), (12), (23), (23)).$$

Besides a choice of a generating system, a Hurwitz system depends on the identification of $\{1, 2, \ldots, d\}$ and the fiber $f^{-1}(*)$.

Two Hurwitz systems present the same (up to equivalence) braid monodromy if and only if they are related by a finite sequence of *Hurwitz moves* and *conjugations*. The *Hurwitz moves* are

$$(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \mapsto (a_1, \ldots, a_{k+1}, a_{k+1}^{-1} a_k a_{k+1}, \ldots, a_n)$$

for $k = 1, \ldots, n-1$ and their inverse moves. *Conjugations* are

$$(a_1, \ldots, a_n) \mapsto (g^{-1} a_1 g, \ldots, g^{-1} a_n g)$$

for $g \in S_d$. When two Hurwitz systems are related by a finite sequence of Hurwitz moves and conjugations, we say that they are *HC-equivalent.* (H and C stand for Hurwitz and conjugation.)

Due to Hurwitz [10], the classification theorem is stated as follows.

Theorem 3.2 *Let $f : M^2 \to S^2$ be a d-fold simple branched covering. Assume that M^2 is connected. Any Hurwitz system of f is HC-equivalent to*

$$((12), \ldots, (12), (23), (23), (34), (34), \ldots, (d-1, d), (d-1, d)).$$

(The number of (12)s is a positive even number, and for each $i = 2, \ldots, d-1$, a pair of $(i, i+1)$ appears.)

In the next section, we will introduce the notion of a *chart*, called a *permutation chart* or an *S_d-chart*, that describes a branched covering or its monodromy. The chart method helps us to construct M^2 from a monodromy, and to understand the classification theorem well.

4 Permutation charts or S_d-charts $(m = 2)$

We denote by τ_i the transposition $(i \ i+1)$. The symmetric group S_d is generated by $\tau_1, \ldots, \tau_{d-1}$, and has a group presentation

$$S_d = \left\langle \tau_1, \ldots, \tau_{d-1} \;\middle|\; \begin{array}{ll} \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j & (|i-j| = 1) \\ \tau_i \tau_j = \tau_j \tau_i & (|i-j| > 1) \\ \tau_i^2 = e & \end{array} \right\rangle.$$

Definition 4.1 A *permutation chart* of degree d or an *S_d-chart* is a labeled graph in S^2 such that each edge is labeled in $\{1, \ldots, d-1\}$ and each vertex is as in Figure 3. We call a vertex a *black vertex*, a *crossing* or a *white vertex* if the valency of the vertex is 1, 4 or 6, respectively.

By the correspondence $i \leftrightarrow \tau_i = (i\ i+1) \in S_d$, the labels of a chart are assumed to be transpositions in S_d (see Figure 1). Figure 4 is an example of an S_4-chart, or a permutation chart of degree 4.

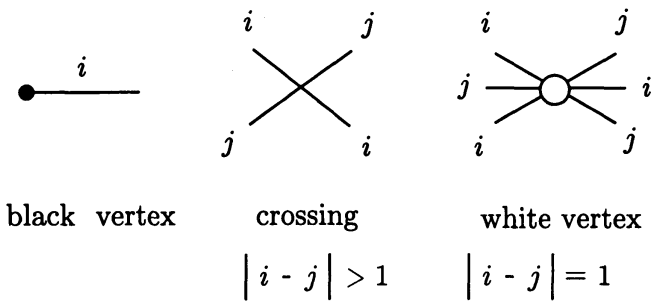

Figure 3: Vertices of a S_d-chart

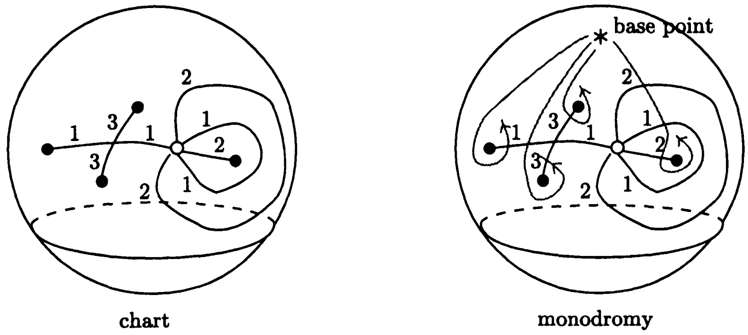

Figure 4: A S_4-chart Γ and the induced monodromy ρ_Γ

For a chart Γ, we consider a monodromy

$$\rho_\Gamma : \pi_1(S^2 \setminus L) \to S_d, \quad [\ell] \mapsto [\text{intersection word of } \ell \text{ w.r.t. } \Gamma],$$

where L ($= L_\Gamma$) is the set of black vertices. An intersection word is a sequence of elements of $\{1, \ldots, d-1\}$, which is regarded as an element of S_d by the correspondence $i \leftrightarrow \tau_i = (i\ i+1) \in S_d$.

Example 4.2 Let Γ be an S_4-chart depicted in the left of Figure 4. When we take a Hurwitz generating system as in the figure, we have a Hurwitz system $(\tau_1,\ \tau_1\tau_3\tau_1,\ \tau_3,\ \tau_2\tau_1\tau_2\tau_1\tau_2)$. It is equal to $(\tau_1,\ \tau_3,\ \tau_3,\ \tau_1)$. And it is Hurwitz equivalent to $(\tau_1,\ \tau_1,\ \tau_3,\ \tau_3)$.

Theorem 4.3 *Let $f : M^2 \to S^2$ be a d-fold simple branched covering, and ρ_f a monodromy of f. There exists a chart Γ such that $\rho_\Gamma = \rho_f$.* (We call Γ a *chart description* of f or ρ_f.)

Local moves on permutation charts illustrated in Figure 5 are called *chart moves.* (Ignore the orientations on edges.) Two charts are said to be *equivalent* or *chart move*

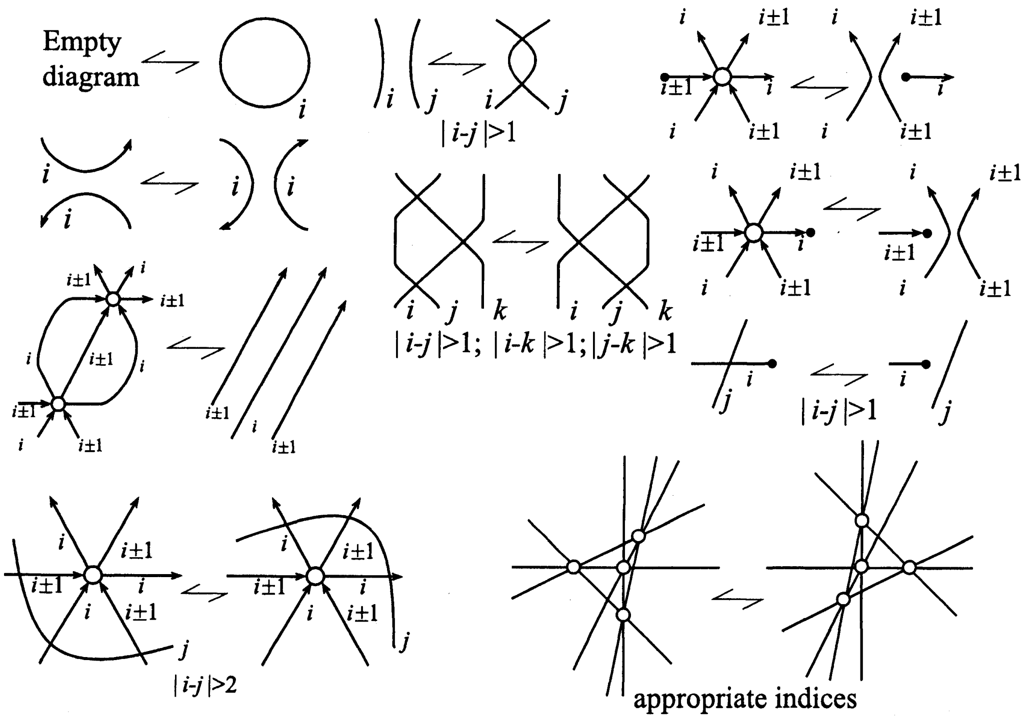

Figure 5: Chart moves

equivalent if they are related by a finite sequence of chart moves and ambient isotopies of S^2.

Theorem 4.4 *Let f and f' be d-fold simple branched covering of S^2, and let Γ and Γ' be their chart descriptions. f is equivalent to f' if and only if Γ is equivalent to Γ'.*

Using an example, we explain how to construct M^2 from a chart description. Let Γ be an S_4-chart depicted in the top of Figure 6. Consider four copies of S^2 labeled by 1, 2, 3 and 4, say S_1^2, S_2^2, S_3^2 and S_4^2, respectively. On the copy S_1^2, draw the edges with label 1 of Γ, on the copy S_2^2, draw the edges with label 1 of Γ and those with label 2, on the copy S_3^2, draw the edges with label 2 of Γ and those with label 3, and on the copy S_4^2, draw the edges with label 3. Cut the four 2-spheres along the edges, and we obtain compact surfaces, say M_1, M_2, M_3 and M_4, as in the bottom of Figure 6. The surface M^2 is obtained from the union $\cup_{i=1}^4 M_i$ by identifying the boundary as follows: Let e be an edge with label 1 on S_1^2, and let e_+ and e_- be the copies of e in ∂M_1. Let e' be the corresponding edge on S_2^2, and let e'_+ and e'_- be the corresponding copies in ∂M_2. Then we identify e_+ with e'_-, and identify e_- with e'_+, respectively. All boundary edges of $\cup_{i=1}^4 M_i$ are identified in this fashion, and we have a closed surface. This is the desired M^2.

At a white vertex, 3 sheets are gathering as in Figure 7.

Theorem 4.5 *Any chart description of $f : M^2 \to S^2$ with connected M is equivalent to a chart as in Figure 8.*

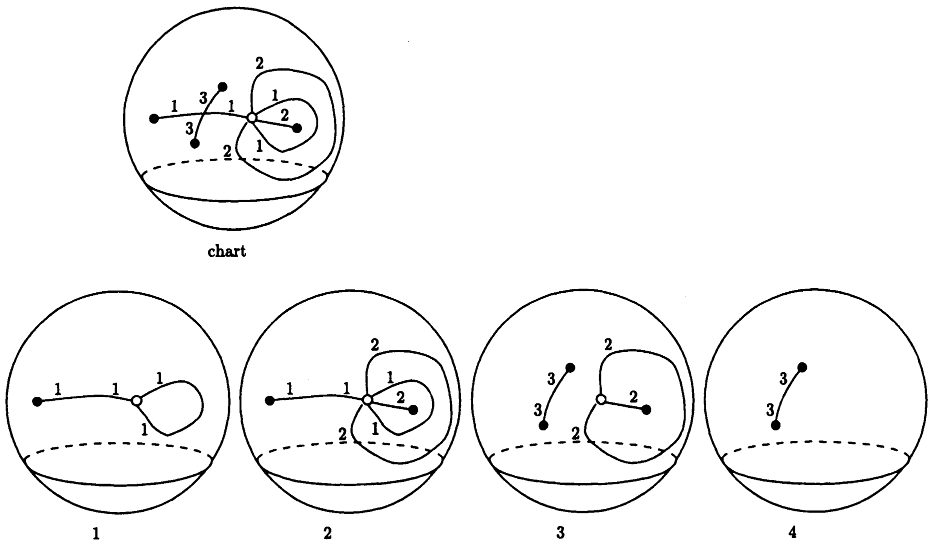

Figure 6: How to construct M^2

This theorem is quite easily proved. As a corollary of this theorem, we have the classification theorem (Theorem 3.1).

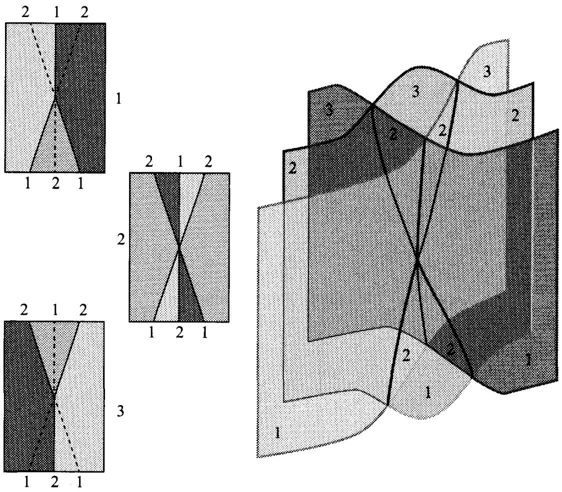

Figure 7: Three sheets gather around a white vertex.

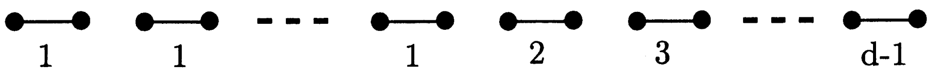

Figure 8: A chart in a normal form

5 Braid charts or B_d-charts $(m = 2)$

Let σ_i $(i = 1, \ldots, d-1)$ be the standard generators of the braid group B_d. Then B_d has a group presentation

$$B_d = \left\langle \sigma_1, \ldots, \sigma_{d-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & (|i-j| = 1) \\ \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| > 1) \end{array} \right\rangle.$$

Definition 5.1 A *braid chart* of degree d or a B_d-*chart* is a labeled and oriented graph in S^2 such that each edge is labeled in $\{1, \ldots, d-1\}$ and each vertex is as in Figure 9. We call a vertex a *black vertex*, a *crossing* or a *white vertex* if the valency of the vertex is 1, 4 or 6, respectively. The arrow at a black vertex in this figure is suppressed since it may either be incoming or outgoing.

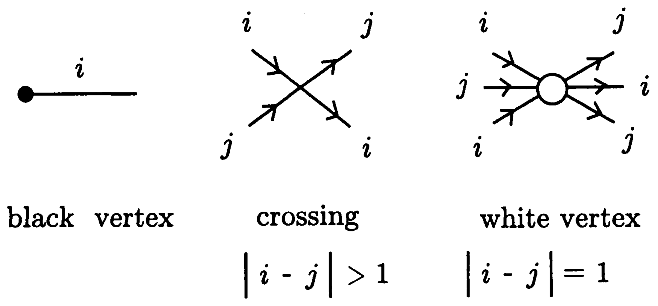

Figure 9: Vertices of a B_d-chart

By the correspondence $i \leftrightarrow \sigma_i = (i\ i+1) \in B_d$, the labels of a chart are assumed to present the standard generators in B_d. Figure 10 is an example of a B_4-chart, or a braid chart of degree 4.

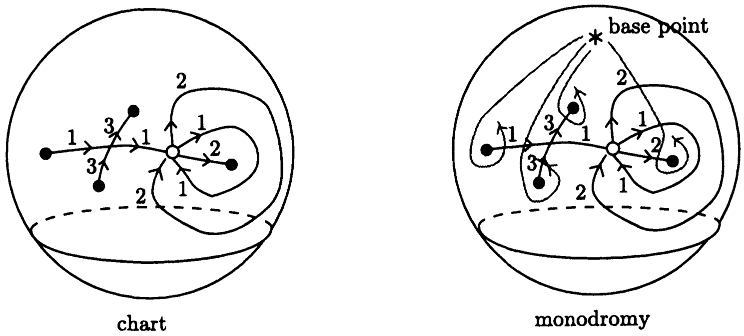

Figure 10: A B_4-chart Γ and the induced monodromy ρ_Γ

Forgetting orientations of the edges from a braid chart, we obtain a permutation chart. Thus we often call a permutation chart an *unoriented chart*, and a braid chart an *oriented chart*.

Definition 5.2 A permutation chart is called *orientable* if one can give orientations to the edges to make it a braid chart. Otherwise it is called *nonorientable.*

For a braid chart Γ of degree d, we consider a monodromy

$$\rho_\Gamma : \pi_1(S^2 \setminus L) \to B_d, \quad [\ell] \mapsto [\text{intersection word of } \ell \text{ w.r.t. } \Gamma],$$

where $L\ (= L_\Gamma)$ is the set of black vertices. An intersection word is a word of $\{1, \ldots, d-1\}$, which is regarded as an element of B_d by the correspondence $i \leftrightarrow \sigma_i = (i\ i+1) \in S_d$.

Example 5.3 Let Γ be a B_4-chart depicted in the left of Figure 10. When we take a Hurwitz generating system as in the right of the figure, we have a Hurwitz system

$$(\sigma_1,\ \sigma_1^{-1}\sigma_3\sigma_1,\ \sigma_3^{-1},\ \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2).$$

It is equal to $(\sigma_1,\ \sigma_3,\ \sigma_3^{-1},\ \sigma_1^{-1})$. And it is Hurwitz equivalent to $(\sigma_1,\ \sigma_1^{-1},\ \sigma_3,\ \sigma_3^{-1})$.

Let $D^2 \times S^2$ be a tubular neighborhood of a standardly embedded 2-sphere in R^4.

Definition 5.4 A PL embedding $g : M^2 \to D^2 \times S^2 \subset R^4$ is a (*simple*) *embedded* 2-*dimensional braid*, or a *surface braid*, of degree d if the composition $M^2 \to D^2 \times S^2 \to S^2$ is a d-fold (simple) branched covering.

For a (simple or nonsimple) embedded 2-dimensional braid $g : M^2 \to D^2 \times S^2 \subset R^4$ of degree m, we can consider a *monodromy* $\rho\,(= \rho_g) : \pi_1(S^2 \setminus L, *) \to B_d$, where $L\,(= L_g)$ is the branch set of the branched covering $M^2 \to D^2 \times S^2 \to S^2$.

Theorem 5.5 *For any simple embedded* 2-*dimensnional braid* $g : M^2 \to D^2 \times S^2 \subset R^4$, *there exists a braid chart* Γ *such that* $\rho_g = \rho_\Gamma$. (Γ is called a *chart description* of g.)

Two charts are *equivalent* or *chart move equivalent* if they are related by a finite sequence of chart moves (Figure 5) and ambient isotopes of S^2.

Theorem 5.6 *Let* Γ *and* Γ' *be chart descriptions of simple embedded* 2-*dimensional braids* g *and* g' *of the same degree.* g *and* g' *are equivalent if and only if* Γ *is equivalent to* Γ'.

Let $\mathrm{pr} : D^2 \times S^2 \to S^2$ be the projection.
Let $f : M^2 \to S^2$ be a simple branched covering, and $g : M^2 \to D^2 \times S^2$ a simple embedded 2-dimensional braid.

Definition 5.7 If $\mathrm{pr} \circ g = f$, then we call g an *embedded lift* of f, and we say that f is *liftable.*

Theorem 5.8 *Any simple branched covering of* S^2 *is liftable.*

Remark 5.9 For any simple branched covering, there exists a chart description that is an orientable permutation chart. Not every chart description of a liftable simple branched covering is orientable.

For further topics related to braid charts and 2-dimensional braids, refer to [4, 5, 13, 14].

6 Three dimensional case ($m = 3$)

We recall the theorem due to H. M. Hilden [8] and J. M. Montesinos [17] again.

Theorem 6.1 (Hilden and Montesinos) *Any closed oriented and connected* 3*-manifold can be represented as a* 3*-fold simple branched covering of* S^3 *branched over a link (or a knot).*

Let $f : M^3 \to S^3$ be a d-fold simple branched covering of S^3 branched along L. Let $\underline{f} : M^3 \setminus f^{-1}(L) \to S^3 \setminus L$ be the associated covering. The covering map $\underline{f}$ is determined by a monodromy $\rho : \pi_1(S^3 \setminus L, *) \to S_d$.

Remark 6.2 The monodromy ρ sends each meridian to a transposition. Conversely, any homomorphism $\rho : \pi_1(S^3 \setminus L, *) \to S_d$ sending each meridian to a transposition is a monodromy of a simple branched covering.

Figure 11 is a knot with a monodromy in S_3. In general, by $(12) \mapsto B =$ blue, $(23) \mapsto R =$ red, $(13) \mapsto G =$ green, we obtain a link with Fox's 3-coloring that represents a 3-manifold. See Figure 12.

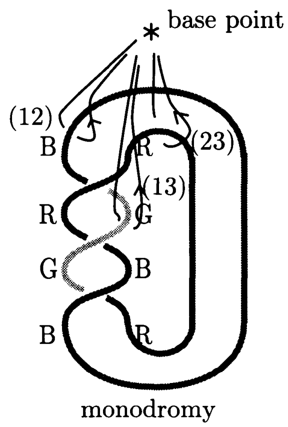

Figure 11: A knot with a monodromy in S_3

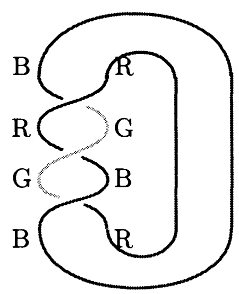

Figure 12: A 3-colored knot

The local move depicted in Figure 13 was introduced by Montesinos, that does not change the 3-manifold.

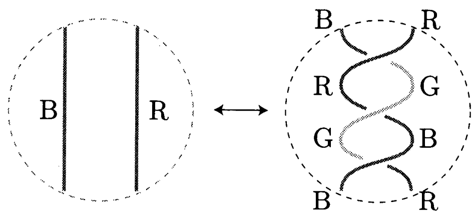

Figure 13: A Montesions move

Applying a Montesions move to the 3-colored knot in Figure 12, we have a 3-colored trivial link as in Figure 14, which represents S^3. Thus it is a nontrivial representation of S^3 as a 3-fold simple branched covering.

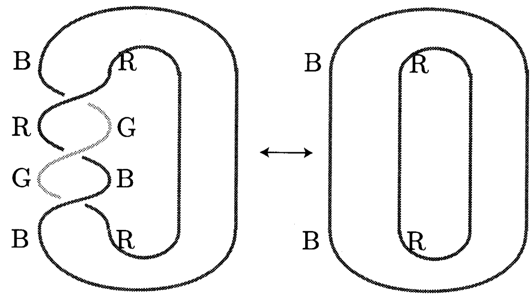

Figure 14: Two representations of S^3 as a 3-fold simple branched covering

Definition 6.3 A homomorphism $\rho : \pi_1(S^3 \setminus L, *) \to S_d$ sending each meridian to a transposition is called a *simple* homomorphism.

A link L with a simple homomorphism $\rho : \pi_1(S^3 \setminus L, *) \to S_d$ induces a d-fold simple branched covering $f : M^3 \to S^3$ branched along L.

Let $D^2 \times S^3$ be a tubular neighborhood of a standardly embedded S^3 in R^5, and let $\mathrm{pr} : D^2 \times S^3 \to S^3$ be the projection.

Definition 6.4 A (*simple*) (*embedded/immersed*) 3-*dimensional braid* is a PL map $g : M^3 \to D^2 \times S^3 \subset R^5$ such that

(1) the composition $\mathrm{pr} \circ g : M^3 \to S^3$ is a (simple) branched covering,

(2) g is an embedding/immersion, and

(3) if g is an immersion, the image of multipoint set under pr is a link in S^3 avoiding the branch set.

Let $f : M^3 \to S^3$ be a branched covering and $g : M^3 \to D^2 \times S^3 \subset R^5$ an embedded/immersed 3-dimensional braid. If $\mathrm{pr} \circ g = f$, then we call g an *embedded/immersed lift* of g.

Theorem 6.5 *For any 2-fold simple branched covering $f : M^3 \to S^3$, there exists an embedded lift $g : M^3 \to D^2 \times S^3 \subset R^5$.*

Theorem 6.6 *For any d-fold simple branched covering $f : M^3 \to S^3$, there exists an immersed lift $g : M^3 \to D^2 \times S^3 \subset R^5$.*

Problem 6.7 When does a simple branched covering $f : M^3 \to S^3$ have an embedded lift?

In terms of groups

Let L be a link in S^3. Recall Definition 6.3 that a homomorphism $f : \pi_1(S^3 \setminus L) \to S_d$ is *simple* if each meridian is mapped to a transposition.

Definition 6.8 A homomorphism $g : \pi_1(S^3 \setminus L) \to B_d$ is *simple* if each meridian is mapped to a conjugate of σ_i or σ_i^{-1}.

Let $\mathrm{pr} : B_d \to S_d$ be the natural projection.

Let $f : \pi_1(S^3 \setminus L) \to S_d$ and $g : \pi_1(S^3 \setminus L) \to B_d$ be simple homomorphisms. If $\mathrm{pr} \circ g = f$, we say that g is a *simple lift* of f.

Problem 6.9 Characterize a simple homomorphism $f : \pi_1(S^3 \setminus L) \to S_d$ that has a simple lift.

In terms of quandles

For an oriented link L in S^3, let $Q(S^3, L)$ denote the fundamental quandle of L ([7, 12, 16]).

Let T_d be the set of transpositions in S_d. Let A_d be the set of conjugates of standard generators of B_d and their inverses. The sets A_d and T_d are regarded as quandles by conjugation. The natural projection $\mathrm{pr} : B_d \to S_d$ induces the projection $\mathrm{pr} : A_d \to T_d$ which is a surjective quandle homomorphism.

Problem 6.10 Characterize a quandle homomorphism $f : Q(S^3, L) \to T_d$ that has a lift $\tilde{f} : Q(S^3, L) \to A_d$, i.e., $\mathrm{pr} \circ \tilde{f} = f$.

In general we are interested in the following problem.

Problem 6.11 Let $p : \widetilde{Q} \to Q$ be a surjective quandle homomorphism. Characterize a quandle homomorphism $f : P \to Q$ that has a lift $\tilde{f} : P \to \widetilde{Q}$ with respect to p, *i.e.*, $f = p \circ \tilde{f}$.

7 2-dimensional charts ($m = 3$)

Permutation charts and braid charts are graphs in S^2 describing simple branched coverings of S^2 and simple 2-dimensional braids. These notions are generalized into higher dimensions. The authors are studying 2-dimensional permutation charts and 2-dimensional braid charts. They are used to describe simple branched coverings of S^3 and simple 3-dimensional braids, respectively.

- A simple embedded branched covering of $S^3 \Leftarrow$ a 2-dimensional permutation chart.
- A simple embedded 3-dimensional braid
$\Leftarrow$ a 2-dimensional braid chart, or a *curtain.*
- A simple immersed 3-dimensional braid
$\Leftarrow$ a 2-dimensional braid chart (or a curtain) with/without *nodal curves.*

A 2-dimensional (permutation or braid) chart is a 2-dimensional subcomplex of S^3 whose faces are (unoriented or oriented), and labeled by integers in $\{1, \ldots, d-1\}$ such that certain conditions around edges are assumed. We show some examples of 2-dimensional charts.

Example 7.1 In Figure 15 a trefoil L with a Seifert surface F is depicted. When we forget the orientation of F, the surface F is regarded as a 2-dimensional permutation chart of degree 2, or a 2-dimensional S_2-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, *) \to S_2$ using intersection words. It describes a simple embedded 2-fold branched covering $f_F : M^3 \to S^3$ with branch set L.

When we use the orientation of F, the surface F is regarded as a 2-dimensional braid chart of degree 2, or a 2-dimensional B_2-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, *) \to B_2$ using intersection words. It describes a simple embedded 3-dimensional braid $g_F : M^3 \to D^2 \times S^3 \subset R^5$.

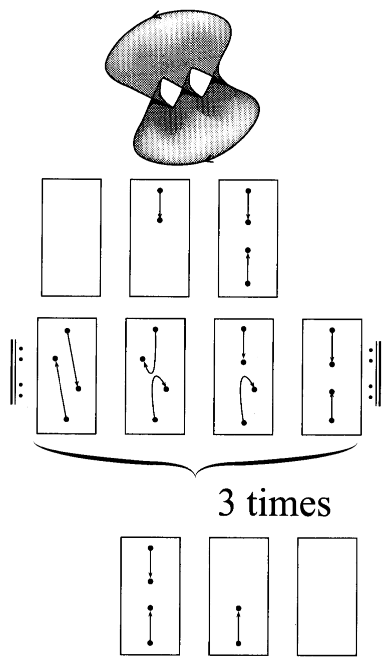

Figure 15: A trefoil with a Seifert surface

Example 7.2 In Figure 16 a knot 5_2, denoted by L here, with a Seifert surface, denoted by F, is depicted. Figure 17 shows a motion picture of L and F.

When we forget the orientation of F, the surface F is regarded as a 2-dimensional permutation chart of degree 2, or a 2-dimensional S_2-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, *) \to S_2$ using intersection words. It describes a simple embedded 2-fold branched covering $f_F : M^3 \to S^3$ with branch set L.

When we use the orientation of F, the surface F is regarded as a 2-dimensional braid chart of degree 2, or a 2-dimensional B_2-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, *) \to B_2$ using intersection words. It describes a simple embedded 3-dimensional braid $g_F : M^3 \to D^2 \times S^3 \subset R^5$.

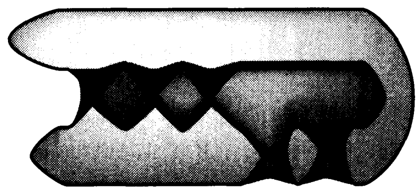

Figure 16: A knot 5_2 with a Seifert surface

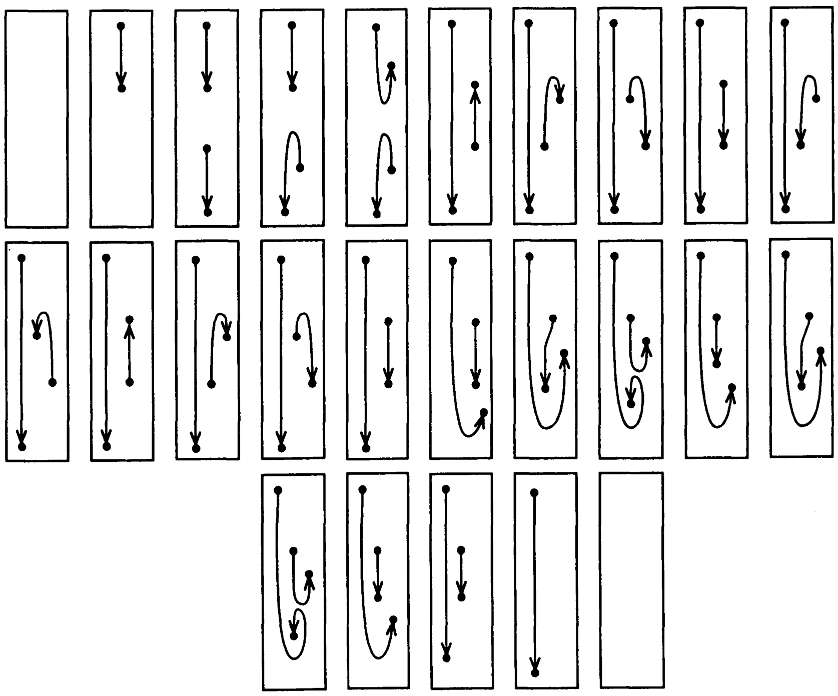

Figure 17: A motion picture

Example 7.3 Figures 18 and 19 show a 3-colored trefoil and a 2-dimensional braid chart. Let L be the trefoil knot depicted on the left of Figure 18. Let $\rho : \pi_1(S^3 \setminus L) \to S_3$ be the monodromy described by the 3-coloring. In the right side of Figures 18 and 19, a motion picture of a 2-dimensional braid chart Γ of degree 3 is depicted. The monodromy induced from Γ is ρ.

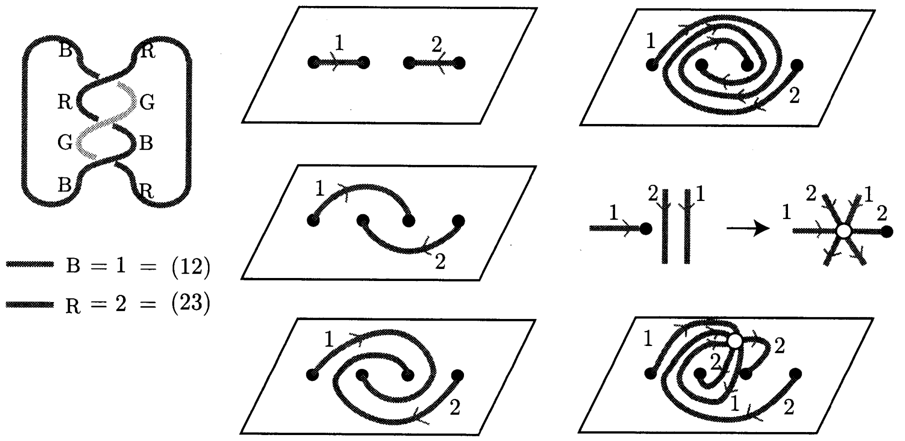

Figure 18: A 3-colored trefoil and a 2-dimensional braid chart

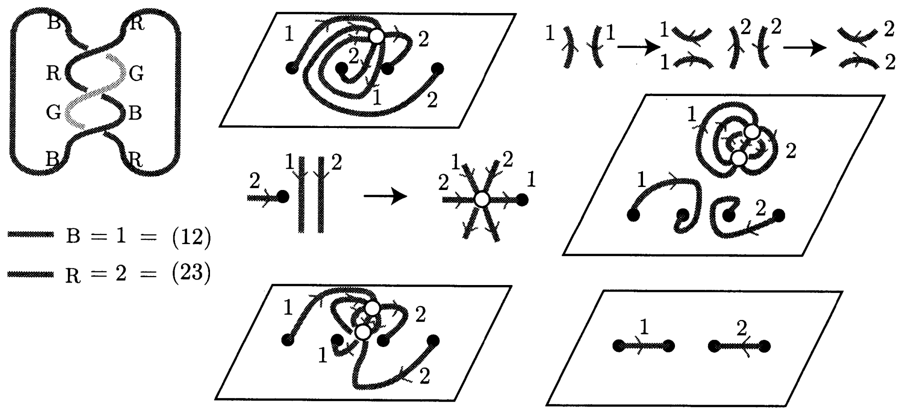

Figure 19: A 3-colored trefoil and a 2-dimensional braid chart

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