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Kyoto University
Braids and branched coverings of dimension three

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1 Introduction

This is on a part of our work in progress, which was introduced at the conference "Intelligence of Low-dimensional Topology" held in RIMS in May, 2012. The purpose of our research is to understand branched coverings and $m$-dimensional braids which are generalizations of classical braids. Here we discuss chart descriptions of branched coverings and braids in dimension $m = 2$ first, and then those for which $m = 3$.

We work in the PL category ([9, 20]). Let $S^m$ denote the $m$-sphere, and let $M^m$ denote a closed oriented $m$-manifold.

2 Preliminaries

We start by giving some definitions and theorems on branched coverings.

Definition 2.1 A PL map $f : M^m \to S^m$ is a branched covering (map) if there exists an $(m - 2)$-subcomplex $L$ of $S^m$ such that the restriction $\underline{f} : M^m \setminus f^{-1}(L) \to S^m \setminus L$ is a covering map.

We denote the covering degree by $d$. We call $f$ a $d$-fold branched covering.

We assume that $L$ is minimum, i.e., $\forall y \in L$, $\#(f^{-1}(y)) < d$. Then we call $L$ the branch set of $f$.

Definition 2.2 A $d$-fold branched covering $f$ is simple if $\forall y \in L$, $\#(f^{-1}(y)) = d - 1$.

Remark 2.3 (1) A branched covering is defined in general as follows (cf. [2, 3]): A PL map between manifolds is called proper if the inverse image of the boundary is the boundary. A proper PL map between manifolds $f : M^m \to N^m$ is called a branched covering if it is finite-to-one and open.

(2) A branched covering $f : M \to N$ is primitive if $f_* : \pi_1(M) \to \pi_1(N)$ is surjective. It is often assumed that a branched covering is primitive.

Note that $M^m$ is closed, oriented and connected in what follows in this section.
Theorem 2.4 (J.W. Alexander [1]) For any closed oriented and connected \( m \)-manifold \( M^m \), there exists a simple branched covering \( f : M^m \to S^m \) for some degree \( d \).

Remark 2.5 (1) A closed oriented and connected 1-manifold \( M^1 \) is homeomorphic to \( S^1 \). Thus there exists a 1-fold covering \( f : M^1 \to S^1 \).

(2) For any closed oriented and connected 2-manifold \( M^2 \), there exists a 2-fold simple branched covering \( f : M^2 \to S^2 \).

Theorem 2.6 (H. M. Hilden [8], J. M. Montesinos [17]) For any closed oriented and connected 3-manifold \( M^3 \), there exists a 3-fold simple branched covering \( f : M^3 \to S^3 \) such that the branch set \( L \) is a link (or a knot).

The following is a conjecture due to Montesinos.

Conjecture 2.7 For any closed oriented and connected 4-manifold \( M^4 \), there exists a 4-fold simple branched covering \( f : M^4 \to S^4 \) such that \( L \) is an embedded surface in \( S^4 \).

Some partial answers to this conjecture are known as follows.

Theorem 2.8 (R. Piergallini [19]) For any closed oriented and connected 4-manifold \( M^4 \), there exists a 4-fold simple branched covering \( f : M^4 \to S^4 \) such that \( L \) is an immersed surface in \( S^4 \).

Theorem 2.9 (M. Iori and R. Piergallini [11]) For any closed oriented and connected 4-manifold \( M^4 \), there exists a 5-fold simple branched covering \( f : M^4 \to S^4 \) such that \( L \) is an embedded surface in \( S^4 \).

3 Two dimensional case \((m = 2)\)

Let \( f : M^2 \to S^2 \) be a \( d \)-fold simple branched covering with branch set \( L \), and let \( f : M^2 \setminus f^{-1}(L) \to S^2 \setminus L \) be the associated covering map.

Take a base point \(*\) of \( S^2 \setminus L \) to consider the fundamental group \( \pi_1(S^2 \setminus L, *) \). The preimage \( f^{-1}(*) \) of the base point \(*\) consists of \( d \) points of \( M^2 \). Then we have a monodromy \( \rho : \pi_1(S^2 \setminus L, *) \to S_d \), where the symmetric group \( S_d \) on letters \( \{1, 2, \ldots, d\} \) is identified with the symmetric group on \( f^{-1}(*) \). (A monodromy \( \rho \) depends on the identification between \( \{1, 2, \ldots, d\} \) and \( f^{-1}(*) \).) The covering \( f \) is determined by the monodromy.

By the Riemann-Hurwitz formula, \( L \) consists of an even number of points.

In Figure 1, a branch set, a monodromy, and a chart are depicted. (A chart description is explained later.)

When a monodromy is described by a chart, it is easy to construct \( M^2 \). We explain it by using an example. Let \( \Gamma \) be the chart depicted on the right of Figure 1. Consider three copies of \( S^2 \) labeled by 1, 2, and 3, say \( S^2_1, S^2_2 \) and \( S^2_3 \), respectively. On the copy \( S^2_1 \), draw the edges with label (12) of \( \Gamma \), on the copy \( S^2_2 \), draw the edges with label (12) of \( \Gamma \) and those with label (23), and on the copy \( S^2_3 \), draw the edges with label (23). Cut the three 2-spheres along these edges, and we obtain three compact surfaces, say \( M_1, M_2 \) and \( M_3 \), as in the bottom of Figure 2. The surface \( M^2 \) is obtained from the union \( M_1 \cup M_2 \cup M_3 \).
by identifying the boundary as follows: Let $e$ be an edge with label $(12)$ on $S_1^2$, and let $e_+$ and $e_-$ be the copies of $e$ in $\partial M_1$. Let $e'_+$ be the corresponding edge on $S_2^2$, and let $e'_-$ be the corresponding copies in $\partial M_2$. Then we identify $e_+$ with $e'_-$, and identify $e_-$ with $e'_+$, respectively. All boundary edges of $M_1 \cup M_2 \cup M_3$ are identified in this fashion, and we have a closed surface. This is the desired $M^2$.

The classification of simple branched coverings was studied by J. Lüroth [15], A. Clebsch [6], A. Hurwitz [10], and others. The classification theorem is stated as follows.

**Theorem 3.1** Let $f : M^2 \to S^2$ and $f' : M'^2 \to S^2$ be d-fold simple branched coverings with branch sets $L$ and $L'$, respectively. We assume that $M^2$ and $M'^2$ are connected. Then $f$ and $f'$ are equivalent if and only if $\#L = \#L'$.

Hurwitz [10] studied branched coverings by using a system of monodromies of meridian elements of the branch set, called a **Hurwitz system**, and studied when two systems...
present the same (up to equivalence) branched coverings.

A Hurwitz system depends on a system of generating set of $\pi_1(S^2 \setminus L, \ast)$. For a generating system depicted in the middle of Figure 1, the Hurwitz system is

$$\alpha = ((12), (12), (12), (12), (23), (23)).$$

Besides a choice of a generating system, a Hurwitz system depends on the identification of $\{1, 2, \ldots, d\}$ and the fiber $f^{-1}(\ast)$.

Two Hurwitz systems present the same (up to equivalence) braid monodromy if and only if they are related by a finite sequence of Hurwitz moves and conjugations. The Hurwitz moves are

$$(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \mapsto (a_1, \ldots, a_{k+1}, a_{k+1}^{-1}a_ka_{k+1}, \ldots, a_n)$$

for $k = 1, \ldots, n - 1$ and their inverse moves. Conjugations are

$$(a_1, \ldots, a_n) \mapsto (g^{-1}a_1g, \ldots, g^{-1}a_ng)$$

for $g \in S_d$. When two Hurwitz systems are related by a finite sequence of Hurwitz moves and conjugations, we say that they are HC-equivalent. ($H$ and $C$ stand for Hurwitz and conjugation.)

Due to Hurwitz [10], the classification theorem is stated as follows.

**Theorem 3.2** Let $f : M^2 \to S^2$ be a $d$-fold simple branched covering. Assume that $M^2$ is connected. Any Hurwitz system of $f$ is HC-equivalent to

$$((12), \ldots, (12), (23), (23), (34), (34), \ldots, (d - 1, d), (d - 1, d)).$$

(The number of $(12)$s is a positive even number, and for each $i = 2, \ldots, d - 1$, a pair of $(i, i + 1)$ appears.)

In the next section, we will introduce the notion of a chart, called a permutation chart or an $S_d$-chart, that describes a branched covering or its monodromy. The chart method helps us to construct $M^2$ from a monodromy, and to understand the classification theorem well.

### 4 Permutation charts or $S_d$-charts ($m = 2$)

We denote by $\tau_i$ the transposition $(i \ i + 1)$. The symmetric group $S_d$ is generated by $\tau_1, \ldots, \tau_{d-1}$, and has a group presentation

$$S_d = \langle \tau_1, \ldots, \tau_{d-1} | \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j \quad (|i - j| = 1), \tau_i\tau_j = \tau_j\tau_i \quad (|i - j| > 1), \tau_i^2 = e \rangle.$$ 

**Definition 4.1** A permutation chart of degree $d$ or an $S_d$-chart is a labeled graph in $S^2$ such that each edge is labeled in $\{1, \ldots, d - 1\}$ and each vertex is as in Figure 3. We call a vertex a black vertex, a crossing or a white vertex if the valency of the vertex is 1, 4 or 6, respectively.
By the correspondence $i \leftrightarrow \tau_i = (i \ i + 1) \in S_d$, the labels of a chart are assumed to be transpositions in $S_d$ (see Figure 1). Figure 4 is an example of an $S_d$-chart, or a permutation chart of degree 4.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

black vertex crossing white vertex

\[
\begin{array}{c}
i - j > 1 \\
i - j = 1
\end{array}
\]

Figure 3: Vertices of a $S_d$-chart

For a chart $\Gamma$, we consider a monodromy

\[\rho_\Gamma : \pi_1(S^2 \setminus L) \to S_d, \quad [\ell] \mapsto [\text{intersection word of } \ell \text{ w.r.t. } \Gamma],\]

where $L (= L_\Gamma)$ is the set of black vertices. An intersection word is a sequence of elements of $\{1, \ldots, d - 1\}$, which is regarded as an element of $S_d$ by the correspondence $i \leftrightarrow \tau_i = (i \ i + 1) \in S_d$.

**Example 4.2** Let $\Gamma$ be an $S_4$-chart depicted in the left of Figure 4. When we take a Hurwitz generating system as in the figure, we have a Hurwitz system $(\tau_1, \tau_1 \tau_3 \tau_1, \tau_3, \tau_3 \tau_1 \tau_2 \tau_1 \tau_2)$. It is equal to $(\tau_1, \tau_3, \tau_3, \tau_1)$. And it is Hurwitz equivalent to $(\tau_1, \tau_1, \tau_3, \tau_3)$.

**Theorem 4.3** Let $f : M^2 \to S^2$ be a $d$-fold simple branched covering, and $\rho_f$ a monodromy of $f$. There exists a chart $\Gamma$ such that $\rho_\Gamma = \rho_f$. (We call $\Gamma$ a chart description of $f$ or $\rho_f$.)

Local moves on permutation charts illustrated in Figure 5 are called chart moves. (Ignore the orientations on edges.) Two charts are said to be equivalent or chart move
equivalent if they are related by a finite sequence of chart moves and ambient isotopies of $S^2$.

**Theorem 4.4** Let $f$ and $f'$ be $d$-fold simple branched covering of $S^2$, and let $\Gamma$ and $\Gamma'$ be their chart descriptions. $f$ is equivalent to $f'$ if and only if $\Gamma$ is equivalent to $\Gamma'$.

Using an example, we explain how to construct $M^2$ from a chart description. Let $\Gamma$ be an $S_4$-chart depicted in the top of Figure 6. Consider four copies of $S^2$ labeled by 1, 2, 3 and 4, say $S^2_1$, $S^2_2$, $S^2_3$ and $S^2_4$, respectively. On the copy $S^2_1$, draw the edges with label 1 of $\Gamma$, on the copy $S^2_2$, draw the edges with label 1 of $\Gamma$ and those with label 2, on the copy $S^2_3$, draw the edges with label 2 of $\Gamma$ and those with label 3, and on the copy $S^2_4$, draw the edges with label 3. Cut the four 2-spheres along the edges, and we obtain compact surfaces, say $M_1$, $M_2$, $M_3$ and $M_4$, as in the bottom of Figure 6. The surface $M^2$ is obtained from the union $\bigcup_{i=1}^{4}M_i$ by identifying the boundary as follows: Let $e$ be an edge with label 1 on $S^2_1$, and let $e_+$ and $e_-$ be the copies of $e$ in $\partial M_1$. Let $e'_+$ and $e'_-$ be the corresponding copies in $\partial M_2$. Then we identify $e_+$ with $e'_-$, and identify $e_-$ with $e'_+$, respectively. All boundary edges of $\bigcup_{i=1}^{4}M_i$ are identified in this fashion, and we have a closed surface. This is the desired $M^2$.

At a white vertex, 3 sheets are gathering as in Figure 7.

**Theorem 4.5** Any chart description of $f : M^2 \to S^2$ with connected $M$ is equivalent to a chart as in Figure 8.
This theorem is quite easily proved. As a corollary of this theorem, we have the classification theorem (Theorem 3.1).
Figure 7: Three sheets gather around a white vertex.

Figure 8: A chart in a normal form
5 Braid charts or $B_d$-charts $(m = 2)$

Let $\sigma_i (i = 1, \ldots, d - 1)$ be the standard generators of the braid group $B_d$. Then $B_d$ has a group presentation

$$B_d = \left\langle \sigma_1, \ldots, \sigma_{d-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \ (|i - j| = 1) \right\rangle.$$  

**Definition 5.1** A *braid chart* of degree $d$ or a *$B_d$-chart* is a labeled and oriented graph in $S^2$ such that each edge is labeled in $\{1, \ldots, d - 1\}$ and each vertex is as in Figure 9. We call a vertex a *black vertex*, a *crossing* or a *white vertex* if the valency of the vertex is 1, 4 or 6, respectively. The arrow at a black vertex in this figure is suppressed since it may either be incoming or outgoing.

![vertices](image)

**Figure 9:** Vertices of a $B_d$-chart

By the correspondence $i \mapsto \sigma_i = (i \ i + 1) \in B_d$, the labels of a chart are assumed to present the standard generators in $B_d$. Figure 10 is an example of a $B_4$-chart, or a braid chart of degree 4.

![chart](image)

**Figure 10:** A $B_4$-chart $\Gamma$ and the induced monodromy $\rho_{\Gamma}$

Forgetting orientations of the edges from a braid chart, we obtain a permutation chart. Thus we often call a permutation chart an *unoriented chart*, and a braid chart an *oriented chart*.
**Definition 5.2** A permutation chart is called *orientable* if one can give orientations to the edges to make it a braid chart. Otherwise it is called *nonorientable*.

For a braid chart $\Gamma$ of degree $d$, we consider a monodromy

$$\rho_{\Gamma}: \pi_{1}(S^{2} \setminus L) \to B_{d}, \quad [\ell] \mapsto \text{intersection word of } \ell \text{ w.r.t. } \Gamma,$$

where $L (= L_{\Gamma})$ is the set of black vertices. An intersection word is a word of $\{1, \ldots, d-1\}$, which is regarded as an element of $B_{d}$ by the correspondence $i \mapsto \sigma_{i} = (i \ i + 1) \in S_{d}$.

**Example 5.3** Let $\Gamma$ be a $B_{d}$-chart depicted in the left of Figure 10. When we take a Hurwitz generating system as in the right of the figure, we have a Hurwitz system

$$(\sigma_{1}, \sigma_{1}^{-1}\sigma_{3}\sigma_{1}, \sigma_{3}^{-1}, \sigma_{2}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{1}\sigma_{2}).$$

It is equal to $(\sigma_{1}, \sigma_{3}, \sigma_{3}^{-1}, \sigma_{1}^{-1})$. And it is Hurwitz equivalent to $(\sigma_{1}, \sigma_{1}^{-1}, \sigma_{3}, \sigma_{3}^{-1})$.

Let $D^{2} \times S^{2}$ be a tubular neighborhood of a standardly embedded 2-sphere in $R^{4}$.

**Definition 5.4** A PL embedding $g: M^{2} \to D^{2} \times S^{2} \subset R^{4}$ is a (simple) embedded 2-dimensional braid, or a surface braid, of degree $d$ if the composition $M^{2} \to D^{2} \times S^{2} \to S^{2}$ is a $d$-fold (simple) branched covering.

For a (simple or nonsimple) embedded 2-dimensional braid $g: M^{2} \to D^{2} \times S^{2} \subset R^{4}$ of degree $m$, we can consider a monodromy $\rho(= \rho_{g}): \pi_{1}(S^{2} \setminus L, \ast) \to B_{d}$, where $L (= L_{g})$ is the branch set of the branched covering $M^{2} \to D^{2} \times S^{2} \to S^{2}$.

**Theorem 5.5** For any simple embedded 2-dimensional braid $g: M^{2} \to D^{2} \times S^{2} \subset R^{4}$, there exists a braid chart $\Gamma$ such that $\rho_{g} = \rho_{\Gamma}$. ($\Gamma$ is called a chart description of $g$.)

Two charts are equivalent or chart move equivalent if they are related by a finite sequence of chart moves (Figure 5) and ambient isotopes of $S^{2}$.

**Theorem 5.6** Let $\Gamma$ and $\Gamma'$ be chart descriptions of simple embedded 2-dimensional braids $g$ and $g'$ of the same degree. $g$ and $g'$ are equivalent if and only if $\Gamma$ is equivalent to $\Gamma'$.

Let $\text{pr}: D^{2} \times S^{2} \to S^{2}$ be the projection.

Let $f: M^{2} \to S^{2}$ be a simple branched covering, and $g: M^{2} \to D^{2} \times S^{2}$ a simple embedded 2-dimensional braid.

**Definition 5.7** If $\text{pr} \circ g = f$, then we call $g$ an embedded lift of $f$, and we say that $f$ is liftable.

**Theorem 5.8** Any simple branched covering of $S^{2}$ is liftable.

**Remark 5.9** For any simple branched covering, there exists a chart description that is an orientable permutation chart. Not every chart description of a liftable simple branched covering is orientable.

For further topics related to braid charts and 2-dimensional braids, refer to [4, 5, 13, 14].
6 Three dimensional case \((m = 3)\)

We recall the theorem due to H. M. Hilden \cite{Hilden} and J. M. Montesinos \cite{Montesinos} again.

**Theorem 6.1 (Hilden and Montesinos)** Any closed oriented and connected 3-manifold can be represented as a 3-fold simple branched covering of \(S^3\) branched over a link (or a knot).

Let \(f : M^3 \rightarrow S^3\) be a \(d\)-fold simple branched covering of \(S^3\) branched along \(L\). Let \(\underline{f} : M^3 \setminus f^{-1}(L) \rightarrow S^3 \setminus L\) be the associated covering. The covering map \(\underline{f}\) is determined by a monodromy \(\rho : \pi_1(S^3 \setminus L, \ast) \rightarrow S_d\).

**Remark 6.2** The monodromy \(\rho\) sends each meridian to a transposition. Conversely, any homomorphism \(\rho : \pi_1(S^3 \setminus L, \ast) \rightarrow S_d\) sending each meridian to a transposition is a monodromy of a simple branched covering.

Figure 11 is a knot with a monodromy in \(S_3\). In general, by (12) \(\mapsto B = \text{blue}\), (23) \(\mapsto R = \text{red}\), (13) \(\mapsto G = \text{green}\), we obtain a link with Fox's 3-coloring that represents a 3-manifold. See Figure 12.

![Figure 11: A knot with a monodromy in \(S_3\)](image)

![Figure 12: A 3-colored knot](image)
The local move depicted in Figure 13 was introduced by Montesinos, that does not change the 3-manifold.

![Figure 13: A Montesions move](image)

Applying a Montesions move to the 3-colored knot in Figure 12, we have a 3-colored trivial link as in Figure 14, which represents $S^3$. Thus it is a nontrivial representation of $S^3$ as a 3-fold simple branched covering.

![Figure 14: Two representations of $S^3$ as a 3-fold simple branched covering](image)

**Definition 6.3** A homomorphism $\rho : \pi_1(S^3 \setminus L, \ast) \to S_d$ sending each meridian to a transposition is called a simple homomorphism.

A link $L$ with a simple homomorphism $\rho : \pi_1(S^3 \setminus L, \ast) \to S_d$ induces a $d$-fold simple branched covering $f : M^3 \to S^3$ branched along $L$.

Let $D^2 \times S^3$ be a tubular neighborhood of a standardly embedded $S^3$ in $R^5$, and let $pr : D^2 \times S^3 \to S^3$ be the projection.

**Definition 6.4** A (simple) (embedded/immersed) 3-dimensional braid is a PL map $g : M^3 \to D^2 \times S^3 \subset R^5$ such that

1. the composition $pr \circ g : M^3 \to S^3$ is a (simple) branched covering,
2. $g$ is an embedding/immersion, and
3. if $g$ is an immersion, the image of multipoint set under $pr$ is a link in $S^3$ avoiding the branch set.

Let $f : M^3 \to S^3$ be a branched covering and $g : M^3 \to D^2 \times S^3 \subset R^5$ an embedded/immersed 3-dimensional braid. If $pr \circ g = f$, then we call $g$ an embedded/immersed lift of $g$. 
Theorem 6.5 For any 2-fold simple branched covering $f : M^3 \to S^3$, there exists an embedded lift $g : M^3 \to D^2 \times S^3 \subset \mathbb{R}^5$.

Theorem 6.6 For any $d$-fold simple branched covering $f : M^3 \to S^3$, there exists an immersed lift $g : M^3 \to D^2 \times S^3 \subset \mathbb{R}^5$.

Problem 6.7 When does a simple branched covering $f : M^3 \to S^3$ have an embedded lift?

In terms of groups

Let $L$ be a link in $S^3$. Recall Definition 6.3 that a homomorphism $f : \pi_1(S^3 \setminus L) \to S_d$ is simple if each meridian is mapped to a transposition.

Definition 6.8 A homomorphism $g : \pi_1(S^3 \setminus L) \to B_d$ is simple if each meridian is mapped to a conjugate of $\sigma_i$ or $\sigma_i^{-1}$.

Let $\text{pr} : B_d \to S_d$ be the natural projection.

Let $f : \pi_1(S^3 \setminus L) \to S_d$ and $g : \pi_1(S^3 \setminus L) \to B_d$ be simple homomorphisms. If $\text{pr} \circ g = f$, we say that $g$ is a simple lift of $f$.

Problem 6.9 Characterize a simple homomorphism $f : \pi_1(S^3 \setminus L) \to S_d$ that has a simple lift.

In terms of quandles

For an oriented link $L$ in $S^3$, let $Q(S^3, L)$ denote the fundamental quandle of $L ([7, 12, 16])$.

Let $T_d$ be the set of transpositions in $S_d$. Let $A_d$ be the set of conjugates of standard generators of $B_d$ and their inverses. The sets $A_d$ and $T_d$ are regarded as quandles by conjugation. The natural projection $\text{pr} : B_d \to S_d$ induces the projection $\text{pr} : A_d \to T_d$ which is a surjective quandle homomorphism.

Problem 6.10 Characterize a quandle homomorphism $f : Q(S^3, L) \to T_d$ that has a lift $\tilde{f} : Q(S^3, L) \to A_d$, i.e., $\text{pr} \circ \tilde{f} = f$.

In general we are interested in the following problem.

Problem 6.11 Let $p : \bar{Q} \to Q$ be a surjective quandle homomorphism. Characterize a quandle homomorphism $f : P \to Q$ that has a lift $\tilde{f} : P \to \bar{Q}$ with respect to $p$, i.e., $f = p \circ \tilde{f}$.

7 2-dimensional charts ($m = 3$)

Permutation charts and braid charts are graphs in $S^2$ describing simple branched coverings of $S^2$ and simple 2-dimensional braids. These notions are generalized into higher dimensions. The authors are studying 2-dimensional permutation charts and 2-dimensional braid charts. They are used to describe simple branched coverings of $S^3$ and simple 3-dimensional braids, respectively.
A simple embedded branched covering of $S^3 \leftarrow$ a 2-dimensional permutation chart.

A simple embedded 3-dimensional braid
$\leftarrow$ a 2-dimensional braid chart, or a curtain.

A simple immersed 3-dimensional braid
$\leftarrow$ a 2-dimensional braid chart (or a curtain) with/without nodal curves.

A 2-dimensional (permutation or braid) chart is a 2-dimensional subcomplex of $S^3$ whose faces are (unoriented or oriented), and labeled by integers in $\{1, \ldots, d-1\}$ such that certain conditions around edges are assumed. We show some examples of 2-dimensional charts.

**Example 7.1** In Figure 15 a trefoil $L$ with a Seifert surface $F$ is depicted. When we forget the orientation of $F$, the surface $F$ is regarded as a 2-dimensional permutation chart of degree 2, or a 2-dimensional $S_2$-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, \ast) \rightarrow S_2$ using intersection words. It describes a simple embedded 2-fold branched covering $f_F : M^3 \rightarrow S^3$ with branch set $L$.

When we use the orientation of $F$, the surface $F$ is regarded as a 2-dimensional braid chart of degree 2, or a 2-dimensional $B_2$-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, \ast) \rightarrow B_2$ using intersection words. It describes a simple embedded 3-dimensional braid $g_F : M^3 \rightarrow D^2 \times S^3 \subset R^5$.

![Figure 15: A trefoil with a Seifert surface](image-url)
**Example 7.2** In Figure 16 a knot $5_2$, denoted by $L$ here, with a Seifert surface, denoted by $F$, is depicted. Figure 17 shows a motion picture of $L$ and $F$.

When we forget the orientation of $F$, the surface $F$ is regarded as a 2-dimensional permutation chart of degree 2, or a 2-dimensional $S_2$-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, \ast) \to S_2$ using intersection words. It describes a simple embedded 2-fold branched covering $f_F : M^3 \to S^3$ with branch set $L$.

When we use the orientation of $F$, the surface $F$ is regarded as a 2-dimensional braid chart of degree 2, or a 2-dimensional $B_2$-chart. (We assume that the sheet has label 1.) It induces a monodromy $\pi_1(S^3 \setminus L, \ast) \to B_2$ using intersection words. It describes a simple embedded 3-dimensional braid $g_F : M^3 \to D^2 \times S^3 \subset \mathbb{R}^5$.

Figure 16: A knot $5_2$ with a Seifert surface

Figure 17: A motion picture

**Example 7.3** Figures 18 and 19 show a 3-colored trefoil and a 2-dimensional braid chart. Let $L$ be the trefoil knot depicted on the left of Figure 18. Let $\rho : \pi_1(S^3 \setminus L) \to S_3$ be the
monodromy described by the 3-coloring. In the right side of Figures 18 and 19, a motion picture of a 2-dimensional braid chart $\Gamma$ of degree 3 is depicted. The monodromy induced from $\Gamma$ is $\rho$.

![Figure 18: A 3-colored trefoil and a 2-dimensional braid chart](image1)

![Figure 19: A 3-colored trefoil and a 2-dimensional braid chart](image2)

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**References**


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