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$\triangle Y$-exchanges and Conway-Gordon type theorems

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1 Intrinsic linkedness, intrinsic knottedness and $\triangle Y$-exchange

Let G be a finite graph and f an embedding of G into the 3-dimensional Euclidean space $\mathbb{R}^3$. Then f is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph.* We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. A subgraph of G which is homeomorphic to a circle is called a *cycle* of G. A cycle of G which contains exactly k edges is called a *k-cycle* of G, and a cycle of G which contains all vertices of G is called a *Hamiltonian cycle* of G. For a positive integer n, $\Gamma^{(n)}(G)$ denotes the set of all cycles of G if $n=1$ and the set of all unions of mutually disjoint n cycles of G if $n\geq 2$. We denote the union of $\Gamma^{(n)}(G)$ over all positive integer n by $\bar{\Gamma}(G)$. If $n=1$, we denote $\Gamma^{(1)}(G)$ by $\Gamma(G)$ simply, and denote the subset of $\Gamma(G)$ consisting of all k-cycles of G by $\Gamma_k(G)$. For an element γ in $\Gamma^{(n)}(G)$ and an element f in $\mathrm{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n=1$ and an n-component link in $f(G)$ if $n\geq 2$. In particular, for a Hamiltonian cycle γ of G, we call $f(\gamma)$ a *Hamiltonian knot* in $f(G)$. A graph H is called a *minor* of a graph G if there exists a subgraph G' of G such that H is obtained from G' by contracting some of the edges. A minor H of G is called a *proper minor* if H does not equal G.

Let K_n be the *complete graph* on n vertices (= 1-skelton of $(n-1)$-simplex if $n\geq 2$), see Fig. 1.1 for $n=6,7$. For spatial embeddings of K_6 and K_7, let us recall the Conway-Gordon theorems which are very famous in spatial graph theory.

Theorem 1.1 (Conway-Gordon [2])

(1) *For any element f in* $\mathrm{SE}(K_6)$, *it follows that*

$$\sum_{\gamma\in\Gamma^{(2)}(K_6)}\mathrm{lk}(f(\gamma))\equiv 1 \pmod 2,$$

where lk *denotes the linking number.*

(2) *For any element f in* $\mathrm{SE}(K_7)$, *it follows that*

$$\sum_{\gamma\in\Gamma_7(K_7)}a_2(f(\gamma))\equiv 1 \pmod 2,$$

where a_i denotes the ith coefficient of the Conway polynomial.

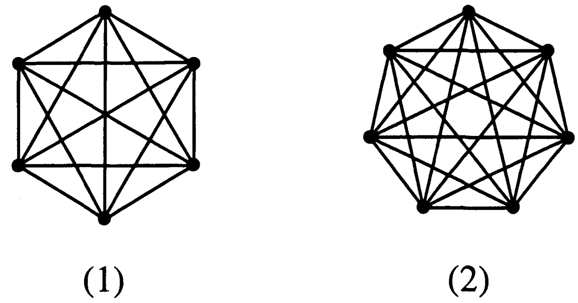


Figure 1.1. The complete graph on n vertices K_n: (1) $n = 6$, (2) $n = 7$

A graph is said to be *intrinsically linked* if for any element f in $\mathrm{SE}(G)$, there exists an element γ in $\Gamma^{(2)}(G)$ such that $f(\gamma)$ is a nonsplittable 2-component link, and to be *intrinsically knotted* if for any element f in $\mathrm{SE}(G)$, there exists an element γ in $\Gamma(G)$ such that $f(\gamma)$ is a nontrivial knot. Theorem 1.1 implies that K_6 is intrinsically linked and K_7 is intrinsically knotted. Moreover, it is known that K_6 (resp. K_7) is *minor-minimal* with respect to the intrinsic linkedness (resp. knottedness), that is, each of the proper minors of K_6 (resp. K_7) is not intrinsically linked [16] (resp. knotted [11]).

We can obtain another intrinsically linked (resp. knotted) graph from K_6 (resp. K_7) in the following way. A *$\triangle Y$-exchange* is an operation to obtain a new graph G_Y from a graph $G_\triangle$ by removing all edges of a 3-cycle $\triangle$ of $G_\triangle$ with the edges uv, vw and wu, and adding a new vertex x and connecting it to each of the vertices u, v and w as illustrated in Fig. 1.2 (we often denote $ux \cup vx \cup wx$ by Y). A *$Y\triangle$-exchange* is the reverse of this operation. We call the set of all graphs obtained from a graph G by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges the *G-family* and denote it by $\mathcal{F}(G)$. In particular, we denote the set of all graphs obtained from G by a finite sequence of $\triangle Y$-exchanges by $\mathcal{F}_\triangle(G)$.

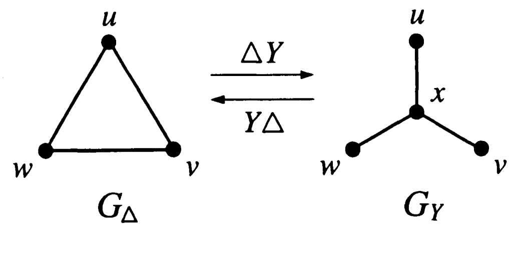


Figure 1.2. $\triangle Y$-exchange and $Y\triangle$-exchange

Example 1.2 (1) The K_6-family consists of seven graphs as illustrated in Fig. 1.3 and $\mathcal{F}_\triangle(K_6) = \mathcal{F}(K_6) \setminus \{P_7\}$. Since P_{10} is isomorphic to the *Petersen graph* which is depicted in Fig. 1.5 (1), the K_6-family is also called the *Petersen family.*

(2) The K_7-family consists of twenty graphs as illustrated in Fig. 1.4 and $\mathcal{F}_\triangle(K_7) = \mathcal{F}(K_7) \setminus \{N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, N'_{12}\}$. Since C_{14} is isomorphic to the *Heawood graph* which is depicted in Fig. 1.5 (2), the K_7-family is also called the *Heawood family.*

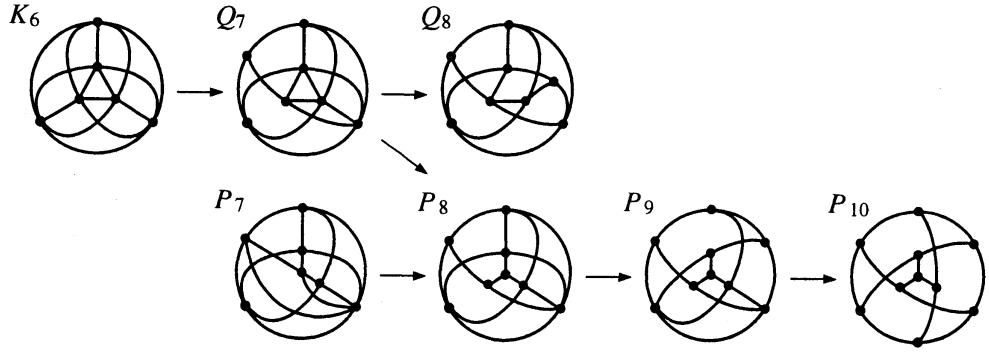


Figure 1.3. K_6-family = Petersen family

The intrinsic linkedness and the intrinsic knottedness behave well under $\triangle Y$-exchanges as follows.

Proposition 1.3 (Motwani-Raghunathan-Saran [11])

(1) *If $G_\triangle$ is intrinsically linked, then G_Y is also intrinsically linked.*

(2) *If $G_\triangle$ is intrinsically knotted, then G_Y is also intrinsically knotted.*

Thus any graph G in $\mathcal{F}_\triangle(K_6)$ (resp. $\mathcal{F}_\triangle(K_7)$) is intrinsically linked (resp. knotted). Moreover, it is also known that G is minor-minomal with respect to the intrinsic linkedness [16] (resp. knottedness [10]).

Now let us give a proof of Proposition 1.3. We denote the set of all elements in $\bar{\Gamma}(G_\triangle)$ containing $\triangle$ by $\bar{\Gamma}_\triangle(G_\triangle)$. Let γ' be an element in $\bar{\Gamma}(G_\triangle)$ which does not contain $\triangle$. Then there exists an element $\bar{\Phi}(\gamma')$ in $\bar{\Gamma}(G_Y)$ such that $\gamma' \setminus \triangle = \bar{\Phi}(\gamma') \setminus Y$. It is easy to see that the correspondence from γ' to $\bar{\Phi}(\gamma')$ defines a surjective map

$$\bar{\Phi} : \bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle) \longrightarrow \bar{\Gamma}(G_Y).$$

Note that if γ' is an element in $\Gamma^{(n)}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$ then $\bar{\Phi}(\gamma')$ is an element in $\Gamma^{(n)}(G_Y)$. For an element γ in $\bar{\Gamma}(G_Y)$, we see that the inverse image of γ by $\bar{\Phi}$ contains at most two elements in $\bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$. In general, the following holds.

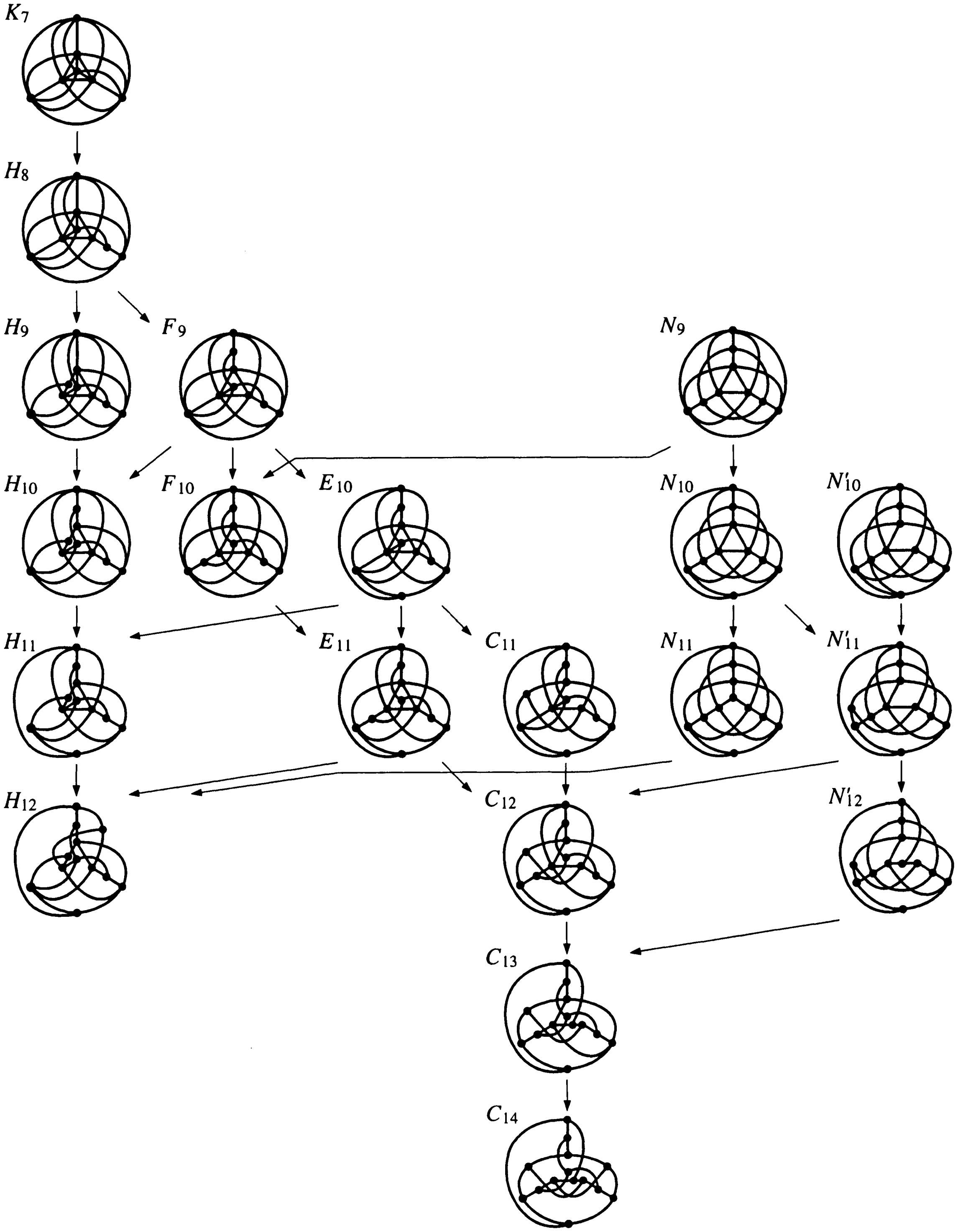
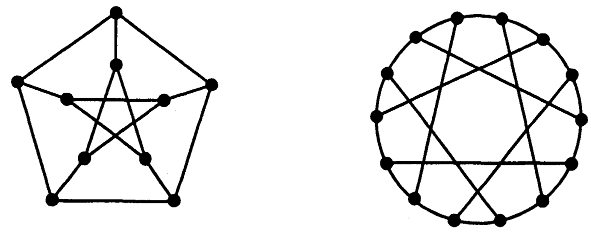


Figure 1.4. K_7-family = Heawood family



(1) (2)

Figure 1.5. (1) Petersen graph, (2) Heawood graph

Proposition 1.4 *Let γ be an element in $\bar{\Gamma}(G_Y)$. Then, the inverse image of γ by $\bar{\Phi}$ consists of exactly one element if and only if γ contains u, v, w and x, or γ does not contain x.*

Let f be an element in $\mathrm{SE}(G_Y)$ and D a 2-disk in $\mathbb{R}^3$ such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $\varphi(f)$ be an element in $\mathrm{SE}(G_\triangle)$ such that $\varphi(f)(x) = f(x)$ for $x \in G_\triangle \setminus \triangle = G_Y \setminus Y$ and $\varphi(f)(G_\triangle) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi : \mathrm{SE}(G_Y) \longrightarrow \mathrm{SE}(G_\triangle).$$

Then we have the following.

Proposition 1.5 *Let f be an element in $\mathrm{SE}(G_Y)$ and γ an element in $\bar{\Gamma}(G_Y)$. Then, $f(\gamma)$ is ambient isotopic to $\varphi(f)(\gamma')$ for each element γ' in the inverse image of γ by $\bar{\Phi}$.*

Proof of Proposition 1.3. We show (2), namely if $G_\triangle$ is intrinsically knotted then G_Y is also intrinsically knotted. For any element f in $\mathrm{SE}(G_Y)$, there exists a element γ' in $\Gamma(G_\triangle)$ such that $\varphi(f)(\gamma')$ is a nontrivial knot because $G_\triangle$ is intrinsically knotted. Note that γ' is not equal to $\triangle$ because $\varphi(f)(\triangle)$ is a trivial knot. Thus $\bar{\Phi}(\gamma')$ belongs to $\Gamma(G_Y)$. Then, by Proposition 1.5, $f\left(\bar{\Phi}(\gamma')\right)$ is ambient isotopic to the nontrivial knot $\varphi(f)(\gamma')$. We can also show (1) in a similar way. $\square$

Remark 1.6 It is known that the converse of Proposition 1.3 (1) is also true [15], but the converse of Proposition 1.3 (2) is not true, see Remark 3.5.

As we see above, $\triangle Y$ exchanges carry the intrinsic linkedness and the intrinsic knottedness for a graph to the one for another graph. Our purpose in this report is to introduce a method to carry not only the intrinsic linkedness and the intrinsic knottedness but also the Conway-Gordon type dependent relation for a graph to the one for another graph by $\triangle Y$-exchanges.

2 $\triangle Y$-exchange and Conway-Gordon theorem

Let A be an additive group and α an A-valued unoriented link invariant. We say that α is *compressible* if $\alpha(L)=0$ for any unoriented link L which have a component K bounding a disk D in $\mathbb{R}^3$ with $D\cap L=\partial D=K$. Namely $\alpha(L)=0$ if L contains a trivial knot as a split component. In particular, $\alpha(L)=0$ when L is a trivial knot. Suppose that for each element γ' in $\bar{\Gamma}(G_\triangle)$, an A-valued unoriented link invariant $\alpha_{\gamma'}$ is assigned. Then for each element γ in $\bar{\Gamma}(G_Y)$, we define an A-valued unoriented link invariant $\tilde{\alpha}_\gamma$ by

$$\tilde{\alpha}_\gamma(L)=\sum_{\gamma'\in\bar{\Phi}^{-1}(\gamma)}\alpha_{\gamma'}(L).$$

Then we have the following lemma.

Lemma 2.1 (Nikkuni-Taniyama [13]) *If $\alpha_{\gamma'}$ is compressible for any element γ' in $\bar{\Gamma}_\triangle(G_\triangle)$, then it follows that*

$$\sum_{\gamma\in\bar{\Gamma}(G_Y)}\tilde{\alpha}_\gamma(f(\gamma))=\sum_{\gamma'\in\bar{\Gamma}(G_\triangle)}\alpha_{\gamma'}(\varphi(f)(\gamma'))$$

for any element f in $\mathrm{SE}(G_Y)$.

Proof. For an element γ' in $\bar{\Gamma}_\triangle(G_\triangle)$, we see that $\varphi(f)(\gamma')$ is the trivial knot if γ' belongs to $\Gamma(G_\triangle)$ and a link containing a trivial knot as a split component if γ' belongs to $\bar{\Gamma}(G_\triangle)\setminus\Gamma(G_\triangle)$. Since $\alpha_{\gamma'}$ is compressible for any element γ' in $\bar{\Gamma}(G_\triangle)$, we see that

$$\sum_{\gamma'\in\bar{\Gamma}(G_\triangle)}\alpha_{\gamma'}(\varphi(f)(\gamma'))=\sum_{\gamma'\in\bar{\Gamma}(G_\triangle)\setminus\bar{\Gamma}_\triangle(G_\triangle)}\alpha_{\gamma'}(\varphi(f)(\gamma')).$$

Note that

$$\bar{\Gamma}(G_\triangle)\setminus\bar{\Gamma}_\triangle(G_\triangle)=\bigcup_{\gamma\in\bar{\Gamma}(G_Y)}\bar{\Phi}^{-1}(\gamma).$$

Then, by Proposition 1.5, we see that

$$\begin{aligned}\sum_{\gamma'\in\bar{\Gamma}(G_\triangle)\setminus\bar{\Gamma}_\triangle(G_\triangle)}\alpha_{\gamma'}(\varphi(f)(\gamma'))&=\sum_{\gamma\in\bar{\Gamma}(G_Y)}\left(\sum_{\gamma'\in\bar{\Phi}^{-1}(\gamma)}\alpha_{\gamma'}(\varphi(f)(\gamma'))\right)\\&=\sum_{\gamma\in\bar{\Gamma}(G_Y)}\left(\sum_{\gamma'\in\bar{\Phi}^{-1}(\gamma)}\alpha_{\gamma'}(f(\gamma))\right)\\&=\sum_{\gamma\in\bar{\Gamma}(G_Y)}\tilde{\alpha}_\gamma(f(\gamma)).\end{aligned}$$

Thus we have the result. $\square$

By Lemma 2.1, we immediately have the following theorem.

Theorem 2.2 *Suppose that $\alpha_{\gamma'}$ is compressible for each element γ' in $\bar{\Gamma}_{\triangle}(G_{\triangle})$. Suppose that there exists a subset A_0 of A such that*

$$\sum_{\gamma' \in \bar{\Gamma}(G_{\triangle})} \alpha_{\gamma'}(g(\gamma')) \in A_0$$

for any element g in $\mathrm{SE}(G_{\triangle})$. *Then we have*

$$\sum_{\gamma \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_{\gamma}(f(\gamma)) \in A_0$$

for any element f in $\mathrm{SE}(G_Y)$.

Proof. Suppose that there exists a subset A_0 of A such that

$$\sum_{\gamma' \in \bar{\Gamma}(G_{\triangle})} \alpha_{\gamma'}(g(\gamma')) \in A_0 \tag{2.1}$$

for any element g in $\mathrm{SE}(G_{\triangle})$. Then by Lemma 2.1 and (2.1), we have

$$\sum_{\gamma \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_{\gamma}(f(\gamma)) = \sum_{\gamma' \in \bar{\Gamma}(G_{\triangle})} \alpha_{\gamma'}(\varphi(f)(\gamma')) \in A_0$$

for any element f in $\mathrm{SE}(G_Y)$. □

As an application of Theorem 2.2, a "Conway-Gordon type" theorem for any element in $\mathcal{F}_{\triangle}(K_6)$ (resp. $\mathcal{F}_{\triangle}(K_7)$) can be produced by the Conway-Gordon theorem for K_6 (resp. K_7).

Example 2.3 Let Q_7 be the graph which is obtained from K_6 by a single $\triangle Y$-exchange. For each element γ' in $\bar{\Gamma}(K_6)$, we define a $\mathbb{Z}_2$-valued unoriented link invariant $\alpha_{\gamma'}$ of an unoriented link L by $\alpha_{\gamma'}(L) \equiv a_1(L) \pmod 2$. Note that $\alpha_{\gamma'}(L) = 0$ if L is not a 2-component link and $\alpha_{\gamma'}(L) \equiv \mathrm{lk}(L) \pmod 2$ if L is a 2-component link. Then by Theorem 1.1 (1), we have

$$\sum_{\gamma' \in \bar{\Gamma}(K_6)} \alpha_{\gamma'}(g(\gamma')) = 1 \tag{2.2}$$

in $\mathbb{Z}_2$ for any element g in $\mathrm{SE}(K_6)$. Note that $\alpha_{\gamma'}$ is compressible for any element γ' in $\bar{\Gamma}(K_6)$. Thus by Theorem 2.2 and (2.2), we have

$$\sum_{\gamma \in \bar{\Gamma}(Q_7)} \tilde{\alpha}_{\gamma}(f(\gamma)) = 1 \tag{2.3}$$

in $\mathbb{Z}_2$ for any element f in $\mathrm{SE}(Q_7)$. Note that each union of mutually disjoint two cycles of Q_7 contains all of the vertices. Thus by Proposition 1.4, for any element γ in $\Gamma^{(2)}(Q_7)$, the inverse image of γ by $\bar{\Phi}$ consists of exactly one element. Therefore we have

$$\tilde{\alpha}_\gamma(L) = \sum_{\gamma' \in \bar{\Phi}^{-1}(\gamma)} \alpha_{\gamma'}(L) \equiv a_1(L) \pmod 2 \tag{2.4}$$

for any element γ in $\Gamma^{(2)}(Q_7)$. Thus by (2.3) and (2.4), we have

$$1 = \sum_{\gamma \in \bar{\Gamma}(Q_7)} \tilde{\alpha}_\gamma(f(\gamma)) \equiv \sum_{\gamma \in \bar{\Gamma}(Q_7)} a_1(f(\gamma)) \equiv \sum_{\gamma \in \Gamma^{(2)}(Q_7)} \mathrm{lk}(f(\gamma)) \pmod 2.$$

3 Conway-Gordon type theorems over integers

Conway-Gordon theorems give dependent relations on the invariants of constituent knots and links in a spatial graph over $\mathbb{Z}_2$. In this section, we consider Conway-Gordon type theorems over integers. It is known that the Conway-Gordon theorems for K_6 and K_7 have integral lifts as follows.

Theorem 3.1 (Nikkuni [12])

(1) For any element f in $\mathrm{SE}(K_6)$, it follows that

$$2\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - 2\sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(K_6)} \mathrm{lk}(f(\gamma))^2 - 1.$$

(2) For any element f in $\mathrm{SE}(K_7)$, it follows that

$$\begin{aligned} & 7\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 6\sum_{\gamma \in \Gamma_6(K_7)} a_2(f(\gamma)) - 2\sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma)) \\ &= 2\sum_{\gamma \in \Gamma^{(2)}_{3,4}(K_7)} \mathrm{lk}(f(\gamma))^2 - 21, \end{aligned}$$

where $\Gamma^{(2)}_{k,l}(G)$ denotes the set of all pairs of two disjoint cycles consisting of a k-cycle and a l-cycle of G.

Note that Theorem 1.1 (1) and (2) can be obtained from Theorem 3.1 (1) and (2) respectively by taking the modulo two reduction. Then, by combining Theorem 2.2 with Theorem 3.1 in a similar way as Example 2.3, it can be shown the following.

Theorem 3.2 (Nikkuni-Taniyama [13])

(1) *Let G be an element in $\mathcal{F}_\triangle(K_6)$. Then, there exist a map ω from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element f in* $\mathrm{SE}(G)$, *it follows that*

$$2\sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 - 1.$$

(2) *Let G be an element in $\mathcal{F}_{\triangle}(K_7)$. Then, there exists a map ω from $\bar{\Gamma}(G)$ to $\mathbb{Z}$ such that for any element f in* SE(G), *it follows that*

$$\sum_{\gamma\in\Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = 2 \sum_{\gamma\in\Gamma^{(2)}(G)} \omega(\gamma)\mathrm{lk}(f(\gamma))^2 - 21.$$

Remark 3.3 Recall that $\mathcal{F}(K_6) \setminus \mathcal{F}_{\triangle}(K_6) = \{P_7\}$. It is known that P_7 is also a minor-minimal intrinsically linked graph [16], and O'Donnol showed that Theorem 3.2 (1) also holds for P_7 [14]. Therefore Theorem 3.2 (1) holds for any graph in the K_6-family.

By taking the modulo two reduction on Theorem 3.2, we immediately have the following.

Corollary 3.4 (1) (Sachs [16], Taniyama-Yasuhara [17]) *Let G be an element in $\mathcal{F}(K_6)$. Then, for any element f in* SE(G), *it follows that*

$$\sum_{\gamma\in\Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma)) \equiv 1 \pmod 2.$$

(2) *Let G be an element in $\mathcal{F}_{\triangle}(K_7)$. Then, there exists a subset Γ of $\Gamma(G)$ such that for any element f in* SE(G), *it follows that*

$$\sum_{\gamma\in\Gamma} a_2(f(\gamma)) \equiv 1 \pmod 2.$$

Remark 3.5 Recall that $\mathcal{F}(K_7) \setminus \mathcal{F}_{\triangle}(K_7) = \{N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, N'_{12}\}$. It is known that any graph in $\mathcal{F}(K_7) \setminus \mathcal{F}_{\triangle}(K_7)$ is not intrinsically knotted [3], [7], [6].

In Theorem 3.2, the proof of the existence of a map ω is constructive. So we can "theoretically" give $\omega(\gamma)$ for each element γ in $\bar{\Gamma}(G)$ concretely (but it is accompanied by a complicated work to carry it out). For a map $\omega : \Gamma(G) \to \mathbb{Z}$ in Theorem 3.2 (1), Hashimoto-Nikkuni gave $\omega(\gamma)$ for each element γ in $\Gamma(G)$ [8].

Example 3.6 In Theorem 3.2 (2), let us consider the case that $G = C_{14}$, namely G is the Heawood graph. We define a map $\omega : \bar{\Gamma}(C_{14}) \to \mathbb{Z}$ by

$$\omega(\gamma) = \begin{cases} 7 & \text{if } \gamma \in \Gamma_{14}(C_{14}) \\ 15 & \text{if } \gamma \in \Gamma_{12}(C_{14}) \\ -6 & \text{if } \gamma \in \Gamma_{10}(C_{14}) \\ -32 & \text{if } \gamma \in \Gamma_{8}(C_{14}) \\ -12 & \text{if } \gamma \in \Gamma_{6}(C_{14}) \\ 2 & \text{if } \gamma \in \Gamma^{(2)}(C_{14}) = \Gamma^{(2)}_{6,6}(C_{14}) \\ 0 & \text{otherwise} \end{cases}$$

for an element γ in $\bar{\Gamma}(C_{14})$ (since C_{14} is bipartite, we have $\Gamma_k(C_{14}) = \emptyset$ if k is odd). Then it can be shown that

$$\sum_{\gamma\in\Gamma(C_{14})} \omega(\gamma)a_2(f(\gamma)) = 2 \sum_{\gamma\in\Gamma^{(2)}(C_{14})} \omega(\gamma)\mathrm{lk}(f(\gamma))^2 - 21$$

for any element f in $\mathrm{SE}(C_{14})$. This implies that

$$\sum_{\gamma\in\Gamma_{12}(C_{14})\cup\Gamma_{14}(C_{14})} a_2(f(\gamma)) \equiv 1 \pmod 2$$

for any element f in $\mathrm{SE}(C_{14})$. Let f and g be two elements in $\mathrm{SE}(C_{14})$ as illustrated in Fig. 3.1. Then it can be shown that $f(C_{14})$ contains exactly one nontrivial knot $f(\gamma_1)$ which is drawn by bold lines, where γ_1 is an element in $\Gamma_{14}(C_{14})$ (such a spatial embedding of C_{14} was exhibited by Kohara-Suzuki first [10]). On the other hand, $g(C_{14})$ contains exactly one nontrivial knot $g(\gamma_2)$ which is drawn by bold lines, where γ_2 is an element in $\Gamma_{12}(C_{14})$. As far as the author knows, g is a first example of a spatial embedding of C_{14} whose image does not contain a nontrivial Hamiltonian knot.

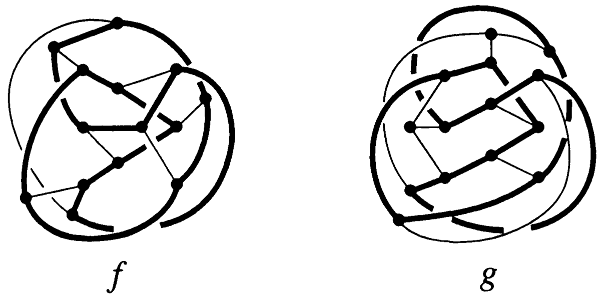


Figure 3.1. Two elements f and g in $\mathrm{SE}(C_{14})$

4 Conway-Gordon type theorem for $K_{3,3,1,1}$

Let $K_{3,3,1,1}$ be the graph as illustrated in Fig. 4.1, which is one of the *complete four-partite graph* on 8 vertices. In [11], Motwani-Raghunathan-Saran claimed that it may be proven that $K_{3,3,1,1}$ is intrinsically knotted by using the same technique of Conway-Gordon theorem for K_7, namely, by showing that for any element in $\mathrm{SE}(K_{3,3,1,1})$, the sum of a_2 over all of the Hamiltonian knots is always congruent to one modulo two. But Kohara-Suzuki showed in [10] that the claim did not hold, that is, the sum of a_2 over all of the Hamiltonian knots is dependent to each element in $\mathrm{SE}(K_{3,3,1,1})$. Actually, they demonstrated the specific two elements f and g in $\mathrm{SE}(K_{3,3,1,1})$ as illustrated in Fig. 4.2. Then $f(K_{3,3,1,1})$ contains exactly one nontrivial knot $f(\gamma_0)$ (= a trefoil knot) which is drawn by bold lines, where γ_0 is a Hamiltonian cycle of $K_{3,3,1,1}$, and $g(K_{3,3,1,1})$ contains exactly two nontrivial knots $g(\gamma_1)$ and $g(\gamma_2)$ (= two trefoil knots) which are drawn by bold lines, where γ_1 and γ_2 are also Hamiltonian cycles of $K_{3,3,1,1}$. Thus the situation of the case of $K_{3,3,1,1}$ is different from the case of K_7.

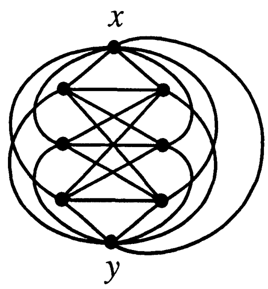


Figure 4.1. $K_{3,3,1,1}$

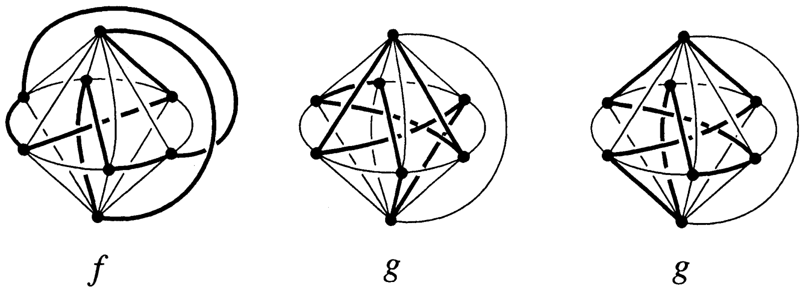


Figure 4.2. Two elements f and g in $\mathrm{SE}(K_{3,3,1,1})$

By using another technique different from Conway-Gordon's, Foisy proved the following.

Theorem 4.1 (Foisy [4]) *For any element f in $\mathrm{SE}(K_{3,3,1,1})$, there exists an element γ in $\cup_{k=4}^{8}\Gamma_k(K_{3,3,1,1})$ such that $a_2(f(\gamma)) \equiv 1 \pmod 2$.*

Corollary 4.2 *$K_{3,3,1,1}$ is intrinsically knotted.*

Proposition 1.3 (2) and Corollary 4.2 implies that any element G in $\mathcal{F}_\triangle(K_{3,3,1,1})$ is also intrinsically knotted. Note that the number of the elements in $\mathcal{F}(K_{3,3,1,1})$ is fifty eight, and the number of the elements in $\mathcal{F}_\triangle(K_{3,3,1,1})$ is twenty six. Since Kohara-Suzuki pointed out that each of the proper minors of G is not intrinsically knotted [10], it follows that any element in $\mathcal{F}_\triangle(K_{3,3,1,1})$ is minor-minimal with respect to the intrinsic knottedness. Note that a $\triangle Y$-exchange does not change the number of edges of a graph. Since K_7 and $K_{3,3,1,1}$ have different numbers of edges, the families $\mathcal{F}_\triangle(K_7)$ and $\mathcal{F}_\triangle(K_{3,3,1,1})$ are disjoint.

On the other hand, Hashimoto-Nikkuni showed the following Conway-Gordon type theorem for $K_{3,3,1,1}$ over integers. Here, x and y denote the exactly two vertices of $K_{3,3,1,1}$ with valency 7 .

Theorem 4.3 (Hashimoto-Nikkuni [9])

(1) *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$, *it follows that*

$$\begin{aligned} & 4\sum_{\gamma\in\Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4\sum_{\substack{\gamma\in\Gamma_7(K_{3,3,1,1})\\ \gamma\not\supset\{x,y\}}} a_2(f(\gamma)) \\ & -4\sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ \gamma\cap\{x,y\}\neq\emptyset}} a_2(f(\gamma)) - 4\sum_{\substack{\gamma\in\Gamma_5(K_{3,3,1,1})\\ \gamma\not\supset\{x,y\}}} a_2(f(\gamma)) \\ = & \sum_{\gamma\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \mathrm{lk}(f(\gamma))^2 + 2\sum_{\gamma\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \mathrm{lk}(f(\gamma))^2 - 18. \end{aligned}$$

(2) *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$, *it follows that*

$$\sum_{\gamma\in\Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \mathrm{lk}(f(\gamma))^2 + 2\sum_{\gamma\in\Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \mathrm{lk}(f(\gamma))^2 \geq 22.$$

By combining Theorem 4.3 (1) and (2), we immediately have the following, which gives an alternative proof of Corollary 4.2.

Corollary 4.4 *For any element f in* $\mathrm{SE}(K_{3,3,1,1})$, *it follows that*

$$\begin{aligned} & \sum_{\gamma\in\Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_7(K_{3,3,1,1})\\ \gamma\not\supset\{x,y\}}} a_2(f(\gamma)) \\ & - \sum_{\substack{\gamma\in\Gamma_6(K_{3,3,1,1})\\ \gamma\cap\{x,y\}\neq\emptyset}} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_5(K_{3,3,1,1})\\ \gamma\not\supset\{x,y\}}} a_2(f(\gamma)) \geq 1. \end{aligned} \tag{4.1}$$

In particular, $K_{3,3,1,1}$ is intrinsically knotted.

From a point of identyfing the place of nontrivial knots in $f(K_{3,3,1,1})$, Corollary 4.4 is a refinement of Theorem 4.1. We also remark here that we see the left side of (4.1) is not always congruent to one modulo two by considering two elements f and g in $\mathrm{SE}(K_{3,3,1,1})$ as illustrated in Fig. 4.2. Thus Corollary 4.4 shows that the argument over integers has a nice advantage.

As an application of Theorem 2.2, a Conway-Gordon type theorem over integers for any element in $\mathcal{F}_{\triangle}(K_{3,3,1,1})$ also can be produced by Corollary 4.4.

Theorem 4.5 (Hashimoto-Nikkuni [9]) *Let G be an element in $\mathcal{F}_{\triangle}(K_{3,3,1,1})$. Then, there exist a map ω from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element f in* SE(G)*, it follows that*

$$\sum_{\gamma\in\Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \geq 1.$$

Remark 4.6 In addition to the elements in $\mathcal{F}_{\triangle}(K_7)\cup\mathcal{F}_{\triangle}(K_{3,3,1,1})$, many minor-minimal intrinsically knotted graph are known [5], [6]. In particular, it has been announced in [6] by Goldberg-Mattman-Naimi that all of the thirty two elements in $\mathcal{F}(K_{3,3,1,1})\setminus\mathcal{F}_{\triangle}(K_{3,3,1,1})$ are minor-minimal intrinsically knotted graphs. Note that their method is based on Foisy's idea in the proof of Theorem 4.1 with the help of a computer.

Remark 4.7 Conway-Gordon type theorems may have applications to molecular topology. A spatial graph is said to be *rectilinear* if each of the edges is a straight line segment in $\mathbb{R}^3$. A rectilinear spatial graph appears in polymer chemistry as a mathematical model for chemical compounds (see [1], for example). For example, as applications of Theorem 3.1 and Theorem 4.3, we can show that the image of a rectilinear spatial embedding of K_7 always contains a nontrivial Hamiltonian knot which is ambient isotopic to a trefoil knot [12], and the image of a rectilinear spatial embedding of $K_{3,3,1,1}$ always contains a nontrivial Hamiltonian knot [9].

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