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Kyoto University
MIXED PENTAGON EQUATION AND DOUBLE SHUFFLE RELATION

HIDEKAZU FURUSHO

ABSTRACT. This paper is a review of the paper [F4] where a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple $L$-values is given. In precise, it is shown that Enriquez’ mixed pentagon equation implies the relations.

0. INTRODUCTION

Multiple $L$-values $L(k_1, \ldots, k_m; \zeta_1, \ldots, \zeta_m)$ are the complex numbers defined by the following series

$$L(k_1, \ldots, k_m; \zeta_1, \ldots, \zeta_m) := \sum_{0<n_1<\cdots<n_m} \frac{\zeta_1^{n_1} \cdots \zeta_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

for $m, k_1, \ldots, k_m \in \mathbb{N}(=\mathbb{Z}_{>0})$ and $\zeta_1, \ldots, \zeta_m \in \mu_N(:the$ group of $N$-th roots of unity in $\mathbb{C}$). They converge if and only if $(k_m, \zeta_m) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have been discussed in several papers [AK, BK, G, R] etc. Multiple $L$-values appear as coefficients of the cyclotomic Drinfel’d associator $\Phi_{KZ}^N$ (5) in $U \mathfrak{F}_{N+1}$: the non-commutative formal power series ring with $N + 1$ variables $A$ and $B(a)$ ($a \in \mathbb{Z}/N\mathbb{Z}$).

The mixed pentagon equation (4) is a geometric equation introduced by Enriquez [E]. The series $\Phi_{KZ}^N$ satisfies the equation, which yields non-trivial relations among multiple $L$-values. The generalised double shuffle relation (the double shuffle relation and the regularization relation) is a combinatorial relation among multiple $L$-values. It is formulated as (6) for $h = \Phi_{KZ}^N$. It is Zhao’s remark [Z] that for specific $N$’s the generalised double shuffle relation does not provide all the possible relations among multiple $L$-values.

Our main theorem is an implication of the generalised double shuffle relation (6) from the mixed pentagon equation (4).

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Theorem 1. Let $U\mathfrak{F}_{N+1}$ be the universal enveloping algebra of the free Lie algebra $\mathfrak{F}_{N+1}$ with variables $A$ and $B(a)$ $(a \in \mathbb{Z}/N\mathbb{Z})$. Let $h$ be a group-like element in $U\mathfrak{F}_{N+1}$ with $c_{B(0)}(h) = 0$ satisfying the mixed pentagon equation (4) with a group-like series $g \in U\mathfrak{F}_{2}$. Then $h$ also satisfies the generalised double shuffle relation (6).

The contents of the article are as follows: We recall the mixed pentagon equation in §1 and the generalised double shuffle relation in §2. In §3 we calculate the 0-th cohomologies of Chen’s reduced bar complex for the Kummer coverings of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Two variable cyclotomic multiple polylogarithms and their associated bar elements there are introduced in §4. By using them, we prove theorem 1 in §5.

1. Mixed pentagon equation

This section is to recall Enriquez’ mixed pentagon equation [E].

Let us fix notations: For $n \geq 2$, the Lie algebra $t_{n}$ of infinitesimal pure braids is the completed $\mathbb{Q}$-Lie algebra with generators $t_{ij}$ ($i \neq j$, $1 \leq i, j \leq n$) and relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$ and $[t_{ij}, t_{kl}] = 0$ for all distinct $i, j, k, l$. We note that $t_{2}$ is the 1-dimensional abelian Lie algebra generated by $t_{12}$. The element $z_{n} = \sum_{1 \leq i < j \leq n} t_{ij}$ is central in $t_{n}$. Put $t_{n}^{0}$ to be the Lie subalgebra of $t_{n}$ with the same generators except $t_{in}$ and the same relations as $t_{n}$. Then we have $t_{n} = t_{n}^{0} \oplus \mathbb{Q} \cdot z_{n}$. Especially when $n = 3$, $t_{3}^{0}$ is a free Lie algebra $\mathfrak{F}_{2}$ of rank 2 with generators $A := t_{12}$ and $B = t_{23}$. For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $t_{n} \to t_{m} : x \mapsto x^{f} = x^{f^{-1}(1), \ldots, f^{-1}(n)}$ is uniquely defined by $(t_{ij})^{f} = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t_{i'j'}$.

For a pair $(\mu, g) \in \mathbb{Q} \times \exp \mathfrak{F}_{2}$ the pentagon equation is the following equation in $\exp t_{4}^{0}$

$$g^{1,2,3,4} g^{12,3,4} = g^{2,3,4} g^{1,23,4} g^{1,2,3}$$

and two hexagon equations the following two equations in $\exp \mathfrak{F}_{2} = \exp t_{4}^{0}$

$$g(A, B) g(B, A) = 1$$

and

$$\exp \{ \frac{\mu A}{2} \} g(C, A) \exp \{ \frac{\mu C}{2} \} g(B, C) \exp \{ \frac{\mu B}{2} \} g(A, B) = 1$$

with $C = -A - B$. These

By our notation, the equation (2) can be read as

$$g(t^{12}, t^{23} + t^{24}) g(t^{13} + t^{23}, t^{34}) = g(t^{23}, t^{34}) g(t^{12} + t^{13}, t^{24} + t^{34}) g(t^{12}, t^{23})$$

Remark 2. It is shown in [F2] that the two hexagon equations (3) are consequences of the pentagon equation (2).
Remark 3. The Drinfel’d associator $\Phi_{KZ} = \Phi_{KZ}(A, B) \in C\langle\langle A, B\rangle\rangle$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where $G_0$ and $G_1$ are the solutions of the formal KZ equation

$$\frac{d}{dz} G(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)$$

such that $G_0(z) \approx z^A$ when $z \to 0$ and $G_1(z) \approx (1 - z)^B$ when $z \to 1$ (cf. [Dr]). The series has the following expression

$$\Phi_{KZ} = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) A^{k_m-1} B \cdots A^{k_1-1} B + \text{(regularized terms)}$$

and the regularised terms are explicitly calculated to be linear combinations of multiple zeta values $\zeta(k_1, \ldots, k_m) = L(k_1, \ldots, k_m; 1, \ldots, 1)$ in [F1] proposition 3.2.3 by Le-Murakami’s method [LM]. It is shown in [Dr] that the pair $(2\pi \sqrt{-1}, \Phi_{KZ})$ satisfies the pentagon equation (2) and the hexagon equations (3).

For $n \geq 2$ and $N \geq 1$, the Lie algebra $t_{n,N}$ is the completed $\mathbb{Q}$-Lie algebra with generators $t^{ij}$ ($2 \leq i \leq n$), $t(a)^{ij}$ ($i \neq j$, $2 \leq i, j \leq n$, $a \in \mathbb{Z}/N\mathbb{Z}$) and relations $t(a)^{ij} = t(-a)^{ji}$, $[t(a)^{ij}, t(a+b)^{jk} + t(b)^{jk}] = 0$, $[t^{ij} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(c)^{ij}, t(a)^{ij}] = 0$, $[t^{ij}, t(a)^{jk}] = 0$ and $[t(a)^{ij}, t(b)^{kl}] = 0$ for all $a, b \in \mathbb{Z}/N\mathbb{Z}$ and all distinct $i, j, k, l$ ($2 \leq i, j, k, l \leq n$). We note that $t_{n,1}$ is equal to $t_n$ for $n \geq 2$. We have a natural injection $t_{n-1,N} \hookrightarrow t_{n,N}$. The Lie subalgebra $f_{n,N}$ of $t_{n,N}$ generated by $t^{1n}$ and $t(a)^{in}$ ($2 \leq i \leq n - 1$, $a \in \mathbb{Z}/N\mathbb{Z}$) is free of rank $(n - 2)N + 1$ and forms an ideal of $t_{n,N}$. Actually it shows that $t_{n,N}$ is a semi-direct product of $f_{n,N}$ and $t_{n-1,N}$. The element $z_{n,N} = \sum_{1 \leq i < j \leq n} t^{ij}$ with $t^{ij} = \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(a)^{ij}$ ($2 \leq i < j \leq n$) is central in $t_{n,N}$. Put $t_{0,N}$ to be the Lie subalgebra of $t_{n,N}$ with the same generators except $t^{1n}$. Then we have $t_{n,N} = t_{0,N} \oplus \mathbb{Q} \cdot z_{n,N}$. Occasionally we regard $t_{0,N}^0$ as the quotient $t_{n,N}/\mathbb{Q} \cdot z_{n,N}$. Especially when $n = 3$, $t_{0,3}^0$ is free Lie algebra $\mathfrak{F}_{N+1}$ of rank $N + 1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ ($a \in \mathbb{Z}/N\mathbb{Z}$).

For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $f(1) = 1$, the Lie algebra morphism $t_{n,N} \to t_{m,N} : x \mapsto x^f = x^{f^{-1}(1)} \cdots f^{-1}(n)$ is uniquely defined by $(t(a)^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t(a)^{i'j'}$ ($i \neq j$, $2 \leq i, j \leq n$) and $(t^{ij})^f = \sum_{j' \in f^{-1}(j)} t^{ij} + \frac{1}{2} \sum_{j', j'' \in f^{-1}(j)} \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(c)^{j'j''}$ + $\sum_{i' \neq 1 \in f^{-1}(1), j' \in f^{-1}(j)} \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(c)^{i'j'}$ ($2 \leq j \leq n$). Again for a partially defined map $g : \{2, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $t_n \to t_{m,N} : x \mapsto x^g = x^{g^{-1}(1)} \cdots g^{-1}(n)$ is uniquely defined by $(t^{ij})^g = \sum_{i' \in g^{-1}(i), j' \in g^{-1}(j)} t(0)^{i'j'}$ ($i \neq j$, $1 \leq i, j \leq n$).
For a pair \((g, h) \in \exp \mathfrak{F}_2 \times \exp \mathfrak{F}_{N+1}\), the \textit{mixed pentagon equation} means the following equation in \(\exp t_{4,N}^{0}\)

\[
(4) \quad h^{1,2,34}h^{12,3,4} = g^{2,3,4}h^{1,23,4}h^{1,2,3}.
\]

By our notation, each term in the equation (4) can be read as

\[
\begin{align*}
    h^{1,2,34} &= h(t^{12}, t^{23}(0) + t^{24}(0), t^{23}(1) + t^{24}(1), \ldots, t^{23}(N-1) + t^{24}(N-1)), \\
    h^{12,3,4} &= h(t^{13} + \sum_c t^{23}(c), t^{34}(0), t^{34}(1), \ldots, t^{34}(N-1)), \\
    g^{2,3,4} &= g(t^{23}(0), t^{34}(0)), \\
    h^{1,23,4} &= h(t^{12} + t^{13} + \sum_c t^{23}(c), t^{24}(0) + t^{34}(0), \ldots, t^{24}(N-1) + t^{34}(N-1)), \\
    h^{1,2,3} &= h(t^{12}, t^{23}(0), t^{23}(1), \ldots, t^{23}(N-1)).
\end{align*}
\]

**Remark 4.** In [E], the cyclotomic analogue \(\Phi_{KZ}^N \in \exp \mathfrak{F}_{N+1}(C)\) of the Drinfel’d associator is introduced to be the renormalised holonomy from 0 to 1 of the KZ-like differential equation

\[
\frac{d}{dz} H(z) = \left( \frac{A}{z} + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \frac{B(a)}{z - \zeta_N^a} \right) H(z)
\]

with \(\zeta_N = \exp\{\frac{2\pi\sqrt{-1}}{N}\}\), i.e., \(\Phi_{KZ}^N = H_1^{-1}H_0\) where \(H_0\) and \(H_1\) are the solutions such that \(H_0(z) \approx z^A\) when \(z \to 0\) and \(H_1(z) \approx (1 - z)^B(0)\) when \(z \to 1\) (cf.[E]). There appear multiple \(L\)-values (1) in each of its coefficient;

\[
(5) \quad \Phi_{KZ}^N = 1 + \sum (-1)^m L(k_1, \cdots, k_m; \xi_1, \ldots, \xi_m) A^{k_m-1}B(a_m) \cdots A^{k_1-1}B(a_1) + \text{(regularized terms)}
\]

with \(\xi_1 = \zeta_N^{a_2-a_1}, \ldots, \xi_{m-1} = \zeta_N^{a_m-a_{m-1}}\) and \(\xi_m = \zeta_N^{-a_m}\), where the regularised terms can be explicitly calculated to combinations of multiple \(L\)-values by the method of Le-Murakami [LM]. In [E] it is shown that the triple \((2\pi\sqrt{-1}, \Phi_{KZ}, \Phi_{KZ}^N)\) satisfies the mixed pentagon equation (4). This is achieved by considering monodromy in the pentagon formed by the divisors \(y = 0, x = 1, \) the exceptional divisor of the blowing-up at \((1, 1), y = 1\) and \(x = 0\) in \(\mathcal{M}_{0,5}^{(N)}\) (see §3).

**Remark 5.** In [EF] it is proved that the mixed pentagon equation (4) implies the distribution relation for a specific case and that the octagon equation follows from the mixed pentagon equation and the special action condition for \(N = 2\).
2. Double shuffle relation

This section is to recall the generalised double shuffle relation in Racinet’s setting [R].

Let us fix notations: Let $\mathcal{F}_{Y_N}$ be the completed graded Lie $Q$-algebra generated by $Y_{n,a}$ ($n \geq 1$ and $a \in \mathbb{Z}/N\mathbb{Z}$) with $\deg Y_{n,a} = n$. Put $U\mathcal{F}_{Y_N}$ its universal enveloping algebra: the non-commutative formal series ring with free variables $Y_{n,a}$ ($n \geq 1$ and $a \in \mathbb{Z}/N\mathbb{Z}$). Let $\pi_Y : U\mathcal{F}_{Y_N} \to U\mathcal{F}_{Y_N}$ be the $Q$-linear map between non-commutative formal power series rings that sends all the words ending in $A$ to zero and the word $A^{n_m-1}B(a_m)\cdots A^{n_1-1}B(a_1)$ ($n_1, \ldots, n_m \geq 1$ and $a_1, \ldots, a_m \in \mathbb{Z}/N\mathbb{Z}$) to

$$(-1)^m Y_{n_m,-a_m}Y_{n_m-1,a_m-a_{m-1}}\cdots Y_{n_1,a_2-a_1}.$$

Define the coproduct $\Delta_*$ of $U\mathcal{F}_{Y_N}$ by $\Delta_* Y_{n,a} = \sum_{k+l=n, b+c=a} Y_{k,b} \otimes Y_{l,c}$ ($n \geq 0$ and $a \in \mathbb{Z}/N\mathbb{Z}$) with $Y_{0,a} := 1$ if $a = 0$ and $0$ if $a \neq 0$. For $h = \sum_{W: \text{word}} c_W(h) W \in U\mathcal{F}_{N+1}$, define the series shuffle regularization $h_* = h_{corr} \cdot \pi_Y(h)$ with the correction term $h_{corr} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{A^{n-1}B(0)}(h) Y_{1,0}^n \right)$.

For a series $h \in \exp \mathcal{F}_{N+1}$ the generalised double shuffle relation stands for the following relation in $U\mathcal{F}_{Y_N}$

$$\Delta_*(h_*) = h_* \otimes h_* \wedge.$$

Remark 6. The series $\Phi_{KZ}^{N} (5)$ satisfies the generalised double shuffle relation (6) because regularised multiple $L$-values satisfy the double shuffle relation.

3. Bar constructions

This section gives a review of the notion of the reduced bar construction and calculates it for $\mathcal{M}_{0,4}^{(N)}$ and $\mathcal{M}_{0,5}^{(N)}$.

We recall the notion of Chen’s reduced bar construction [C]. Let $(A^* = \oplus_{q=0}^{\infty} A^q, d)$ be a differential graded algebra (DGA). The reduced bar complex $\bar{B}^*(A)$ is the tensor algebra $\oplus_{r=0}^{\infty} (\bar{A})^\otimes r$ with $\bar{A} = \oplus_{i=0}^{\infty} \bar{A}^i$ where $\bar{A}^0 = A^1/dA^0$ and $\bar{A}^i = A^{i+1}$ ($i > 0$). We denote $a_1 \otimes \cdots \otimes a_r$ ($a_i \in \bar{A}^* = \bar{A}$) by $[a_1] \cdots [a_r]$. The degree of elements in $\bar{B}^*(A)$ is given by the total degree of $\bar{A}^*$. Put $Ja = (-1)^{p-1}a$ for $a \in \bar{A}^p$. Define

$$d'[a_1] \cdots [a_k] = \sum_{i=1}^{k} (-1)^i [Ja_1] \cdots [Ja_{i-1}]d_a[a_i][a_{i+1}] \cdots [a_k]$$
and
\[ d''[a_1| \cdots |a_k] = \sum_{i=1}^{k} (-1)^{i-1} [Ja_1| \cdots |Ja_{i-1}|Ja_{i+1}|a_{i+2}| \cdots |a_k]. \]

Then \( d' + d'' \) forms a differential. The differential and the shuffle product (loc.cit.) give \( \overline{B}^*(A) \) a structure of commutative DGA. Actually it also forms a Hopf algebra, whose coproduct \( \Delta \) is given by
\[ \Delta([a_1| \cdots |a_r]) = \sum_{s=0}^{r} [a_1| \cdots |a_s] \otimes [a_{s+1}| \cdots |a_r]. \]

For a smooth complex manifold \( \mathcal{M} \), \( \Omega^*(\mathcal{M}) \) means the de Rham complex of smooth differential forms on \( \mathcal{M} \) with values in \( \mathbb{C} \). We denote the 0-th cohomology of the reduced bar complex \( \overline{B} \cdot (\Omega(\mathcal{M})) \) with respect to the differential by \( H^0 \overline{B}(\mathcal{M}) \).

Let \( \mathcal{M}_{0,4} \) be the moduli space \( \{(x_1, \cdots, x_4) \in (\mathbb{P}^1)^4 | x_i \neq x_j (i \neq j) \}/\text{PGL}_2(\mathbb{C}) \) of 4 different points in \( \mathbb{P}^1 \). It is identified with \( \{z \in \mathbb{P}^1_{\mathbb{C}} | z \neq 0, 1, \infty \} \) by sending \( [(0, z, 1, \infty)] \) to \( z \). Denote its Kummer \( N \)-covering \( G_{m} \backslash \mu_{N} = \{z \in \mathbb{P}^1_{\mathbb{C}} | z^{N} \neq 0, 1, \infty \} \) by \( \mathcal{M}_{0,4}^{(N)} \).

The space \( H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \) is generated by \( \omega_0 := d\log(z) \) and \( \omega_\zeta := d\log(z-\zeta) \) \( (\zeta \in \mu_{N}) \).

We have an identification \( H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \simeq U\mathfrak{F}_{N+1} \otimes \mathbb{C} \) by \( \text{Exp} \Omega_{4}^{(N)} := \sum X_{i_{m}} \cdots X_{i_{1}} \otimes [\omega_{i_{m}}| \cdots |\omega_{i_{1}}] \in U\mathfrak{F}_{N+1} \otimes_{\mathbb{Q}} H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \). Here the sum is taken over \( m \geq 0 \) and \( i_1, \cdots, i_m \in \{0\} \cup \mu_{N} \) and \( X_0 = A \) and \( X_\zeta = B(a) \) when \( \zeta = \zeta_{N}^{a} \). It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product \( l_1 \cdot l_2 \in H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \) for \( l_1, l_2 \in H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \) is given by \( l_1 \cdot l_2(f) := \sum_i l_1(f^{(i)}_1)l_2(f^{(i)}_2) \) for \( f \in U\mathfrak{F}_{N+1} \otimes \mathbb{C} \) with \( \Delta(f) = \sum_i f^{(i)}_1 \otimes f^{(i)}_2 \). Occasionaly we regard \( H^0 \overline{B}(\mathcal{M}_{0,4}^{(N)}) \) as the regular function ring of \( F_{N+1}(\mathbb{C}) = \{g \in U\mathfrak{F}_{N+1} \otimes \mathbb{C} | g : \text{group-like} \} \). Let \( \mathcal{M}_{0,5} \) be the moduli space \( \{(x_1, \cdots, x_5) \in (\mathbb{P}^1)^5 | x_i \neq x_j (i \neq j) \}/\text{PGL}_2(\mathbb{C}) \) of 5 different points in \( \mathbb{P}^1 \). It is identified with \( \{(x, y) \in \mathbb{G}^2_{m} | x \neq 1, y \neq 1, xy \neq 1 \} \) by sending \( [(0, xy, y, 1, \infty)] \) to \( (x, y) \). Denote its Kummer \( N^2 \)-covering
\[ \{(x, y) \in \mathbb{G}^2_{m} | x^N \neq 1, y^N \neq 1, (xy)^N \neq 1 \} \]
by $\mathcal{M}_{0,5}^{(N)}$. It is identified with $W_N/C^\times$ by $(x, y) \mapsto (xy, y, 1)$ where

$$W_N = \{(z_2, z_3, z_4) \in G_m| z_i^N \neq z_j^N (i \neq j)\}.$$

The space $H^0\tilde{B}(\mathcal{M}_{0,5}^{(N)})$ is a subspace of the tensor coalgebra generated by

$$\omega_{1,i} := d\log z_i$$

and $\omega_{i,j}(a) := d\log(z_i - \zeta_N^a z_j)$ $(2 \leq i, j \leq 4, a \in \mathbb{Z}/N\mathbb{Z})$.

**Proposition 7.** We have an identification

$$H^0\tilde{B}(\mathcal{M}_{0,5}^{(N)}) \simeq (Ut_{4,N}^0)^* \otimes \mathbb{C}.$$

**Proof.** By [K], $H^0\tilde{B}(W_N)$ can be calculated to be the 0-th cohomology $H^0\tilde{B}^*(S)$ of the reduced bar complex of the Orlik-Solomon algebra $S^*$. The algebra $S^*$ is the (trivial-)differential graded $\mathbb{C}$-algebra $S^* = \oplus_{q=0}^\infty S^q$ defined by generators

$$\omega_{1,i} = d\log z_i$$

and $\omega_{i,j}(a) = d\log(z_i - \zeta_N^a z_j)$ $(2 \leq i, j \leq 4, a \in \mathbb{Z}/\mathbb{Z})$ in degree 1 and relations

$$\omega_{i,j}(a) = \omega_{j,i}(-a), \quad \omega_{ij}(a) \wedge \{\omega_{ik}(a + b) + \omega_{kj}(b)\} = 0,$$

$$\{\omega_{1i} + \omega_{1j} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \omega(c)_{ij}\} \wedge \omega(a)_{ij} = 0,$$

$$\omega_{1i} \wedge \{\omega_{1j} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \omega(c)_{ij}\} = 0,$$

$$\omega_{1i} \wedge \omega(a)_{jk} = 0 \quad \text{and} \quad \omega(a)_{ij} \wedge \omega(b)_{kl} = 0$$

for all $a, b \in \mathbb{Z}/N\mathbb{Z}$ and all distinct $i, j, k, l$ $(2 \leq i, j, k, l \leq n)$. By direct calculation, the element

$$\sum_{i=2}^4 t_{1i} \otimes \omega_{1i} + \sum_{2 \leq i < j \leq 4, a \in \mathbb{Z}/N\mathbb{Z}} t_{ij}(a) \otimes \omega_{ij}(a) \in (t_{4,N})^{\deg=1} \otimes S^1$$

yields a Hopf algebra identification of $H^0\tilde{B}(W_N)$ with $(Ut_{4,N}^0)^* \otimes \mathbb{C}$ since both are quadratic.

By the long exact sequence of cohomologies induced from the $G_m$-bundle $W_N \to \mathcal{M}_{0,5}^{(N)} = W_N/C^\times$, we get

$$0 \to H^1(\mathcal{M}_{0,5}^{(N)}) \to H^1(W_N) \to H^1(G_m) \to 0$$

and

$$H^i(\mathcal{M}_{0,5}^{(N)}) \simeq H^i(W_N) \quad (i \geq 2).$$

It yields the identification of the subspace $H^0\tilde{B}(\mathcal{M}_{0,5}^{(N)})$ of $H^0\tilde{B}(W_N)$ with $(Ut_{4,N}^0)^* \otimes \mathbb{C}$. $\square$
The above identification is induced from
\[
\text{Exp } \Omega_{5}^{(N)} := \sum t_{J_{m}} \cdots t_{J_{1}} \otimes [\omega_{J_{m}} | \cdots | \omega_{J_{1}}] \in Ut_{4,N}^{0} \otimes_{Q} H^{0} \bar{B}(M_{0,5}^{(N)})
\]
where the sum is taken over \( m \geq 0 \) and \( J_{1}, \cdots, J_{m} \in \{(1, i)|2 \leq i \leq 4\} \cup \{(i, j, a)|2 \leq i < j \leq 4, a \in \mathbb{Z}/N\mathbb{Z}\} \).

Especially the identification between degree 1 terms is given by
\[
\Omega_{5}^{(N)} = t_{12} d \log (xy) + t_{13} d \log y + \sum_{a} t_{23}(a) d \log (y(x - \zeta_{N}^{a})) + \sum_{a} t_{24}(a) d \log (xy - \zeta_{N}^{a}) + \sum_{a} t_{34}(a) d \log (y - \zeta_{N}^{a})
\]
In terms of the coordinate \((x, y)\),
\[
\Omega_{5}^{(N)} = t_{12} d \log (xy) + t_{13} d \log y + \sum_{a} t_{23}(a) d \log (y(x - \zeta_{N}^{a}))
\]
\[
+ \sum_{a} t_{24}(a) d \log (xy - \zeta_{N}^{a}) + \sum_{a} t_{34}(a) d \log (y - \zeta_{N}^{a})
\]
It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product \( l_{1} \cdot l_{2} \in H^{0} \bar{B}(M_{0,5}^{(N)}) \) for \( l_{1}, l_{2} \in H^{0} \bar{B}(M_{0,5}^{(N)}) \) is given by \( l_{1} \cdot l_{2}(f) := \sum_{i} f_{1}^{(i)} l_{2}(f_{2}^{(i)}) \) for \( f \in Ut_{4,N}^{0} \otimes \mathbb{C} \) with \( \Delta(f) = \sum_{i} f_{1}^{(i)} \otimes f_{2}^{(i)} \) (\( \Delta \) : the coproduct of \( Ut_{4,N}^{0} \)).

Occasionally we also regard \( H^{0} \bar{B}(M_{0,5}^{(N)}) \) as the regular function ring of \( K_{4}^{N}(\mathbb{C}) = \{ g \in Ut_{4,N}^{0} \otimes \mathbb{C}| g : \text{group-like}\} \).

By a generalization of Chen’s theory [C] to the case of tangential basepoints, especially for \( M = M_{0,4}^{(N)} \) or \( M_{0,5}^{(N)} \), we have an isomorphism
\[
\rho : H^{0} \bar{B}(M) \simeq I_{0}(M)
\]
as algebras over \( \mathbb{C} \) which sends \( \sum_{I=(i_{m},\cdots,i_{1})} c_{I}[\omega_{i_{m}} | \cdots | \omega_{i_{1}}] \) (\( c_{I} \in \mathbb{C} \)) to \( \sum_{I} c_{I} \int_0 \omega_{i_{m}} \cdots \omega_{i_{1}} \). Here \( \sum_{I} c_{I} \int_0 \omega_{i_{m}} \cdots \omega_{i_{1}} \) means the iterated integral defined by
\[
(7) \sum_{I} c_{I} \int_{0<t_{1}<\cdots<t_{m-1}<t_{m}<1} \omega_{i_{m}}(\gamma(t_{m})) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdots \omega_{i_{1}}(\gamma(t_{1}))
\]
for all analytic paths \( \gamma : (0, 1) \to M(\mathbb{C}) \) starting from the tangential basepoint \( o \) (defined by \( \frac{d}{dx} \) for \( M = M_{0,4}^{(N)} \) and defined by \( \frac{d}{dx} \) and \( \frac{d}{dy} \) for \( M = M_{0,5}^{(N)} \) at the origin in \( M \) (for its treatment see also [De]§15)).
and $I_{o}(\mathcal{M})$ stands for the C-algebra generated by all such homotopy invariant iterated integrals with $m \geq 1$ and $\omega_{1}, \ldots, \omega_{m} \in H_{DR}^{1}(\mathcal{M})$.

4. TWO VARIABLE CYCLOTOMIC MULTIPLE POLYLOGARITHMS

We introduce cyclotomic multiple polylogarithms, $Li_{a}(\overline{\zeta}(z))$ and $Li_{a,b}(\overline{\zeta}(x), \overline{\eta}(y))$, and their associated bar elements, $l^{\bar{a}}_{a}$ and $l^{\bar{a,b}}_{a,b}$, which play important roles to prove our main theorems.

For a pair $(a, \overline{\zeta})$ with $a = (a_{1}, \ldots, a_{k}) \in \mathbb{Z}_{>0}^{k}$ and $\overline{\zeta} = (\zeta_{1}, \ldots, \zeta_{k})$ with $\zeta_{i} \in \mu_{N}$: the group of roots of unity in $\mathbb{C}$ $(1 \leq i \leq k)$, its weight and its depth are defined to be $wt(a, \overline{\zeta}) = a_{1} + \cdots + a_{k}$ and $dp(a, \overline{\zeta}) = k$ respectively. Put $\overline{\zeta}(x) = (\zeta_{1}, \ldots, \zeta_{k-1}, \zeta_{k}x)$. Put $z \in \mathbb{C}$ with $|z| < 1$. Consider the following complex analytic function, one variable cyclotomic multiple polylogarithm

$$Li_{a}(\overline{\zeta}(z)) := \sum_{0 < m_{1} < \cdots < m_{k}} \frac{\zeta_{1}^{m_{1}} \cdots \zeta_{k-1}^{m_{k-1}} (\zeta_{k}z)^{m_{k}}}{m_{1}^{a_{1}} \cdots m_{k-1}^{a_{k-1}} m_{k}^{a_{k}}}.$$ 

It satisfies the following differential equation

$$\frac{d}{dz}Li_{a}(\overline{\zeta}(z)) = \begin{cases} \frac{1}{z}Li_{(a_{1}, \ldots, a_{k-1}, a_{k}-1)}(\overline{\zeta}(z)) & \text{if } a_{k} \neq 1, \\ \frac{1}{\zeta_{1}^{-1}-z} & \text{if } a_{k} = 1, k \neq 1, \\ \frac{1}{z} & \text{if } a_{k} = 1, k = 1. \end{cases}$$

It gives an iterated integral starting from $o$, which lies on $I_{o}(\mathcal{M}_{0,4}^{(N)})$. Actually by the map $\rho$ it corresponds to an element of the $\mathbb{Q}$-structure $U_{F_{N+1}}^{*}$ of $V(\mathcal{M}_{0,4}^{(N)})$ denoted by $l^{\bar{a}}_{a}$. It is expressed as

$$l^{\bar{a}}_{a} = (-1)^{k} c_{\text{word}} \left[ \zeta_{1}^{m_{1}} \cdots \zeta_{k-1}^{m_{k-1}} (\zeta_{k}z)^{m_{k}} \right] \frac{\omega_{1} \cdots \omega_{k-1} \omega_{k} \cdots \omega_{k-1} \omega_{k-1} \cdots \omega_{1}}{m_{1}^{a_{1}} \cdots m_{k-1}^{a_{k-1}} m_{k}^{a_{k}}}.$$ 

By the standard identification $\mu \simeq \mathbb{Z}/NZ$ sending $\zeta_{N} = \exp\{\frac{2\pi \sqrt{-1}}{N}\}$ $\mapsto 1$, for a series $\varphi = \sum_{W: \text{word}} c_{W}(\varphi)W$ it is calculated by

$$l^{\bar{a}}_{a}(\varphi) = (-1)^{k} c_{\text{word}} \left[ \zeta_{N}^{m_{1}} \cdots \zeta_{N}^{m_{k-1}} (\zeta_{N}z)^{m_{k}} \right] \frac{\omega_{1}^{n_{1}} \cdots \omega_{k-1}^{n_{k-1}} \omega_{k}^{n_{k-1}} \cdots \omega_{1}^{n_{1}}}{m_{1}^{a_{1}} \cdots m_{k-1}^{a_{k-1}} m_{k}^{a_{k}}}.$$ 

With $\zeta_{i} = \zeta_{N}^{e_{i}}$ $(e_{i} \in \mathbb{Z}/NZ)$.

For $a = (a_{1}, \ldots, a_{k}) \in \mathbb{Z}_{>0}^{k}$, $b = (b_{1}, \ldots, b_{l}) \in \mathbb{Z}_{>0}^{l}$, $\overline{\zeta} = (\zeta_{1}, \ldots, \zeta_{k})$, $\overline{\eta} = (\eta_{1}, \ldots, \eta_{l})$ with $\zeta_{i}, \eta_{j} \in \mu_{N}$ and $x, y \in \mathbb{C}$ with $|x| < 1$ and $|y| < 1$, consider the following complex function, the two variables multiple polylogarithm

$$Li_{a,b}(\overline{\zeta}(x), \overline{\eta}(y)) := \sum_{0 < n_{1} < \cdots < m_{k} < n_{l}} \frac{\zeta_{1}^{m_{1}} \cdots \zeta_{k-1}^{m_{k-1}} (\zeta_{k}x)^{m_{k}} \cdot \eta_{1}^{n_{1}} \cdots \eta_{l-1}^{n_{l-1}} (\eta_{l}y)^{n_{l}}}{m_{1}^{a_{1}} \cdots m_{k-1}^{a_{k-1}} m_{k}^{a_{k}} \cdot n_{1}^{b_{1}} \cdots n_{l-1}^{b_{l-1}} n_{l}^{b_{l}}}.$$
It satisfies the following differential equations.

\[
\frac{d}{dx} Li_{a,b}(\zeta(x), \eta(y)) = \begin{cases} 
\frac{1}{x} Li_{a_1, \ldots, a_k-1, a_k-1}(\zeta(x), \eta(y)) & \text{if } a_k \neq 1, \\
\frac{1}{\zeta_k - 1} Li_{a_1, \ldots, a_k-1, b}(\zeta_1, \ldots, \zeta_{k-2}, \zeta_{k-1} x, \eta(y)) - \left( \frac{1}{x} + \frac{1}{\zeta_k - 1} \right) & \text{if } a_k = 1, k \neq 1, l \neq 1, \\
\frac{1}{\zeta_k - 1} Li_{a_1, \ldots, a_k-1, b_1}(\zeta_1, \ldots, \zeta_{k-1} \eta_1 x, \eta_2, \ldots, \eta_{l-1}, \eta l y) & \text{if } a_k = 1, k = 1, l \neq 1, \\
\frac{1}{\zeta_k - 1} Li_{a_1, \ldots, a_k-1, b_1}(\zeta_1, \ldots, \zeta_{k-1} x, \eta_1 y) - \left( \frac{1}{x} + \frac{1}{\zeta_k - 1} \right) & \text{if } a_k = 1, k = 1, l = 1, \\
\frac{1}{\zeta_1 - 1} Li_{a_1, \ldots, a_k-1, b_1}(\zeta_1, \ldots, \zeta_{k-1} \eta_1 x y) & \text{if } a_k = 1, k \neq 1, l = 1, \\
\frac{1}{\zeta_1 - 1} Li_{a_1, \ldots, a_k-1, b_1}(\zeta_1, \ldots, \zeta_{k-1} \eta_1 x y) & \text{if } a_k = 1, k = 1, l = 1.
\end{cases}
\]

\[
\frac{d}{dy} Li_{a,b}(\zeta(x), \eta(y)) = \begin{cases} 
\frac{1}{y} Li_{a_1, \ldots, a_k-1, b}(\zeta(x), \eta(y)) & \text{if } b_l \neq 1, \\
\frac{1}{\eta_l - 1} Li_{a_1, \ldots, a_k-1, b}(\zeta, \ldots, \eta_{l-1} x, \eta_l y) & \text{if } b_l = 1, l \neq 1, \\
\frac{1}{\eta_l - 1} Li_{a_1, \ldots, a_k-1, b}(\zeta, \ldots, \eta_{l-1} x) & \text{if } b_l = 1, l = 1.
\end{cases}
\]

By analytic continuation, the functions \( Li_{a,b}(\zeta(x), \eta(y)), Li_{b,a}(\eta(y), \zeta(x)), Li_{a}(\zeta(x)), Li_{a}(\eta(y)) \) and \( Li_{a}(\zeta x y) \) give iterated integrals starting from 0, which lie on \( I_o(\mathcal{M}_{0,5}^{(N)}) \). They correspond to elements of the \( \mathbb{Q} \)-structure \((U^{0}_{4,N})^* \) of \( V(\mathcal{M}_{0,5}^{(N)}) \) by the map \( \rho \) denoted by \( l_{a_1, \ldots, a_k, b}(\zeta(x), \eta(y)), l_{b_1, \ldots, b_l, a}(\zeta(x), \eta(y)) \) and \( l_{a}(\zeta x y) \) respectively. Note that they are expressed as

\[
\sum_{I=(i_m, \ldots, i_1)} c_I \left[ \omega_{i_m} \cdots \omega_{i_1} \right]
\]

for some \( m \in \mathbb{N} \) with \( c_I \in \mathbb{Q} \) and \( \omega_{i_j} \in \{ \frac{dx}{x}, \frac{dx}{\zeta-x}, \frac{dy}{y}, \frac{dy}{\zeta-y}, \frac{xdy+ydx}{\zeta xy} \} \) for \( \zeta \in \mu_N \).

5. PROOF OF MAIN THEOREMS

This section gives a proof of theorem 1.

Proof of theorem 1. Let \( a = (a_1, \ldots, a_k) \in \mathbb{Z}_{>0}^k, b = (b_1, \ldots, b_l) \in \mathbb{Z}_{>0}^l, \zeta = (\zeta_1, \ldots, \zeta_k) \) and \( \eta = (\eta_1, \ldots, \eta_l) \) with \( \zeta_i, \eta_j \in \mu_N \subset \mathbb{C} \) (1 ≤ \( i \leq k \) and 1 ≤ \( j \leq l \)). Put \( \zeta(x) = (\zeta_1 x, \ldots, \zeta_{k-1} x, \zeta_k x) \) and
\[ \bar{\eta}(y) = (\eta_1, \ldots, \eta_{l-1}, \eta_l y). \]

Recall that multiple polylogarithms satisfy the following analytic identity, the series shuffle formula in \( I_o(\mathcal{M}_{0,5}^{(N)}) \):

\[
L_i^a(\bar{\zeta}(x)) \cdot L_i^b(\bar{\eta}(y)) = \sum_{\sigma \in Sh^{\leq}(k, l)} L_{\sigma(a, b)}^\sigma(\bar{\zeta}(x), \bar{\eta}(y)).
\]

Here \( Sh^{\leq}(k, l) := \bigcup_{N=1}^{\infty} \{ \sigma : \{1, \ldots, k+l\} \rightarrow \{1, \ldots, N\} | \sigma \) is onto, \( \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l) \}, \sigma(a, b) := (c_1, \ldots, c_N) \) with

\[
c_i = \begin{cases} 
  a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\
  a_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\
  b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k,
\end{cases}
\]

and \( \sigma(\bar{\zeta}(x), \bar{\eta}(y)) := (z_1, \ldots, z_N) \) with

\[
z_i = \begin{cases} 
  x_s y_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\
  x_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\
  y_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k,
\end{cases}
\]

for \( x_i = \zeta_i (i \neq k), \ z_{k}x (i = k) \) and \( y_j = \eta_j (j \neq l), \ y_l y (j = l) \). Since \( \rho \) is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in the \( Q \)-structure \( (Ut_{4,N}^{0})^* \) of \( V(\mathcal{M}_{0,5}^{(N)}) \)

\[
(8) \quad l_{a}^{\bar{\zeta}}(x) \cdot l_{b}^{\bar{\eta}}(y) = \sum_{\sigma \in Sh^{\leq}(k, l)} l_{\sigma(a, b)}^{\sigma(\bar{\zeta}(x), \bar{\eta}(y))}.
\]

Let \((g, h)\) be a pair in theorem 1. By the group-likeness of \( h \), i.e. \( h \in \exp \mathfrak{F}_{N+1} \), the product \( h^{1,23,4} h^{1,2,3} \) is group-like, i.e. belongs to \( \exp t_{4,N}^2 \). Hence \( \Delta(h^{1,23,4} h^{1,2,3}) = (h^{1,23,4} h^{1,2,3}) \otimes (h^{1,23,4} h^{1,2,3}) \), where \( \Delta \) is the standard coproduct of \( Ut_{4,N}^2 \). Therefore

\[
l_{a}^{\bar{\zeta}}(x) \cdot l_{b}^{\bar{\eta}}(y) (h^{1,23,4} h^{1,2,3}) = (l_{a}^{\bar{\zeta}}(x) \otimes l_{b}^{\bar{\eta}}(y))(\Delta(h^{1,23,4} h^{1,2,3})) \\
= l_{a}^{\bar{\zeta}}(x) (h^{1,23,4} h^{1,2,3}) \cdot l_{b}^{\bar{\eta}}(y) (h^{1,23,4} h^{1,2,3}).
\]

Evaluation of the equation (8) at the group-like element \( h^{1,23,4} h^{1,2,3} \) gives the series shuffle formula

\[
(9) \quad l_{a}^{\bar{\zeta}}(h) \cdot l_{b}^{\bar{\eta}}(h) = \sum_{\sigma \in Sh^{\leq}(k, l)} l_{\sigma(a, b)}^{\sigma(\bar{\zeta}, \bar{\eta})}(h)
\]

for admissible pairs \( (a, \bar{\zeta}) \) and \( (b, \bar{\eta}) \) by the results in \([F4]\) because the group-likeness and (4) for \( h \) implies \( c_0(h) = 1 \) and \( c_A(h) = 0 \).

A pair \((a, \bar{\zeta})\) with \( a = (a_1, \ldots, a_k) \) and \( \bar{\zeta} = (\zeta_1, \ldots, \zeta_k) \) is called \textit{admissible} if \( (a_k, \zeta_k) \neq (1, 1) \).
By putting $l_{1}^{1,S}(h) := -T$ and $l_{a}^{\overline{\zeta},S}(h) := l_{a}^{\overline{\zeta}}(h)$ for all admissible pairs $(a, \overline{\zeta})$, the series regularized value $l_{a}^{\overline{\zeta},S}(h)$ in $Q[T]$ ($T$: a parameter which stands for $\log z$, cf. [R]) for a non-admissible pair $(a, \overline{\zeta})$ is uniquely determined in such a way (cf. [AK]) that the above series shuffle formulae remain valid for $l_{a}^{\overline{\zeta},S}(h)$ with all pairs $(a, \overline{\zeta})$.

Define the integral regularized value $l_{a}^{\overline{\zeta},I}(h)$ in $Q[T]$ for all pairs $(a, \overline{\zeta})$ by $l_{a}^{\overline{\zeta},I}(h) = l_{a}^{\overline{\zeta}}(e^{TB(0)}h)$. Equivalently $l_{a}^{\overline{\zeta},I}(h)$ for any pair $(a, \overline{\zeta})$ can be uniquely defined in such a way that the iterated integral shuffle formulae (loc.cit) remain valid for all pairs $(a, \overline{\zeta})$ with $l_{1}^{1,I}(h) := -T$ and $l_{a}^{\overline{\zeta},I}(h) := l_{a}^{\overline{\zeta}}(h)$ for all admissible pairs $(a, \overline{\zeta})$ because they hold for admissible pairs by the group-likeness of $h$ (cf. loc.cit).

Let $\mathbb{L}$ be the $Q$-linear map from $Q[T]$ to itself defined via the generating function:

$$\mathbb{L}(\exp Tu) = \sum_{n=0}^{\infty} \mathbb{L}(T^{n}) \frac{u^{n}}{n!} = \exp \left\{ -\sum_{n=1}^{\infty} l_{n}^{1,I}(h) \frac{u^{n}}{n} \right\}.$$

**Proposition 8.** Let $h$ be an element as in theorem 1. Then the regularization relation holds, i.e. $l_{a}^{\overline{\zeta},S}(h) = \mathbb{L}(l_{a}^{\overline{\zeta},I}(h))$ for all pairs $(a, \overline{\zeta})$.

**Proof.** We may assume that $(a, \overline{\zeta})$ is non-admissible because the proposition is trivial if it is admissible. Put $1^{n} = (1, 1, \cdots, 1)$. When $a = 1^{n}$ and $\overline{\zeta} = \overline{1^{n}}$, the proof is given by the same argument to [F3] as follows: By the series shuffle formulae,

$$\sum_{k=0}^{m} (-1)^{k} l_{k+1}^{1,S}(h) \cdot l_{1^{m-k}}^{1^{m-k},S}(h) = (m+1) l_{1^{m+1},S}(h)$$

for $m \geq 0$. Here we put $l_{0}^{1,S}(h) = 1$. This means

$$\sum_{k,l \geq 0} (-1)^{k} l_{k+1}^{1,S}(h) \cdot l_{1^{l}}^{1^{l},S}(h) u^{k+l} = \sum_{m \geq 0} (m+1) l_{1^{m+1},S}(h) u^{m}.$$

Put $f(u) = \sum_{n \geq 0} l_{1^{n},S}(h) u^{n}$. Then the above equality can be read as

$$\sum_{k \geq 0} (-1)^{k} l_{k+1}^{1,S}(h) u^{k} = \frac{d}{du} \log f(u).$$

Integrating and adjusting constant terms gives

$$\sum_{n \geq 0} l_{1^{n},S}(h) u^{n} = \exp \left\{ -\sum_{n \geq 1} (-1)^{n} l_{1^{n},S}(h) \frac{u^{n}}{n} \right\} = \exp \left\{ -\sum_{n \geq 1} (-1)^{n} l_{1^{n}}^{1,I}(h) \frac{u^{n}}{n} \right\}.$$
because \( l^S_n(h) = l^I_n(h) = l^I_n(h) \) for \( n > 1 \) and \( l^S_1(h) = l^I_1(h) = -T \).

Since \( l^S_m(h) = (-T)^m \frac{m!}{m!} \), we get \( l^S_m(h) = \mathbb{L}(l^I_m(h)) \).

When \((a, \bar{\zeta})\) is of the form \((a'1^l, \bar{\zeta}'1^l)\) with \((a', \bar{\zeta}')\) admissible, the proof is given by the following induction on \( l \). By (8),

\[
l'^{(a, \bar{\zeta})}(h') \cdot l^I_{1^l}(h') = \sum_{(\sigma \in Sh \leq k, l)} l^I_{\sigma(a',1^l)}(\sigma(\bar{\zeta}',1^l)(h'))
\]

for \( h' = e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}} h^{1,23,4} h^{1,2,3} \) with \( k = dp(a') \). Then by our induction assumption, taking the image by the map \( \mathbb{L} \) gives

\[
l'^{(a, \bar{\zeta})}(h) \cdot l^I_{1^l}(h) = \sum_{\sigma \neq id \in Sh \leq k, l} l^I_{\sigma(a',1^l)}(\sigma(\bar{\zeta},1^l)(h)).
\]

Then by our induction assumption, taking the image by the map \( \mathbb{L} \) gives

\[
l'^{(a, \bar{\zeta})}(h) \cdot l^S_{1^l}(h) = \mathbb{L}(l^I_{1^l}(h)) + \sum_{\sigma \neq id \in Sh \leq k, l} l^I_{\sigma(a',1^l)}(\sigma(\bar{\zeta},1^l)(h)).
\]

Since \( l'^{(a, \bar{\zeta})}(h) \cdot l^S_{1^l}(h) \) satisfy the series shuffle formula, \( \mathbb{L}(l'^{(a, \bar{\zeta})}(h)) \) must be equal to \( l'^{(a, \bar{\zeta})}(h) \), which concludes proposition 8.

Embed \( U\mathfrak{Y}_N \) into \( U\mathfrak{Y}_{N+1} \) by sending \( Y_{m,a} \) to \(-A^{m-1}B(-a)\). Then by the above proposition,

\[
l'^{(a, \bar{\zeta})}(h) = \mathbb{L}(l'^{(a, \bar{\zeta})}(h)) = \mathbb{L}(l'^{(a, \bar{\zeta})}(e^{TB(0)}h)) = \mathbb{L}(l'^{(a, \bar{\zeta})}(e^{TB(0)}\pi_Y(h)))
\]

for all \((a, \bar{\zeta})\) because \( l^I_1(h) = 0 \). As for the third equality we use \((\mathbb{L} \otimes Q id) \circ (id \otimes Q l'^{(a, \bar{\zeta})}) = (id \otimes Q l^I_{(a, \bar{\zeta})}) \circ (\mathbb{L} \otimes Q id) \) on \( Q[T] \otimes Q U\mathfrak{Y}_{N+1} \). All \( l'^{(a, \bar{\zeta})}(h) \)'s satisfy the series shuffle formulae (9), so the \( l'^{(a, \bar{\zeta})}(e^{-TY_{1,0}}h) \)'s do also. By putting \( T = 0 \), we get that \( l'^{(a, \bar{\zeta})}(h) \)'s also satisfy the series shuffle formulae for all \( a \). Therefore \( \Delta_\ast(h) = h_\ast \otimes h_\ast \). This completes the proof of theorem 1.
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GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA, 464-8602, JAPAN
E-mail address: furusho@math.nagoya-u.ac.jp