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Combinatorial Games -A Research Project by High School Students Using Computer Algebra Systems III-

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Abstract

The authors have studied bitter chocolate problems, and they have published their results in [6], [7] and [8]. These chocolate problems are equivalent to the variants of games of nim conditioned by the inequalities such as $y \leq \lfloor \frac{x+z}{3} \rfloor$ and $y \leq x+z$, where non-negative integers x, y, z stand for the numbers of stones in piles. In this article the authors are going to study a variant of nim conditioned by the inequalities $y \leq \lfloor \frac{min(x,z)}{2} \rfloor$. The authors have presented this problem in [5], but here we are going to use a new method of proof that is very different from the one used in [5]. This method is very simple and easier to understand. The authors have used Mathematica when they discovered the formula for P-positions of this chocolate problem.

The authors are also going to introduce some predictions on chocolate problems. These predictions have been made by Mathematica, too. Almost all the ideas used in the research has been proposed by high school students and freshmen in a university. The authors present a list of the previous results of their research in the references to show the possibility of the research project by high school students.

1 The game that satisfies the inequality $2y \leq min(x, z)$

In this article we are going to study combinatorial games. Let Z^+ be the set of non-negative integers. There is a brief introduction of chocolate problems in [8]. Therefore please consult [8], if you are not familiar with these combinatorial games.

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In a combinatorial game there are two kinds of positions. One kind is a P-position, a previous-player-winning position. The other is an N-position, a next-player-winning position. One of the most important topics of combinatorial games is to find all the P-positions of a game.

First we are going to define the game of Nim.

Definition 1

There are 3 piles. Two players alternate by taking all or some of the counters in a single heap. The player taking the last counter or stack of counters is the winner.

Remark 1.1

This definition is a special case of the definition of the game of Nim in [1].

We are going to define the nim-sum that is very important for the theory of combinatorial games.

Definition 2

Let x, y be non-negative integers, and write them in base 2, so $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$. We define the nim-sum $x \oplus y$ by

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i, \tag{1}$$

where $w_i = x_i + y_i \ (mod\ 2)$.

Definition 3

Let $A = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, x \oplus y \oplus z = 0\}$ and $B = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, x \oplus y \oplus z \neq 0\}$.

Theorem 4

The lists A and B are the list of P-positions and the list of N-positions of the game of Definition 1 respectively.

Proof For a proof see [2] and [3]. ∎

Here we are going to study bitter chocolate problems that are variants of the game of Nim.

Definition 5

Given a pieces of chocolate, where the light gray parts are sweet and the dark gray part is very bitter. This game is played by two players in turn. Each player breaks the chocolate (in a straight line along the grooves) and eats the piece he breaks off. The player to leave his opponent with the single bitter part is the winner.

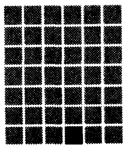
Remark 1.2

Note that in Definition 5 we are not considering misere play, since the player who breaks off the last groove wins. This is a very important point in understanding chocolate problems.

We can make many kinds of chocolate problems.

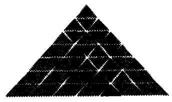
Example 1

Here we have 3 types of chocolate problems. The problem in Graph 1.1 has been introduced in [3]. It is easy to see that it is equivalent to the game of Nim with 3 piles. Other problems have been proposed by the authors. You can see the interesting facts of problems of Graph 1.2 and Graph 1.3 in [31] and [7].



Graph 1.1

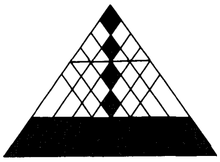
Graph 1.2



Graph 1.3

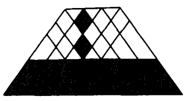
In this article we are going to study other types of chocolate problems.

Example 2
Here we have the chocolate problem in Graph 1.4.
The light gray parts are sweet, and the dark gray part is very bitter. We do not have anything in the white parts. We can cut this chocolate in 3 ways. Therefore it is proper to represent this chocolate with 3 non-negative integers {5, 2, 5}.



Graph 1.4

Example 3
Here we have three examples of positions of the chocolates that appear in the game of chocolate of Graph 1.4.

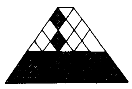


{4, 1, 5}

Graph 1.5

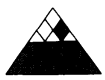
{5, 0, 3}

Graph 1.6



{3,1,4}

Graph 1.7



{3,1,2}

Graph 1.8

It is clear that the chocolate problem of Graph 1.4 satisfies the inequality $2y \leq min(x, z)$, *and this is equivalent to the inequality*

$$y \leq \lfloor \frac{min(x,z)}{2} \rfloor, \tag{2}$$

where $\lfloor \ \rfloor$ *is the floor function.*

We can define the problem in Graph 1.4 without a chocolate.
This is a variant of the game of Nim conditioned by an inequality.

Definition 6
We are going to define a variant of the game of Nim. In this game there are 3 piles and the numbers of each piles can be represented by 3 non-negative integers $\{x, y, z\}$.

These numbers $\{x,y,z\}$ satisfy Inequality (2), and hence the reduction of numbers satisfies the following conditions.
(a). You can reduce y to any non-negative number $v < y$.
(b). You can reduce x to any non-negative number $u < x$, and then the second coordinate will be $Min(y, \lfloor \frac{Min(u,z)}{2} \rfloor)$.
(c). You can reduce z to any non-negative number $w < z$, and then the second coordinate will be $Min(y, \lfloor \frac{Min(x,w)}{2} \rfloor)$.
Note that these conditions look quite natural when you look at the structure of the chocolate in Graph1.4.

Remark 1.3
Some people may think that the shapes of the chocolates in Graph 1.4 is not natural. It might be good to tell about how the authors came up with these chocolates. After they studied some chocolate problems in Example 1, they began to study chocolate problem conditioned by other types of inequalities. They chose Inequality (2), and by calculation of Mathematica they found out that the chocolate problem with this inequality produces a simple formula for the list of P-positions. When they looked for the chocolate that satisfies the inequality, they discovered the chocolate of Graph 1.4.

Next we are going to define the function $move1(\{x,y,z\})$ for a position $\{x,y,z\}$. $move1(\{x,y,z\})$ is the list of all positions that can be reached from the position $\{x,y,z\}$ in one step (directly).

Definition 7
For $x,y,z \in Z^+$ we define $move1(\{x,y,z\})$
$= \{\{x', Min(y, \lfloor \frac{Min(x',z)}{2} \rfloor), z\}; x' < x\}$
$\cup \{\{x, Min(y, \lfloor \frac{Min(x,z')}{2} \rfloor), z'\}; z' < z\} \cup \{\{x,y',z\}; y' < y\}$.

Example 4
In Example 3 $\{3,1,2\} \in move1(\{3,1,4\})$. and $\{5,0,3\} \notin move1(\{4,1,5\})$. Note that $\{3,1,4\} \notin move1(\{4,1,5\})$, since it will take 2 steps to reach $\{3,1,4\}$ from $\{4,1,5\}$.

Definition 8
Let $A_1 = \{\{x,y,z\}; x,y,z \in Z^+, y \leq \lfloor \frac{min(x,z)}{2} \rfloor$ and $x \oplus y \oplus z = 0\}$, $B_1 = \{\{x,y,z\}; x,y,z \in Z^+, y \leq \lfloor \frac{min(x,z)}{2} \rfloor$ and $x \oplus y \oplus z \neq 0\}$.

We are going to show that A_1 and B_1 are the lists of P-positions and N-positions of the game of Definition 6 respectively. We have to prove the following (a) and (b).
(a). If you start with a P-position (an element of A_1), then any move leads to an N-position (an element of B_1).
(b). If you start with an N-position (an element of B_1), then there is a proper move that leads to a P-position (an element of A_1).
Therefore we are going to prove the following (3) and (4).

$$move1(\{x,y,z\}) \subset B_1 \tag{3}$$

for any $\{x,y,z\} \in A_1$.

$$move1(\{x,y,z\}) \cap A_1 \neq \emptyset \tag{4}$$

for any $\{x,y,z\} \in B_1$.

Lemma 9
Let a, b be non-negative integers such that $2a \leq b$, and write them in base 2, so $a = \sum_{i=0}^{n-1} a_i 2^i$ and $b = \sum_{i=0}^{n} b_i 2^i$ with $a_i, b_i \in \{0,1\}$.
Then for any non-negative integer k we have $2\sum_{i=k}^{n-1} a_i 2^i \leq \sum_{i=k+1}^{n} b_i 2^i$.

Proof Suppose that there exists k such that $2\sum_{i=k}^{n-1} a_i 2^i > \sum_{i=k+1}^{n} b_i 2^i$. Then it is clear that $2^{k+1} \leq 2\sum_{i=k}^{n-1} a_i 2^i - \sum_{i=k+1}^{n} b_i 2^i$.
Then $2a > 2\sum_{i=k}^{n-1} a_i 2^i \geq 2^{k+1} + \sum_{i=k+1}^{n} b_i 2^i > \sum_{i=0}^{n} b_i 2^i = b$, which contradicts to the hypothesis of this lemma. ∎

Theorem 10
For any $\{x, y, z\} \in A_1$ we have $move1(\{x, y, z\}) \subset B_1$.

Proof Let $\{x, y, z\} \in A_1$, then we have

$$x \oplus y \oplus z = 0 \tag{5}$$

and

$$y \leq \lfloor \frac{min(x, z)}{2} \rfloor. \tag{6}$$

Suppose that we move from $\{x, y, z\}$ to $\{p, q, r\}$, and we are going to prove that $\{p, q, r\} \in B_1$.
Let $n = \lfloor log_2 x \rfloor$. Then by (5) and (6) we have
$n = \lfloor log_2 z \rfloor > \lfloor log_2 y \rfloor$.
We can write x, y and z in base 2, so

$$x = \sum_{i=0}^{n} x_i 2^i, y = \sum_{i=0}^{n-1} y_i 2^i \ , \ z = \sum_{i=0}^{n} z_i 2^i, \tag{7}$$

where $x_i, y_i, z_i \in \{0,1\}$. We can also write p, q and r in base 2, so

$$p = \sum_{i=0}^{n} p_i 2^i, q = \sum_{i=0}^{n-1} q_i 2^i \ , \ r = \sum_{i=0}^{n} r_i 2^i, \tag{8}$$

where $p_i, q_i, r_i \in \{0,1\}$.
[1]. We are going to reduce y to q. Then we are not going to affect the value of y. Let k be the largest integer such that $q_k = 0 < 1 = y_k$. By (5) we have $x_k + y_k + z_k = 0$, and hence by the fact that $q_k < y_k$ we have $x_k + q_k + z_k \neq 0 \ (mod\ 2)$.
This shows that $p \oplus q \oplus r = x \oplus q \oplus z \neq 0$, and hence we have $\{p, q, r\} = \{x, q, z\} \in B_1$.
[2]. We are going to reduce z to r. By (6) we have

$$\frac{x}{2} \geq y. \tag{9}$$

Let k be the largest integer such that $r_k = 0 < 1 = z_k$.
Then we have

$$r_i = z_i \tag{10}$$

for $i = k+1, k+2, ..., n$. By (6) we have $2y \leq z$, and hence by Lemma 9 we have

$$2\sum_{i=k}^{n-1} y_i 2^i \leq \sum_{i=k+1}^{n} z_i 2^i. \tag{11}$$

By (10) and (11) we have $r \geq \sum_{i=k+1}^{n} z_i 2^i \geq 2\sum_{i=k}^{n-1} y_i 2^i$, and hence by (9) and (c) of Definition 6 $q = Min(y, \lfloor \frac{Min(x,r)}{2} \rfloor) \geq \sum_{i=k}^{n-1} y_i 2^i$, which shows that $q_k = y_k$. By (5) we have $x_k + y_k + z_k = 0$, and hence by the fact that $q_k = y_k$ and $r_k < z_k$ we have $p_k + q_k + r_k = x_k + y_k + r_k \neq 0$ $(mod\ 2)$.
This shows that $p \oplus q \oplus r = x \oplus q \oplus r \neq 0$, and hence we have $\{p, q, r\} = \{x, q, r\} \in B_1$.
[3]. When we reduce x to p, then we can use a method that is very similar to the one used in [2] to prove this theorem.

∎

Lemma 11
Suppose that

$$2\sum_{i=0}^{n-1} a_i 2^i \leq \sum_{i=k+1}^{n} b_i 2^i + \sum_{i=0}^{k} 2^i \tag{12}$$

and

$$2\sum_{i=0}^{n-1} a_i 2^i > \sum_{i=k}^{n} b_i 2^i + \sum_{i=0}^{k-1} 2^i \tag{13}$$

,with $a_i, b_i \in \{0, 1\}$. Then we have $b_k = 0$, $a_{k-1} = 1$ and $2\sum_{i=k}^{n-1} a_i 2^i = \sum_{i=k+1}^{n} b_i 2^i$.

Proof If we compare (12) and (13), it is easy to see that $b_k = 0$. By Lemma 9 we have $2\sum_{i=k}^{n-1} a_i 2^i \leq \sum_{i=k+1}^{n} b_i 2^i$, and hence

$$2\sum_{i=k}^{n-1} a_i 2^i + 2\sum_{i=0}^{k-2} 2^i \leq \sum_{i=k+1}^{n} b_i 2^i + 2\sum_{i=0}^{k-2} 2^i < \sum_{i=k+1}^{n} b_i 2^i + \sum_{i=0}^{k-1} 2^i = \sum_{i=k}^{n} b_i 2^i + \sum_{i=0}^{k-1} 2^i, \tag{14}$$

where the last equation is deprived from the fact that $b_k = 0$.
If $a_{k-1} = 0$, then

$$2\sum_{i=0}^{n-1} a_i 2^i \leq 2\sum_{i=k}^{n-1} a_i 2^i + 2\sum_{i=0}^{k-2} 2^i. \tag{15}$$

(14) and (15) contradict to (13). Therefore $a_{k-1} = 1$.
We suppose that $2\sum_{i=k}^{n-1} a_i 2^i < \sum_{i=k+1}^{n} b_i 2^i$, then we have

$$2\sum_{i=k}^{n-1} a_i 2^i + 2^{k+1} \leq \sum_{i=k+1}^{n} b_i 2^i. \tag{16}$$

Since $2\sum_{i=0}^{n-1} a_i 2^i < 2\sum_{i=k}^{n-1} a_i 2^i + 2^{k+1}$, (16) contradicts to (13). Therefore we have $2\sum_{i=k}^{n-1} a_i 2^i = \sum_{i=k+1}^{n} b_i 2^i$.

∎

Theorem 12
Let $\{x, y, z\} \in B_1$, then $move1(\{x, y, z\}) \cap A_1 \neq \emptyset$.

Proof Let $\{x, y, z\} \in B_1$. Then we have

$$x \oplus y \oplus z \neq 0 \tag{17}$$

and

$$y \leq \lfloor \frac{min(x, z)}{2} \rfloor. \tag{18}$$

We are going to reduce $\{x, y, z\}$ to $\{p, q, r\} \in move1(\{x, y, z\}) \cap A_1$. Let

$$x = \sum_{i=0}^{m} x_i 2^i, y = \sum_{i=0}^{m} y_i 2^i \ , \ z = \sum_{i=0}^{m} z_i 2^i, \tag{19}$$

where $x_i, y_i, z_i \in \{0, 1\}$.
Suppose that $x_i + y_i + z_i = 0 \ (mod\ 2)$ for i = n+1, n+2, ...,m and

$$x_n + y_n + z_n \neq 0 \ (mod\ 2) \tag{20}$$

for some $n \in Z^+$.
[1]. We assume that $y_n = 1$, then we let $q_n = 0$, $q_i = x_i + z_i \ (mod\ 2)$ for $i = 1, 2, 3, ..., n-1$ and $q_i = y_i$ for $i = n+1, ..., m$. Let $q = \sum_{i=0}^{m} q_i 2^i$, $p = x$ and $r = z$. Then we have $q < y$, and hence $\{p, q, r\} = \{x, q, z\} \in move1(\{x, y, z\})$. By the definition of q it is easy to see that $p \oplus q \oplus r = x \oplus q \oplus z = 0$, and hence $\{p, q, r\} \in A_1$.
[2]. We assume that $y_n = 0$, then we have $x_n = 1$ or $z_n = 1$.
[2.1] We can assume without any loss of generality that $z_n = 1$, and we are going to reduce z to r. First we let $r_n = x_n + y_n$, then by $y_n = 0$, $z_n = 1$ and (20) we have $r_n = 0 < 1 = z_n$. Clearly we have $x_n + y_n + r_n = 0 \ (mod\ 2)$.
By (18) we have

$$2\sum_{i=0}^{m} y_i 2^i \leq \sum_{i=0}^{m} z_i 2^i \leq \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=0}^{n} 2^i. \tag{21}$$

Therefore we have the following [2.1.1] or [2.1.2] or [2.1.3].
[2.1.1]. Suppose that we define $r_i = x_i + y_i \ (mod\ 2)$ for $i = 0, 1, 2, ..., n-1$, $r_i = z_i$ for $i = n+1, n+2, ..., m$ and we get

$$2\sum_{i=0}^{m} y_i 2^i \leq \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=0}^{n} r_i 2^i. \tag{22}$$

Let $p = x$, $q = y$ and $r = \sum_{i=0}^{m} r_i 2^i$. Since $r_n = 0 < 1 = z_n$, $r < z$. By (22) we can reduce z to r without affecting y, and hence we have $\{p, q, r\} = \{x, y, r\} \in move1(\{x, y, r\})$. Clearly $p \oplus q \oplus r = x \oplus y \oplus r = 0$, and hence $\{p, q, r\} = \{x, y, r\} \in A_1$.
[2.1.2]. Suppose that there exists k such that we can define

$$r_i = x_i + y_i \tag{23}$$

for $i = k, k+1, ...n-1$ so that we have

$$2\sum_{i=0}^{m} y_i 2^i \leq \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=k+1}^{n} r^i 2^i + \sum_{i=0}^{k} 2^i. \tag{24}$$

and

$$2\sum_{i=0}^{m} y_i 2^i > \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=k}^{n} r^i 2^i + \sum_{i=0}^{k-1} 2^i. \tag{25}$$

By (24), (25) and Lemma 11 we have $r_k = 0$ and $y_{k-1} = 1$.

We are going to define the values of q_i and r_i. Let $q_{k-1} = r_k$ and $r_{k-1} = x_{k-1} + q_{k-1} \ (mod\ 2)$. After that we let $q_i = r_{i+1}$ and $r_i = x_i + q_i \ (mod\ 2)$ for $i = 0, 1, 2, 3, ..., k-2$. We let $r_i = z_i$ for

$i = n+1, n+2, ..., m$ and $r = \sum_{i=0}^{m} r^i 2^i$. By Lemma 11, (24) and (25) we have

$$2\sum_{i=k}^{m} y_i 2^i = \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=k+1}^{n} r_i 2^i = \sum_{i=k+1}^{m} r_i 2^i. \tag{26}$$

Let $q_i = y_i$ for $i = k, k+1, ..., m$ and $q = \sum_{i=0}^{m} q_i 2^i$. Since $q_{k-1} = r_k = 0 < y_{k-1}$, $q < y$. By (26) and the fact that $q_i = r_{i+1}$ for $i = 0, 1, 2, 3, ..., k-1$ we have $q = \lfloor \frac{r}{2} \rfloor$, and hence by $q < y$, $p = x$ and $r < z$ we have $\{p, q, r\} = \{x, q, r\} \in movel(\{x, y, z\})$. By the definition of q, r it is easy to see that $p \oplus q \oplus r = x \oplus q \oplus r = 0$, and hence $\{p, q, r\} \in A_1$.
[2.1.3]. Suppose that we have

$$2\sum_{i=0}^{m} y_i 2^i > \sum_{i=n+1}^{m} z_i 2^i + r_n 2^n + \sum_{i=0}^{n-1} 2^i. \tag{27}$$

By (21)

$$2\sum_{i=0}^{m} y_i 2^i \le \sum_{i=0}^{m} z_i 2^i \le \sum_{i=n+1}^{m} z_i 2^i + \sum_{i=0}^{n} 2^i. \tag{28}$$

Then by (27), (28) and Lemma 11 we have $y_{n-1} = 1$. We let $q_i = r_{i+1}$ and $r_i = x_i + q_i \ (mod\ 2)$ for $i = 0, 1, 2, 3, ..., n-1$. Let $q_i = y_i$ for $i = n, n+1, ..., m$ and $q = \sum_{i=0}^{m} q_i 2^i$. Since $q_{n-1} = r_n = 0 < 1 = y_{n-1}$, $q < y$. We let $r_i = z_i$ for i = n+1, n+2,...,m and $r = \sum_{i=0}^{m} r^i 2^i$. Since $r_n = 0 < 1 = z_n$, $r < z$. By Lemma 11, (28) and (27) we have

$$2\sum_{i=n}^{m} y_i 2^i = \sum_{i=n+1}^{m} z_i 2^i, \tag{29}$$

and hence by the fact that $q_i = r_{i+1}$ for $i = 0, 1, 2, 3, ..., n-1$ we have $q = \lfloor \frac{r}{2} \rfloor$, and hence by $q < y$, $p = x$ and $r < z$ we have $\{p, q, r\} = \{x, q, r\} \in movel(\{x, y, z\})$. By the definition of q, r it is easy to see that $p \oplus q \oplus r = x \oplus q \oplus r = 0$, and hence $\{p, q, r\} \in A_1$. ∎

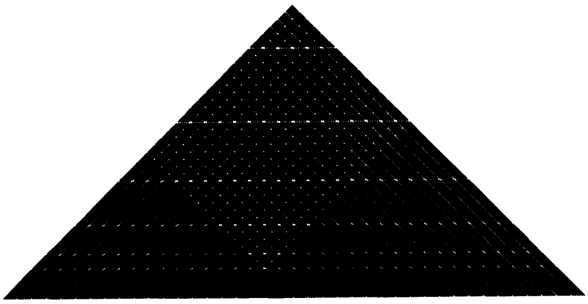
Theorem 13
A_1 is the list of P-positions and B_1 is the list of N-positions of the game of Definition 6.

Proof If we start the game with a position $\{x, y, z\} \in A_1$, then by Theorem 10 any option leads to a position in B_1. From this position by Theorem 12 a proper option leads to a position in A_1. Note that any option reduces some of the numbers in the coordinates. In this way a previous player can always reach a position in A_1, and finally reaches $\{0, 0, 0\} \in A_1$.
If we start the game with a position $\{x, y, z\} \in B_1$, then by Theorem 12 a proper option leads to a position in A_1. In this way the next player can always reach $\{0, 0, 0\}$. ∎

2 Some Unsolved Chocolate Problems

It is not difficult to calculate the list of P-positions for a chocolate game with an inequality by computer programs if the size of the chocolate is small enough for the computer that is used in calculation.

Example 5
The chocolate problem with the inequality $y \le Round(x + z)$ has a very interesting property.



Graph 2.1

This is the Mathematica program to present all the P-positions of the game for $\{\{x, y, z\}x, z \leq 200$ and $y \leq 20\}$.

```
k = 1; ss = 20; al =
 Flatten[Table[{a, b, c}, {a, 0, 10 ss}, {b, 0, ss}, {c,
    0, 10 ss}], 2];
allcases =
  Select[al, (1/k) #[[2]] <= Round[Sqrt[#[[1]] + #[[3]]]] &];
move[z_] := Block[{p}, p = z;
  Union[Table[{t1, Min[Floor[Round[Sqrt[(t1 + p[[3]])]]], p[[2]]],
     p[[3]]}, {t1, 0, p[[1]] - 1}],
   Table[{p[[1]], t2, p[[3]]}, {t2, 0, p[[2]] - 1}],
   Table[{p[[1]], Min[Floor[Round[Sqrt[(t3 + p[[1]])]]], p[[2]]],
     t3}, {t3, 0, p[[3]] - 1}]
   ]
  ]; Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, move[pos]]];
pposition = Select[allcases, Gr[#] == 0 &];
```

By the above Mathematica program we could make a prediction. It seems to be very difficult to prove this prediction.

Prediction 2.1
The list of P-positions of this game is $\{\{x-3, y, z-3\}; x \oplus y \oplus z = 0$ *and* $y \leq Round(x+z)\}$

Example 6
The chocolate problem with the inequality $y \leq \lfloor \frac{x!+y!}{8} \rfloor$ *has a very interesting property.*

We omit the Mathematica program to present all the P-positions of the game. By the above Mathematica program we could make a prediction. It seems to be very difficult to prove this prediction.

Prediction 2.2
The list of P-positions of this game is $\{x+3, y, z+3\}; x \oplus y \oplus z = 0$ *and* $y \leq \lfloor \frac{x!+y!}{8} \rfloor\}$

Example 7
The chocolate problem with the inequality $y \leq \lfloor \frac{(x+y)^2}{k} \rfloor$ *has a very interesting property.*

This is the Mathematica program to look for an integer k and h such that the list of P-positions of the game with inequality $y \leq \lfloor \frac{(x+y)^2}{k} \rfloor$ is $\{x+h, y, z+h\}; x \oplus y \oplus z = 0$ and $y \leq \lfloor \frac{(x+y)^2}{k} \rfloor\}$. Then the 3D graph made by the list of P-positions produces the Sierpinski sieve. See [7].

```
Do[Clear[ss, al, allcases, move, Mex, Gr, pposition, ggg];
 ss = 10;
 al = Flatten[Table[{a, b, c}, {a, 0, ss}, {b, 0, 5 ss}, {c, 0, ss}],
   2];
```

```
allcases = Select[al, Floor[(#[[1]] + #[[3]])^2/k] >= #[[2]] &];
Print[k];
Print[allcases // Length]
 Clear[move, Gr, Mex]; move[z_] := Block[{p}, p = z;
  Union[Table[{t1, Min[Floor[(t1 + p[[3]])^2/k], p[[2]]],
     p[[3]]}, {t1, 0, p[[1]] - 1}],
   Table[{p[[1]], t2, p[[3]]}, {t2, 0, p[[2]] - 1}],
   Table[{p[[1]], Min[Floor[(p[[1]] + t3)^2/k], p[[2]]], t3}, {t3, 0,
      p[[3]] - 1}]
   ]];
Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, move[pos]]];
pposition = Select[allcases, Gr[#] == 0 &];
If[Complement[Map[{#[[2]], #[[1]] - #[[3]]} &, pposition],
   sierpin] == {}, Print["Sierpinski"]; k = 0;
 While[k <
    10 && ! (Union[
       Map[nimsum,
        Map[{#[[1]] + k, #[[2]], #[[3]] + k} &, pposition]]] == {0}),
   k = k + 1]; If[k < 9, Print[k],
  kk = 0;
  While[kk <
     10 && ! (Union[
        Map[nimsum,
         Map[{#[[1]] - kk, #[[2]], #[[3]] - kk} &,
          pposition]]] == {0}), kk = kk + 1];
  If[kk < 9, Print[-kk],]], Print["No"]];
Print[ListPointPlot3D[pposition, Background -> Black,
  ColorFunction -> "Rainbow", PlotStyle -> PointSize[0.01]]];
ggg = Map[{#[[2]], #[[3]] - #[[1]]} &, pposition];
Print[ListPlot[ggg, Background -> Black]], {k, 1, 10}]
```

By the above Mathematica program we could make a prediction. It seems to be very difficult to prove this prediction.

Prediction 2.3

For k = 1,2,3,4 the list of P-positions of this game is $\{x+1, y, z+1\}; x \oplus y \oplus z = 0$ *and* $y \leq \lfloor \frac{(x+y)^2}{k} \rfloor\}$, *and for k = 5,6,7,8,9,10 the list of P-positions of this game is* $\{x+2, y, z+2\}; x \oplus y \oplus z = 0$ *and* $y \leq \lfloor \frac{(x+y)^2}{k} \rfloor\}$.

Remark 2.1

As to other results of the research by high school students of the group see [5], ...,[31]. Almost all of these results are based on the ideas of high school students, and some ideas have been proposed by university students. After a good idea was proposed by students, a mathematician who was their teacher helped students to formulate theories. The authors presented this long list of articles in the references to show that high school students can discover many kinds of facts and theorems when they can use computer algebra systems properly.

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