On the problem of Goldberg for the rational maps (Computer Algebra: Design of Algorithms, Implementations and Applications)

Author(s)
Fujimura, Masayo

Citation
数理解析研究所講究録 (2012), 1815: 90-99

Issue Date
2012-10

URL
http://hdl.handle.net/2433/194568

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
On the problem of Goldberg for the rational maps

Masayo FUJIMURA*

Department of Mathematics
National Defense Academy, Yokosuka 239-8686, JAPAN
E-mail: masayo@nda.ac.jp

Abstract
In this paper, we solve a problem of Goldberg that determine the number of equivalence classes of rational maps corresponding to each critical set, when the degree is small and ∞ is critical.

1 Introduction
In [3], Goldberg suggested a problem that determine the number of equivalence classes of rational maps corresponding to each critical set. This problem is based on her theorem (Theorem 1.3 in [3]), and it is known that the theorem deeply concern with B. and M. Shapiro conjecture (see [1]).

By using algebraic computation system, we solve a problem of Goldberg when the degree is small and ∞ is critical, and this gives a complete answer to this problem together with our results in [2]. This work is joint work with M. Karima and M. Taniguchi (Nara Women’s Univ.).

A rational map of degree $d$ is a map with the following form,

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P$ and $Q$ are coprime polynomials with $\max\{\deg P, \deg Q\} = d$.

**Definition 1.** Two rational maps $R_1$ and $R_2$ are said to be Möbius equivalent if there is a Möbius transformation $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $R_2 = M \circ R_1$.

Let $X_d$ be the set of all equivalence classes of rational maps of degree $d$, and $X_d^{(k)}$ be the subset of $X_d$ consisting of all equivalence classes of rational maps with $k$-hold critical point at $\infty$, where $k = 0$ means the rational maps that $\infty$ is non-critical.

*The author is partially supported by Grant-in-Aid for Scientific Research (C) 22540240.
Remark 1. A rational map $R$ of degree $d$ has $2d - 2$ critical points counted including multiplicity. The set of critical points of $R$ is invariant under taking a Möbius conjugate.

Every set of critical points of $R$ is admissible, i.e., every critical point has multiplicity at most $d - 1$. Therefore, the space $X_d$ is the disjoint union of $X^{(0)}_d, X^{(1)}_d, \cdots,$ and $X^{(d-1)}_d$.

Goldberg showed the following theorem.

Theorem (Goldberg [3]). A $(2d - 2)$-tuple $B$ is the critical set of at most $C(d)$ classes in $X_d$, where $C(d)$ means the $d$-th Catalan number $\frac{1}{d}\left(\begin{array}{l}2d-2 \\ d-1\end{array}\right)$.

The maximal is attained by a Zariski open subset of the space $\hat{\mathbb{C}}^{2d-2}$ of all $B$.

The map $\Phi_d : X_d \to \hat{\mathbb{C}}^{2d-2}$ is defined by sending an equivalence class to the set of critical points, and the restriction of $\Phi_d$ to $X^{(k)}_d$ is denoted by $\Phi^{(k)}_d$.

Then Goldberg's problem (see [3]) is written as follows.

Problem

- Describe in detail the ramification sets of the maps $\Phi_d$.
- For every point $c \in \hat{\mathbb{C}}^{2d-2}$, determine the number of points in the preimage $\Phi_d^{-1}(c)$.

We give the complete answer to this problem for the case of $d = 3$ and $4$.

2 The case that $\infty$ is non-critical

Theorem 2 (Fujimura, Karima and Taniguchi [2]).

For each class in $X^{(0)}_d$, there is a unique representative $R$ of the form

$$R(z) = \frac{P(z)}{Q(z)} = z + \frac{a_{d-2}z^{d-2} + \cdots + a_0}{z^{d-1} + b_{d-2}z^{d-2} + \cdots + b_0}.$$

For each $R = \frac{P}{Q}$ in the above form, the critical points of $R$ is obtained by the equation

$$P'(z)Q(z) - P(z)Q'(z) = z^{2d-2} + c_{2d-3}z^{2d-3} + \cdots + c_0 = 0.$$

Then, the map $\Phi^{(0)}_d$ is defined as follows,

$$\Phi^{(0)}_d : \mathbb{C}^{2d-2} \cup \mathbb{C}^{2d-2} \mapsto \mathbb{C}^{2d-2}$$

$$(a_{d-2}, \cdots, a_0, b_{d-2}, \cdots, b_0) \mapsto (c_{2d-3}, \cdots, c_0).$$

We give the complete answer to this problem for the case of $d = 3$ and $4$. 
The defining equation of the ramification locus of $\Phi_d^{(0)}$ gives the answer to a problem of Goldberg for the case that $\infty$ is non-critical. For the details, see [2].

Thereafter, we consider the case that $\infty$ is critical.

3 The case that $\infty$ is critical

3.1 The case of degree 3

Proposition 3.

1. For each class in $X_3^{(1)}$, there is a unique representative in $CB_3^{(1)}$, where

$$CB_3^{(1)} = \left\{ R(z) = z^2 + az + \frac{c}{z+b} \ (c \neq 0) \right\}.$$ 

2. For each class in $X_3^{(2)}$, there is a unique representative in $CB_3^{(2)}$, where

$$CB_3^{(2)} = \{ R(z) = z^3 + az^2 + bz \}.$$ 

3.1.1 The case that $\infty$ is simple critical point

Let $R = \frac{P}{Q}$ be a rational map in $CB_3^{(1)}$, and $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_3^{(1)} : CB_3^{(1)} \to \mathbb{C}^3$ is defined by sending $(a, b, c)$ to $(c_0, c_1, c_2)$.

Proposition 4. The ramification locus of $\Phi_3^{(1)}$ is given by $a = 0$, $\Phi_3^{(1)}(CB_3^{(1)}) = \mathbb{C}^3 \setminus E^{(1)}(3)$ and $\Phi_3^{(1)}$ is 2-valent on the the set of the points in $\mathbb{C}^3 \setminus E^{(1)}(3)$ satisfying that

$$4c_0c_2^3 - c_1^2c_2 - 4c_0c_1c_2 + c_1^3 + c_0^2 \neq 0 \text{ or } 2c_2^3 - 2c_1c_2 + c_0 \neq 0.$$ 

Proof. The map $\Phi_3^{(1)}$ is defined by

$$(a, b, c) \mapsto (c_0, c_1, c_2) = \left( \frac{ab^2 - c}{2}, ab + b^2, b + \frac{a}{2} \right).$$ 

For $c = (c_0, c_1, c_2) \in \mathbb{C}^3 \setminus E^{(1)}(3)$, every $(\Phi_3^{(1)})^{-1}(c)$ is given by

$$\begin{cases} B = b^2 - 2c_2b + c_1 = 0 \\ C = (4c_2^2 - 2c_1)b + c - 2c_1c_2 + 2c_0 = 0 \\ A = a + 2b - 2c_2 = 0, \end{cases} \quad (1)$$

which has exactly 2 solutions except for discriminant$_b(B) = c_2^2 - c_1 = 0$. 

92
The map $\Phi^{(1)}_{3}$ is not defined on $\{(a, b, c) \mid c = 0\}$ where
\[
\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.
\]
From (1), for each $(c_0, c_1, c_2)$, the coefficient $c$ is determined by
\[
-c^2 + (-8c_2^3 + 8c_1c_2 - 4c_0)c - 16c_0c_1c_2 - 4c_1^3 - 4c_0^2 = 0. \tag{2}
\]
Therefore, the exceptional set $E^{(1)}(3)$ corresponds to the condition that the equation (2) has 0 as a unique solution. Thus we have
\[
E^{(1)}(3) = \{4c_0c_2^3 - c_1^2c_2^2 - 4c_0c_1c_2 + c_1^3 + c_0^2 = 0 \text{ and } 2c_2^3 - 2c_1c_2 + c_0 = 0\}.
\]

### 3.1.2 The case that $\infty$ is double critical point

Let $R(z) = z^3 + az + b$ be a polynomial map in $CB^{(2)}_{3}$, and $z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi^{(2)}_{3} : CB^{(2)}_{3} \rightarrow \mathbb{C}^2$ is defined by sending $(a, b)$ to $(c_0, c_1)$.

**Proposition 5.** The map $\Phi^{(2)}_{3}$ is bijective.

**Proof.** Since the map $\Phi^{(2)}_{3}$ is given by $(a, b) \mapsto (c_0, c_1) = \left(\frac{2a}{3}, \frac{b}{3}\right)$, the assertion follows.

### 3.2 The case of degree 4

**Proposition 6.**

1. For each class in $X^{(1)}_{4}$, there is a unique representative in $CB^{(1)}_{4}$, where
\[
CB^{(1)}_{4} = \left\{ R(z) = z^2 + cz + \frac{a_1z + a_0}{z^2 + b_1z + b_0} \mid (a_0a_1b_1 - b_0a_1^2 - a_0^2 \neq 0) \right\}.
\]

2. For each class in $X^{(2)}_{4}$, there is a unique representative in $CB^{(2)}_{4}$, where
\[
CB^{(2)}_{4} = \left\{ R(z) = z^3 + a_2z^2 + a_1z + \frac{c}{z + b} \mid (c \neq 0) \right\}.
\]

3. For each class in $X^{(3)}_{4}$, there is a unique representative in $CB^{(3)}_{4}$, where
\[
CB^{(3)}_{4} = \left\{ R(z) = z^4 + a_3z^3 + a_2z^2 + a_1z \right\}.
\]
3.2.1 The case that $\infty$ is simple critical point

Let $R = \frac{P}{Q}$ be a rational map in $CB_4^{(1)}$, and $z^5 + c_4z^4 + \cdots + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_4^{(1)} : CB_4^{(1)} \rightarrow \mathbb{C}^5$ is defined by sending $(a_0, a_1, b_0, b_1, c)$ to $(c_0, \cdots, c_4)$.

**Proposition 7.** The ramification locus of the map $\Phi_4^{(1)}$ is given by

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0b_1 - a_1)c + 4b_0b_1^2 + 2a_1b_1 - 2a_0 - 16b_0^2 = 0,$$

$\Phi_4^{(1)}(CB_4^{(1)}) = \mathbb{C}^5 \setminus E^{(1)}(4)$, and $\Phi_4^{(1)}$ is 5-valent on the set of points in $\mathbb{C}^5 \setminus E^{(1)}(4)$, where defining equation of $E^{(1)}(4)$ is given in the proof.

**Proof.** The five critical points of $R$ is given as the solution of the following equation,

$$2z^5 + (c + 4b_1)z^4 + (2b_1c + 2b_1^2 + 4b_0)z^3 + ((b_1^2 + 2b_0)c + 4b_0b_1 - a_1)z^2 + (2b_0b_1c - 2a_0 + 2b_0^2)z + b_0^2c - a_0b_1 + b_0a_1 = 0. \quad (3)$$

Therefore, the map $\Phi_4^{(1)}$ is defined by $(a_0, a_1, b_0, b_1, c) \mapsto (c_0, \cdots, c_4)$, where

$$c_0 = (b_0^2c - a_0b_1 + b_0a_1)/2,$$

$$c_1 = (2b_0b_1c - 2a_0 + 2b_0^2)/2,$$

$$c_2 = ((b_1^2 + 2b_0)c + 4b_0b_1 - a_1)/2,$$

$$c_3 = (2b_1c + 2b_1^2 + 4b_0)/2,$$

$$c_4 = (c + 4b_1)/2. \quad (4)$$

The ramification locus is obtained from the Jacobian of the map $\Phi_4^{(1)}$,

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0b_1 - a_1)c + 4b_0b_1^2 + 2a_1b_1 - 2a_0 - 16b_0^2 = 0.$$

For $c \in \mathbb{C}^5 \setminus E^{(1)}(4)$, every $(\Phi_4^{(1)})^{-1}(c)$ is given by,

$$\begin{cases} B_1 = 81b_1^5 - 162c_4b_1^4 + (108c_4^2 + 54c_3)b_1^3 + (-24c_4^3 - 72c_3c_4 + 12c_2)b_1^2 \\ + (2c_3^2c_4^2 - 8c_2c_4 + 9c_3^2 - 4c_1)b_1 - 6c_3^2c_4 + 4c_2c_3 + 8c_0 \\ B_0 = -3b_1^2 + 2c_4b_1 + 2b_0 - c_3 \\ A_1 = -10b_1^4 + 12c_4b_1^3 + (-4c_4^2 - 2c_3)b_1 - a_1 + 2c_3c_4 - 2c_2 \\ A_0 = 15b_1^4 - 16c_4b_1^3 + (4c_4^2 + 2c_3)b_1^2 + 4a_0 - c_3^2 + 4c_1 \\ C = c + 4b_1 - 2c_4, \end{cases}$$

which has exactly 5 solutions except for discriminant $b_1(B_1) = 0.$
\[1296c_0c_1^2c_4^7 + ((-1296c_0c_1c_2 - 324c_1^3)c_3 + 384c_0c_2^3 + 108c_2^2c_2^2 - 7776c_0^2c_1)c_4^5 + (324c_0c_1c_3^2 + (-108c_0c_2^2 + 324c_1^2c_2)c_3^2 + (-204c_1c_2^2 + 888c_0c_2c_2 - 7452c_0c_1)c_3 + 32c_2^2 - 936c_0c_1c_2^2 + 108c_1c_2 - 11664c_0c_3)c_4^3 + (-81c_1c_3^2 + (54c_1c_2^2 - 972c_0c_2)c_3^2 + (-9c_3^3 + 8316c_0c_1c_2 + 2106c_2^2)c_3 + (-2412c_0c_3 - 738c_1c_4^2 + 49572c_0c_1c_3)c_3^2 + 8c_1c_4^2 + 108c_0c_2^2 + 4284c_0c_1c_2 + 27c_1c_4)c_4^4 + (-1944c_0c_1c_3 + (648c_0c_2^2 - 2052c_1c_2)c_3^2 + (1296c_1c_3^2 - 2462c_0c_2^2 + 9288c_0c_1c_2)c_3^3 + (-204c_3^2 + 1512c_0c_1c_2 - 1800c_2^3) + 72900c_3^2)c_3 + 1320c_0c_4^2 + 368c_2^2c_4^2 - 26460c_0c_1c_3c_2 + 3396c_0c_1c_3)c_4^4 + (486c_0c_2^3 + (-324c_0c_2^2 + 5832c_2^2)c_3^2 + (54c_2^3 - 13608c_0c_1c_2 - 3834c_0c_2)c_3^3 + (3672c_0c_3 + 2592c_0c_2^2 - 86670c_0c_1c_3c_2 + (-738c_1c_3^2 + 12690c_0c_1c_3^2 - 13284c_0c_0c_1c_3^2 - 984c_1c_4)c_3 + 108c_2^2 - 2124c_0c_1c_3^2 + 634c_2^3c_2^2 + 45050c_0c_3c_2 + 49950c_0^2c_1)c_4^4 + (2916c_0c_1c_3^2 + (-972c_0c_2^2 + 3240c_0c_2^2)c_3^2 + (-2052c_0c_3 + 38880c_0c_2^2 + 6156c_0c_1c_2)c_3^3 + (324c_0^2 + 3024c_0c_2^2 + 5544c_0c_1c_3^2 + 121500c_0c_1c_2^2 + (-3888c_0c_1c_3^2 - 1800c_2^3c_2 + 118800c_0c_1c_3^2 + 12240c_0c_2^2) + 108c_1c_2^2 - 8100c_0c_3c_2 - 5220c_0c_1c_3c_2 + 352c_0c_3c_2 + 202500c_0c_2c_1)c_4 - 729c_0c_2^3c_2^2 + 486c_0c_2^3c_2^2 - 8748c_0c_2^3c_2^2 + (-81c_1c_2^3 + 972c_0c_1c_2 + 972c_1) + (324c_0c_3^2 + 3834c_0c_2^2 + 12150c_0c_1c_3^2 + (2106c_1c_3^2 - 36450c_0c_1c_3^2 - 18360c_0c_1c_3^2 - 432c_1c_3^2 + (-324c_2^3 + 14580c_0c_1c_3^2 - 984c_2^3c_2^2 - 202500c_0c_2c_1c_2 - 27000c_1c_3^3)c_3 - 648c_0c_2^2 + 27c_1c_2^2 + 20250c_0c_1c_2^2 - 24000c_0c_1c_2^2 + 64c_1^5 - 253125c_2^4 = 0.\]

The map \(\Phi_4^{(1)}\) is not defined on

\[r := \text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = -a_0a_1b_1 + b_0a_1^2 + a_0^2 = 0.\]

From (4), for each \((c_0, \cdots, c_4)\), \(r\) is determined by the equation of the form,

\[8503056r^5 + P_4r^4 + P_3r^3 + P_2r^2 + P_1r + P_0 = 0\]

\[(P_k \in \mathbb{C}[c_0, c_1, c_2, c_3, c_4], \ k = 0, 1, 2, 3, 4). \ (5)\]

Therefore, the exceptional set \(E^{(1)}(4)\) corresponds to the condition that this equation has 0 as a unique solution. Thus we have

\[E^{(1)}(4) = \{P_0 = P_1 = P_2 = P_3 = P_4 = 0\},\]

where

\[P_0 = -256(256c_0^3c_3^5 + (-192c_2^2c_3c_4 - 128c_0c_2^3c_2 - 144c_0c_2^2c_2 - 27c_1)c_4^4 + ((144c_0c_2^2 - 6c_0c_1^2)c_5^2 + (-800c_0c_0c_2^2 + 18c_0c_1c_2 - 1600c_0)c_3 + 16c_0c_1c_2 + 4c_0c_0c_2^2 + 160c_0c_1c_2 - 36c_0c_0c_2^2)c_4^4 + (-27c_0c_2^4 + (18c_0c_1c_2 - 4c_0c_0)c_3^3 + (-4c_0c_0^2 + c_1c_2^2 + 1020c_0c_1)c_3^2 + (560c_0^2c_2^2 - 746c_0c_1c_2 + 144c_0c_2^2)c_3 + 24c_0c_0c_1c_2 - 6c_0c_1c_2 + 2000c_0c_2 - 50c_0c_1c_2)c_4^4 + ((-630c_0c_2 + 24c_0c_0c_2^2 + (356c_0c_1c_2 - 80c_0c_2 + 2250c_0)c_3^2 + (-72c_0c_2 + 18c_0c_2^2 - 2050c_0c_1c_2 + 160c_0c_1c_2 - 900c_0c_2^2 + 1020c_0c_2^2 - 192c_1c_2 - 2500c_0c_1)c_4 + 108c_0c_3 + (72c_0c_1c_2 + 16c_0c_2^2 - 2c_0c_0c_2 - 900c_0c_1c_2 + 135c_0c_1c_2 + 560c_0c_1c_2 - 128c_1c_2^2 + (-630c_0c_1c_2 + 144c_0c_2^2 - 375c_0c_2^2 + 200c_0c_2^2)c_3 + 108c_0c_3^2 - 27c_1^2c_2^2 + 2250c_0c_1c_2 - 1600c_0c_1c_2 + 256c_1^2 + 3125c_0^4).\]
$$P_1 = 256((144c_0c_2 - 54c_0^2)c_4^2 + (-54c_0c_2^2 + 18c_1c_2c_3 - 4c_2^3 - 36c_0c_1)c_4^2 + ((-702c_0c_2 + 279c_0^2)c_3 - 6c_1c_2^2 - 1350c_1c_2c_3^2 + (243c_0^2c_3 - 81c_1c_2^3c_3 + (18c_2^3 + 810c_0c_1)c_3 + 1440c_0c_2 - 624c_1c_2c_4 + (-405c_0c_2 - 216c_1c_2^2) + (279c_1c_2 + 3375c_2^2) + 54c_0^3 - 3600c_0c_1c_2 + 1120c_1c_1^2c_3 + (-192c_2c_1 - 128c_0c_2c_1^2 + 144c_0c_1c_2c_2 - 27c_0c_2^2) + ((144c_0c_2 - 6c_0c_1c_2^2 + 18c_1c_2 - 1600c_0^3)c_3 + (160c_2^4 - 4c_2^3c_1 + 160c_2c_1c_2 - 36c_0c_1)c_4^2 + (-27c_2c_1^2 + (18c_0c_1c_2 - 4c_1^3)c_4^3 + (-4c_0c_2^3 + (c_2^2 + 1020c_0c_1)c_2^2 + (560c_0^2c_2^2 - 7486c_0c_1c_2 + 144c_0^4)c_3 + 24c_0c_1c_2 - 6c_0c_2^2 + 2000c_0c_1c_2 - 50c_0c_1^2c_2^4 + ((-600c_0c_2 + 24c_0c_1)c_3^3 + (356c_0c_1c_2 - 80c_1^2c_2 + 2250c_0^3)c_3^2 + (-720c_0^4 + 18c_2^3c_0c_2 - 2050c_0c_1c_3^2 + 900c_0^2c_2^3 + 1020c_0c_1c_2c_3^2 - 1920c_0c_1c_2 - 2500c_0c_1c_2c_3^2 + 108c_0^2c_2^3 + (-720c_0c_1c_2 + 16c_0c_1c_2 - 4c_1^2c_2 - 900c_0^2c_1c_2 + (825c_0^2c_2^2 + 560c_0c_1c_2^2 - 128c_0^2c_2) + (-630c_0c_1c_2^2 + 144c_0^2c_2 - 3750c_0c_2c_3^2 + 2000c_0c_2c_3^2)c_3 + 108c_0^5c_2 - 27c_2^4 + 2250c_0c_1c_2c_2 + 256c_1^3 + 3125c_0^2),)

$$

$$P_2 = 864(2048c_0c_1c_3 - 1024c_0^2c_2c_4 + (1152c_0c_2 - 48c_0^2)c_3^3 + (640c_0c_1c_3 + 144c_0^2c_2c_4 - 23040c_0^3)c_3 + 1280c_0^2c_2 - 32c_1c_2c_3 + 1280c_0c_1c_2c_4 - 288c_0c_1c_3^2 + (-216c_0c_1c_2c_4 - (1440c_0c_1c_2 - 32c_1c_2c_3 + 8c_0^2c_2 + 15840c_0c_1c_2c_4 + (9600c_0^2c_1^2 - 117280c_0c_1c_2c_4 + 2232c_1c_2c_3^2 + 900c_0c_1c_2c_3 - 1020c_0c_1c_2c_3c_2 - 1920c_0c_1c_2c_3c_2 - 2500c_0c_1c_2c_3c_2 + 108c_0^2c_1c_2c_3 + (-720c_0c_1c_2c_3^2 + 16c_0c_1c_2c_3^2 - 4c_1^2c_2c_3 - 900c_0^2c_1c_2c_3^2 + (825c_0^2c_2^2 + 560c_0c_1c_2^2 - 128c_0^2c_2c_4 + (-630c_0c_1c_2^2 + 144c_0^2c_2c_4 - 3750c_0c_2c_3^2 + 2000c_0c_2c_3^2)c_3 + 108c_0^5c_2 - 27c_2^4 + 2250c_0c_1c_2c_2 + 256c_1^3 + 3125c_0^2),)

$$
\[ P_3 = 11664(768c_0^2c_1^4 + (-384c_0c_1c_3 - 256c_0^2c_2 + 144c_1^2c_2)c_4^5 + ((-54c_0c_1 + 18c_1c_2)c_3 + (-4c_2^2 + 216c_0c_1)c_2^3 + ((-1350c_0c_2 - 9c_1^2)c_3^3 + (402c_1c_2^2 + 9450c_0^2)c_3^2 + ((-80c_1^3 - 9720c_0c_1c_2 + 2500c_0^3)c_3 - 3040c_0c_2^2 + 1416c_1^2c_2 - 360c_1^3)c_4^3 + ((-4c_2^{3} - 900c_0c_1c_2 - 2240c_1^2)c_3^3 + (3960c_0c_2^2 + 2500c_1^2c_2^2 + 22500c_0^2c_1)c_3 - 360c_1c_2^4 + 30000c_0^{2}c_2^2 - 40000c_0c_1c_2^2 + 9600c_1^4),
\]

\[ P_4 = 19683(768c_0c_4^3 + (-192c_1c_3 - 128c_2^2)c_4^2 + (144c_2c_3^2 - 45000c_0c_3 + 1024c_1c_2)c_4 - 27c_3^4 + 216c_1c_3^2 - 192c_2^2c_3 + 4800c_0c_2 - 2480c_1^4),
\]

### 3.2.2 The case that \( \infty \) is double critical point

Let \( R = \frac{P}{Q} \) be a rational map in \( CB_{4}^{(2)} \), and \( z^4 + c_3z^3 + \cdots + c_0 = 0 \) be the equation defined by \( P'(z)Q(z) - P(z)Q'(z) = 0 \).

Then, the map \( \Phi_{4}^{(2)} : CB_{4}^{(2)} \to \mathbb{C}^4 \) is defined by sending \((a_1, a_2, b, c)\) to \((c_0, \cdots, c_3)\).

**Proposition 8.** The ramification locus of \( \Phi_{4}^{(2)} \) is given by \( 3b^2 - 2a_2b + a_1 = 0 \), \( \Phi_{4}^{(2)}(CB_{4}^{(2)}) = \mathbb{C}^4 \setminus E^{(2)}(4) \), and \( \Phi_{4}^{(2)} \) is 3-valent on the set of the points in \( \mathbb{C}^4 \setminus E^{(2)}(4) \), where the defining equation of \( E^{(2)}(4) \) is given in the proof.

**Proof.** The four critical points of \( R \) in \( \mathbb{C} \) is given as the solution of

\[ 3z^4 + (6b + 2a_2)z^3 + (3b^2 + 4a_2b + a_1)z^2 + (2a_2b^2 + 2a_1b)z + a_1b^2 - c = 0. \]

Therefore, the map \( \Phi_{4}^{(2)} \) is defined by \((a_1, a_2, b, c) \mapsto (c_0, \cdots, c_3)\), where

\[
\begin{align*}
    c_0 &= (a_1b^2 - c)/3, \\
    c_1 &= (2a_2b^2 + 2a_1b)/3, \\
    c_2 &= (3b^2 + 4a_2b + a_1)/3, \\
    c_3 &= (6b + 2a_2)/3. \\
\end{align*}
\]

The ramification locus is obtained from the Jacobian of the map \( \Phi_{4}^{(2)} \),

\[ 3b^2 - 2a_2b + a_1 = 0. \]

For \( c \in \mathbb{C}^4 \setminus E^{(2)}(4) \), every \((\Phi_{4}^{(2)})^{-1}(c)\) is given by,

\[
\begin{align*}
    B &= 4b^3 - 3c_3b^2 + 2c_2b - c_1 \\
    A_1 &= -9b^2 + 6c_3b + a_1 - 3c_2 \\
    A_2 &= 6b + 2a_2 - 3c_3 \\
    C &= (9c_3^2 - 24c_2)b^2 + (-6c_2c_3 + 36c_1)b - 16c + 3c_1c_3 - 48c_0,
\end{align*}
\]
which has exactly 2 solutions except for discriminant$_b(B) = 0$.

The map $\Phi_{4}^{(2)}$ is not defined on

$$\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.$$ 

From (6), for each $(c_0, \cdots, c_3)$, $c$ is determined by the equation,

$$256c^3 - 3(27c_3^4 - 144c_2c_3^2 + 192c_1c_3 + 128c_2^2 - 768c_0)c^2$$

$$- 18(27c_0c_3^4 - 9c_1c_2c_3^3 + (2c_2^3 - 144c_0c_2 + 3c_1^2)c_3 + (40c_1c_2^2 + 192c_0c_1)c_3$$

$$- 8c_2^4 + 128c_0c_2^2 - 72c_1^2c_2 - 384c_0^2)c$$

$$- 27(27c_0c_3^4 + (-18c_0c_1c_2 + 4c_1^3)c_3^3 + (4c_0c_2^3 - c_1^2c_2^2 - 144c_0^2c_2 + 6c_0c_1^2)c_3^2$$

$$+ (80c_0c_1c_2^2 - 18c_1^3c_2 + 192c_2^2c_1)c_3 - 16c_0c_2^4 + 4c_1^2c_2^3 + 128c_0c_1^2c_2$$

$$+ 27c_1^4 - 256c_0^3 = 0.$$ 

Therefore, the exceptional set $E^{(2)}(4)$ corresponds to the condition that this equation has 0 as a unique solution.

Hence, the defining equation of $E^{(2)}(4)$ is

$$P_0 = P_1 = P_2 = 0,$$

where

$$P_0 = -729c_0c_1c_2 - 108c_1^3)c_3^3$$

$$+ (-108c_0c_2^2 + 27c_1^2c_2 + 3888c_0c_2 - 162c_0c_1^2)c_3^3$$

$$+ (-2160c_0c_1c_2 + 486c_3^3c_2 - 5184c_0^2c_1)c_3 + 432c_0c_2^4 - 108c_1^2c_2^3$$

$$- 3456c_0c_2^2 + 3888c_0c_1^2c_2 - 729c_1^4 + 6912c_0^3;$$

$$P_1 = -486c_0c_1c_2^3 + 162c_1c_2c_3^3 + (-36c_2^3 + 2592c_0c_2 - 54c_1^2)c_3^3$$

$$+ (-720c_1c_2^2 - 3456c_0c_1)c_3 + 144c_3^4 - 2304c_0c_2^2 + 1296c_1^2c_2 + 6912c_0^3;$$

$$P_2 = -81c_3^4 + 432c_2c_3^2 - 576c_1c_3 - 384c_2^2 + 2304c_0.$$ 

\[ \square \]

3.2.3 The case that $\infty$ is triple critical point

Let $R$ be a polynomial map in $CB_{4}^{(3)}$, $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi_{4}^{(3)} : CB_{4}^{(3)} \to \mathbb{C}^3$ is defined by sending $(a_1, a_2, a_3)$ to $(c_0, c_1, c_2)$.

**Proposition 9.** The map $\Phi_{4}^{(3)}$ is bijective.
Proof. The three critical points of \( R \) in \( \mathbb{C} \) is given as the solution of the following equation
\[
4z^3 + 3a_3z^2 + 2a_2z + a_1 = 0.
\]
Therefore, the map \( \Phi_4^{(3)} \) is defined by
\[
(a_1, a_2, a_3) \mapsto (c_0, c_1, c_2) = \left(\frac{a_1}{4}, \frac{2a_2}{4}, \frac{3a_3}{4}\right),
\]
and the assertion follows.

For \( d = 3, 4 \), the complete answer for the problem of Goldberg is obtained.

References


