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Kyoto University

京都大学
THE REAL REPRESENTATION ASSOCIATED WITH COPRIME NORMAL SUBGROUPS

Masaharu Morimoto

Graduate School of Natural Science and Technology, Okayama University

Dedicated to Professor K. Shimakawa on his 60th birthday

Abstract. Let $G$ be a finite group. In this article, we introduce a mutually coprime family of normal subgroups of $G$ and the real $G$-module associated with the family, and we report interesting results on the real $G$-module.

1. PRELIMINARY

Throughout this paper, $G$ is a finite group. We mean by a real $G$-module a real $G$-representation space of finite dimension. Let $S(G)$ denote the set of all subgroups of $G$.

In the study of smooth $G$-actions on disks and spheres, there are important families of normal subgroups of $G$: for examples, $\{G\}$, $\{G^{(2)}\}$, $\{G^{ni1}\}$, $\mathcal{K}(G) = \{G^{\{p\}} \mid p \text{ is a prime}\}$, and $\mathcal{N}_p(G) = \{H \leq G \mid |G/H| = 1 \text{ or } p\}$,

where $G^{\{p\}}$ is the smallest normal subgroup $H$ such that $G/H$ has order of $p$-power (possibly $|G/H| = 1$), and $G^{ni1}$ is the smallest normal subgroup $N$ such that $G/N$ is nilpotent.

Let $\mathcal{L}$ be a set of subgroups of $G$ such that each minimal element of $\mathcal{L}$ is a normal subgroup of $G$. Let $\mathbb{R}[G]$ denote the regular representation of $G$ and let $\mathbb{R}[G]^{\mathcal{L}}$ denote the smallest $G$-submodule of $\mathbb{R}[G]$ containing all $\mathbb{R}[G]^L$ with $L \in \mathcal{L}$. Let $\mathbb{R}[G]_{\mathcal{L}}$ be
the orthogonal complement of $\mathbb{R}[G]^\mathcal{L}$ in $\mathbb{R}[G]$ with respect to some $G$-invariant inner product on $\mathbb{R}[G]$, i.e.

$$\mathbb{R}[G]_\mathcal{L} = \mathbb{R}[G] - \mathbb{R}[G]^\mathcal{L}.$$ 

In this paper we call $\mathbb{R}[G]_\mathcal{L}$ the real $G$-module associated with $\mathcal{L}$.

**Definition 1.1.** A nonempty family $\mathcal{K}$ of normal subgroups of $G$ is called *mutually coprime* if either

1. $\mathcal{K} = \{G\}$, or
2. $G \notin \mathcal{K}$ and $|G/K|$'s are mutually prime integers, i.e.

$$\left(|G/K|, |G/K'|\right) = 1 \text{ for all } K, K' \in \mathcal{K} \text{ such that } K \neq K'.$$

If $\mathcal{K}$ is a mutually coprime family of normal subgroups of $G$, then the equality

(1.1) $$\mathbb{R}[G]_{\mathcal{K}} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{K \in \mathcal{K}} (\mathbb{R}[G/K] - \mathbb{R})$$

holds, where $\mathbb{R}$ is the 1-dimensional trivial real $G$-module.

**Definition 1.2.** Let $\mathcal{L}$ be a set of subgroups of $G$. Then we define the *upper closure* $\overline{\mathcal{L}}$ of $\mathcal{L}$ by

(1.2) $$\overline{\mathcal{L}} = \{H \in S(G) \mid H \supset L \text{ for some } L \in \mathcal{L}\},$$

and the *exterior* $\underline{\mathcal{L}}$ of $\mathcal{L}$ by

(1.3) $$\underline{\mathcal{L}} = S(G) \setminus \overline{\mathcal{L}}.$$

With this notation, we have $\mathcal{L}(G) = \overline{\mathcal{K}(G)}$ and $\mathcal{M}(G) = \mathcal{K}(G)$, cf. E. Laitinen-M. Morimoto [1].

**Definition 1.3.** Let $V$ be a real $G$-module and $\mathcal{H}$ a family of subgroups of $G$. We say that $V$ is *$\mathcal{H}$-complete* if for each $H \in \mathcal{H}$, any irreducible real $H$-module is isomorphic to a submodule of $\text{res}_H^G V$.

The main results which will be reported in this article are Theorems 2.1, 2.2 and 3.2. The proofs will appear somewhere else.
2. Completeness and gap property

Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. We introduce two practically important properties of $\mathbb{R}[G]_{\mathcal{K}}$ as the theorems below.

**Theorem 2.1.** Let $G$ be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. Then for any $H \in \mathcal{K}$, $\text{res}_{H}^{G}\mathbb{R}[G]_{\mathcal{K}}$ contains a real $H$-submodule isomorphic to $\mathbb{R}[H]$. Hence the real $G$-module $\mathbb{R}[G]_{\mathcal{K}}$ is $\mathcal{K}$-complete.

**Theorem 2.2.** Let $G$ be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$. Then the real $G$-module $\mathbb{R}[G]_{\mathcal{K}}$ possesses the following properties.

1. $\mathbb{R}[G]_{\mathcal{K}}^{H} \neq 0$ if and only if $H \in \mathcal{K}$.
2. Let $p$ be a prime and $H < K \leq G$ with $|K : H| = p$. Then
   \[ \dim \mathbb{R}[G]_{\mathcal{K}}^{H} \geq p \dim \mathbb{R}[G]_{\mathcal{K}}^{K} \]
   holds; the equality holds if and only if there exists $K_{k} \in \mathcal{K}$ such that $p||G : K_{k}|$, $|KK_{k} : HK_{k}| = p$, and $HK_{i} = G$ for all $K_{i} \in \mathcal{K} \setminus \{K_{k}\}$.
3. Let $H < K \leq G$. Then
   \[ \dim \mathbb{R}[G]_{\mathcal{K}}^{H} \geq 2 \dim \mathbb{R}[G]_{\mathcal{K}}^{K} \]
   holds; the equality holds if and only if
   (a) $H \in \mathcal{K}$, or
   (b) $K \in \mathcal{K}$, $|K : H| = 2$, there exists $K_{k} \in \mathcal{K}$ such that $2||G : K_{k}|$, $|KK_{k} : HK_{k}| = 2$ and $HK_{i} = G$ for all $K_{i} \in \mathcal{K} \setminus \{K_{k}\}$.

The next proposition has been used in the induction argument of the equivariant surgery theory, cf. [1, 4, 5].

**Proposition 2.3.** Let $G$ be an Oliver group, and let $P$, $H_{1}$, $H_{2}$ be subgroups of $G$ such that $P \in \mathcal{P}(G)$, $P < H_{1}$, and $P < H_{2}$. If the equality

$$2\dim \mathbb{R}[G]_{\mathcal{L}(G)}^{H_{i}} = \dim \mathbb{R}[G]_{\mathcal{L}(G)}^{P}$$

holds for each $i = 1$ and 2, then the smallest subgroup $K$ containing $H_{1}$ and $H_{2}$ belongs to $\mathcal{M}(G) = \mathcal{S}(G) \setminus \mathcal{L}(G)$. 
3. Canonical line bundle of real projective space

Let $V$ be a real $G$-module (of finite dimension). The real projective space $P(V)$ is the space of all 1-dimensional real vector subspaces of $V$, and $P(V)$ has the canonically induced $G$-action. Let $\gamma_M$, where $M = P(V)$, denote the canonical line bundle of $M$.

**Lemma 3.1.** Let $V$ be a real $G$-module and $M = P(V)$. Then the following equalities hold as real $G$-vector bundles via canonical isomorphisms.

1. $\text{Hom}(\gamma_M, \gamma_M) = \epsilon_M(\mathbb{R})$.
2. $\text{Hom}(\gamma_M, \epsilon_M(\mathbb{R})) = \gamma_M$.
3. $T(M) = \text{Hom}(\gamma_M, \gamma_M^\perp)$.
4. $T(M) \oplus \epsilon_M(\mathbb{R}) = \text{Hom}(\gamma_M, \epsilon_M(V))$.
5. $\text{Hom}(\gamma_M, \epsilon_M(V)) = \gamma_M \otimes V$.

The equalities (1)–(4) above follow from the proof of [3, Lemma 4.4]. The equality (5) holds because

$$\text{Hom}(\gamma_M, \epsilon_M(V)) = \text{Hom}(\gamma_M, \epsilon_M(\mathbb{R})) \otimes_{\mathbb{R}} V = \gamma_M \otimes_{\mathbb{R}} V.$$

**Theorem 3.2.** Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of $G$ and let $V$ be a real $G$-module such that $V = V^\mathcal{K}$. Then for $K_i \in \mathcal{K},$

\[
P(V)^{K_i} = \begin{cases} 
P(V^{K_i}) & \text{if } 2 \| G : K_i \\
\prod_{L \in \mathcal{A}_i} P(V^{L_{G/L}}) & \text{if } 2 \int |G : K_i|
\end{cases}
\]

and

\[
\gamma_{P(V)}|_{P(V)^{K_i}} = \begin{cases} 
\gamma_{P(V^{K_i})} & \text{if } 2 \| G : K_i \\
\prod_{L \in \mathcal{A}_i} \gamma_{P(V^{L_{G/L}})} & \text{if } 2 \int |G : K_i|,
\end{cases}
\]

where $\mathcal{A}_i$ is the set of all subgroups $L$ such that $|G : L| = 2$ and $|K_i : K_i \cap L| = 2$.

In addition

\[
(\gamma_{P(V)} \otimes_{\mathbb{R}} V)^{K_i} = \begin{cases} 
\gamma_{P(V^{K_i})} \otimes_{\mathbb{R}} V^{K_i} & \text{if } 2 \| G : K_i \\
\prod_{L \in \mathcal{A}_i} \gamma_{P(V^{L_{G/L}})} \otimes_{\mathbb{R}} V^{L_{G/L}} & \text{if } 2 \int |G : K_i|
\end{cases}
\]

$$= T(P(V)^{K_i}) \oplus \epsilon_{P(V)^{K_i}}(\mathbb{R}).$$
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Graduate School of Natural Science and Technology
Okayama University
Tsushimanaka 3-1-1
Okayama, 700-8530 Japan
E-mail address: morimoto@ems.okayama-u.ac.jp