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THE REAL REPRESENTATION ASSOCIATED WITH COPRIME NORMAL SUBGROUPS

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Dedicated to Professor K. Shimakawa on his 60th birthday

Abstract. Let G be a finite group. In this article, we introduce a mutually coprime family of normal subgroups of G and the real G-module associated with the family, and we report interesting results on the real G-module.

1. Preliminary

Throughout this paper, G is a finite group. We mean by a *real G-module* a real G-representation space of finite dimension. Let $\mathcal{S}(G)$ denote the set of all subgroups of G.

In the study of smooth G-actions on disks and spheres, there are important families of normal subgroups of G: for examples, $\{G\}$, $\{G^{\{2\}}\}$, $\{G^{\mathrm{nil}}\}$,

$$\mathcal{K}(G) = \{G^{\{p\}} \mid p \text{ is a prime}\}, \text{ and}$$
$$\mathcal{N}_p(G) = \{H \trianglelefteq G \mid |G/H| = 1 \text{ or } p\},$$

where $G^{\{p\}}$ is the smallest normal subgroup H such that G/H has order of p-power (possibly $|G/H| = 1$), and G^{nil} is the smallest normal subgroup N such that G/N is nilpotent.

Let $\mathcal{L}$ be a set of subgroups of G such that each minimal element of $\mathcal{L}$ is a normal subgroup of G. Let $\mathbb{R}[G]$ denote the regular representation of G and let $\mathbb{R}[G]^{\mathcal{L}}$ denote the smallest G-submodule of $\mathbb{R}[G]$ containing all $\mathbb{R}[G]^L$ with $L \in \mathcal{L}$. Let $\mathbb{R}[G]_{\mathcal{L}}$ be

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the orthogonal complement of $\mathbb{R}[G]^{\mathcal{L}}$ in $\mathbb{R}[G]$ with respect to some G-invariant inner product on $\mathbb{R}[G]$, i.e.

$$\mathbb{R}[G]_{\mathcal{L}} = \mathbb{R}[G] - \mathbb{R}[G]^{\mathcal{L}}.$$

In this paper we call $\mathbb{R}[G]_{\mathcal{L}}$ the *real G-module associated with $\mathcal{L}$.*

Definition 1.1. A nonempty family $\mathcal{K}$ of normal subgroups of G is called *mutually coprime* if either

(1) $\mathcal{K} = \{G\}$, or

(2) $G \notin \mathcal{K}$ and $|G/K|$'s are mutually prime integers, i.e.

$$(|G/K|, |G/K'|) = 1 \text{ for all } K,\ K' \in \mathcal{K} \text{ such that } K \neq K'.$$

If $\mathcal{K}$ is a mutually coprime family of normal subgroups of G, then the equality

$$\mathbb{R}[G]_{\mathcal{K}} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{K \in \mathcal{K}} (\mathbb{R}[G/K] - \mathbb{R}) \tag{1.1}$$

holds, where $\mathbb{R}$ is the 1-dimensional trivial real G-module.

Definition 1.2. Let $\mathcal{L}$ be a set of subgroups of G. Then we define the *upper closure* $\overline{\mathcal{L}}$ of $\mathcal{L}$ by

$$\overline{\mathcal{L}} = \{H \in \mathcal{S}(G) \mid H \supset L \text{ for some } L \in \mathcal{L}\}, \tag{1.2}$$

and the *exterior* $\underline{\mathcal{L}}$ of $\mathcal{L}$ by

$$\underline{\mathcal{L}} = \mathcal{S}(G) \smallsetminus \overline{\mathcal{L}}. \tag{1.3}$$

With this notation, we have $\mathcal{L}(G) = \overline{\mathcal{K}(G)}$ and $\mathcal{M}(G) = \underline{\mathcal{K}(G)}$, cf. E. Laitinen-M. Morimoto [1].

Definition 1.3. Let V be a real G-module and $\mathcal{H}$ a family of subgroups of G. We say that V is *$\mathcal{H}$-complete* if for each $H \in \mathcal{H}$, any irreducible real H-module is isomorphic to a submodule of $\mathrm{res}^G_H V$.

The main results which will be reported in this article are Theorems 2.1, 2.2 and 3.2. The proofs will appear somewhere else.

2. Completeness and gap property

Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of G. We introduce two practically important properties of $\mathbb{R}[G]_{\mathcal{K}}$ as the theorems below.

Theorem 2.1. *Let G be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of G. Then for any $H \in \underline{\mathcal{K}}$, $\mathrm{res}^G_H \mathbb{R}[G]_{\mathcal{K}}$ contains a real H-submodule isomorphic to $\mathbb{R}[H]$. Hence the real G-module $\mathbb{R}[G]_{\mathcal{K}}$ is $\underline{\mathcal{K}}$-complete.*

Theorem 2.2. *Let G be a finite group and let $\mathcal{K}$ be a mutually coprime family of normal subgroups of G. Then the real G-module $\mathbb{R}[G]_{\mathcal{K}}$ possesses the following properties.*

(1) ${\mathbb{R}[G]_{\mathcal{K}}}^H \neq 0$ *if and only if* $H \in \underline{\mathcal{K}}$.

(2) *Let p be a prime and $H < K \leq G$ with $|K : H| = p$. Then*

$$\dim {\mathbb{R}[G]_{\mathcal{K}}}^H \geq p \dim {\mathbb{R}[G]_{\mathcal{K}}}^K$$

holds; the equality holds if and only if there exists $K_k \in \mathcal{K}$ such that $p \mid |G : K_k|$, $|KK_k : HK_k| = p$, and $HK_i = G$ for all $K_i \in \mathcal{K} \smallsetminus \{K_k\}$.

(3) *Let $H < K \leq G$. Then*

$$\dim {\mathbb{R}[G]_{\mathcal{K}}}^H \geq 2 \dim {\mathbb{R}[G]_{\mathcal{K}}}^K$$

holds; the equality holds if and only if

(a) $H \in \overline{\mathcal{K}}$, *or*

(b) $K \in \underline{\mathcal{K}}$, $|K : H| = 2$, *there exists $K_k \in \mathcal{K}$ such that $2 \mid |G : K_k|$, $|KK_k : HK_k| = 2$ and $HK_i = G$ for all $K_i \in \mathcal{K} \smallsetminus \{K_k\}$.*

The next proposition has been used in the induction argument of the equivariant surgery theory, cf. [1, 4, 5].

Proposition 2.3. *Let G be an Oliver group, and let P, H_1, H_2 be subgroups of G such that $P \in \mathcal{P}(G)$, $P < H_1$, and $P < H_2$. If the equality*

$$2 \dim {\mathbb{R}[G]_{\mathcal{L}(G)}}^{H_i} = \dim {\mathbb{R}[G]_{\mathcal{L}(G)}}^P \tag{2.1}$$

holds for each $i = 1$ and 2, then the smallest subgroup K containing H_1 and H_2 belongs to $\mathcal{M}(G) = \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$.

3. Canonical line bundle of real projective space

Let V be a real G-module (of finite dimension). The real projective space $P(V)$ is the space of all 1-dimensional real vector subspaces of V, and $P(V)$ has the canonically induced G-action. Let γ_M, where $M = P(V)$, denote the canonical line bundle of M.

Lemma 3.1. *Let V be a real G-module and $M = P(V)$. Then the following equalities hold as real G-vector bundles via canonical isomorphisms.*

(1) $\mathrm{Hom}(\gamma_M, \gamma_M) = \varepsilon_M(\mathbb{R})$.
(2) $\mathrm{Hom}(\gamma_M, \varepsilon_M(\mathbb{R})) = \gamma_M$.
(3) $T(M) = \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$.
(4) $T(M) \oplus \varepsilon_M(\mathbb{R}) = \mathrm{Hom}(\gamma_M, \varepsilon_M(V))$.
(5) $\mathrm{Hom}(\gamma_M, \varepsilon_M(V)) = \gamma_M \otimes V$.

The equalities (1)–(4) above follow from the proof of [3, Lemma 4.4]. The equality (5) holds because

$$\mathrm{Hom}(\gamma_M, \varepsilon_M(V)) = \mathrm{Hom}(\gamma_M, \varepsilon_M(\mathbb{R})) \otimes_{\mathbb{R}} V = \gamma_M \otimes_{\mathbb{R}} V.$$

Theorem 3.2. *Let $\mathcal{K}$ be a mutually coprime family of normal subgroups of G and let V be a real G-module such that $V = V^{\mathcal{K}}$. Then for $K_i \in \mathcal{K}$,*

$$P(V)^{K_i} = \begin{cases} P(V^{K_i}) & \text{if } 2 \mid |G : K_i| \\ P(V^{K_i}) \amalg \coprod_{L \in \mathcal{A}_i} P({V^L}_{G/L}) & \text{if } 2 \nmid |G : K_i| \end{cases} \tag{1}$$

and

$$\gamma_{P(V)}|_{P(V)^{K_i}} = \begin{cases} \gamma_{P(V^{K_i})} & \text{if } 2 \mid |G : K_i| \\ \gamma_{P(V^{K_i})} \amalg \coprod_{L \in \mathcal{A}_i} \gamma_{P({V^L}_{G/L})} & \text{if } 2 \nmid |G : K_i|, \end{cases} \tag{2}$$

where $\mathcal{A}_i$ is the set of all subgroups L such that $|G : L| = 2$ and $|K_i : K_i \cap L| = 2$. In addition

$$\begin{aligned} &(\gamma_{P(V)} \otimes_{\mathbb{R}} V)^{K_i} \\ &= \begin{cases} \gamma_{P(V^{K_i})} \otimes_{\mathbb{R}} V^{K_i} & \text{if } 2 \mid |G : K_i| \\ \gamma_{P(V^{K_i})} \otimes_{\mathbb{R}} V^{K_i} \amalg \coprod_{L \in \mathcal{A}_i} \gamma_{P({V^L}_{G/L})} \otimes_{\mathbb{R}} {V^L}_{G/L} & \text{if } 2 \nmid |G : K_i| \end{cases} \\ &= T(P(V)^{K_i}) \oplus \varepsilon_{P(V)^{K_i}}(\mathbb{R}). \end{aligned} \tag{3}$$

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