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ON BORSUK-ULAM GROUPS

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ABSTRACT. A Borsuk-Ulam group is a group for which the isovariant Borsuk-Ulam theorem holds. A fundamental question is: which groups are Borsuk-Ulam groups? In this article, we shall recall some properties and previous results on a Borsuk-Ulam group. After that, we provide a new family of Borsuk-Ulam groups. We also pose some open questions.

1. NOTATION AND TERMINOLOGY

Let $G$ be a compact Lie group and $V$ an (orthogonal or unitary) representation space of $G$. We denote by $SV$ the unit sphere of $V$, called a $G$-representation sphere. A $G$-equivariant map (or $G$-map for short) $f : X \to Y$ is a continuous map between $G$-spaces satisfying

$$f(gx) = gf(x), \ \forall x \in X, g \in G.$$ 

It is easy to see that if $f$ is $G$-equivariant, then

1. $f(X^H) \subset Y^H$, so we have the restriction map $f^H : X^H \to Y^H$.

2. $G_x \leq G_{f(x)} \ (\forall x \in X)$.

Definition. A continuous map $f : X \to Y$ is called a $G$-isovariant map if $f$ is a $G$-equivariant map satisfying $G_x = G_{f(x)} \ (\forall x \in X)$.

It is easy to see that $f : X \to Y$ is $G$-isovariant if and only if $f$ is a $G$-equivariant map such that $f|_{G(x)} : G(x) \to Y$ is injective for any $x \in X$, where $G(x)$ is the orbit of $x$. Similarly we define an isovariant homotopy as follows.

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**Definition.** Let $f$, $g$ be $G$-isovariant maps. We call $f$ and $g$ isovariantly $G$-homotopic if there exists a $G$-isovariant map $H : X \times I \to Y$, called a $G$-isovariant homotopy, such that $H(-, 0) = f$ and $H(-, 1) = g$.

Let $[X, Y]_G^{\text{isov}}$ denote the set of $G$-isovariant homotopy classes of $G$-isovariant maps.

By the definition of isovariance, we easily see the following.

1. Let $X$ and $Y$ be free $G$-spaces. Then $G$-equivariance is equivalent to $G$-isovariance.
2. If $f : X \to Y$ is an injective $G$-map, then $f$ is $G$-isovariant.
3. If there exists a $G$-isovariant map $f : X \to Y$, then $\text{Iso}(X) \subset \text{Iso}(Y)$, where $\text{Iso}(X)$ is the set of isotropy subgroups of $X$.

**Example 1.1.** Let $X = G/H$ and $Y = G/K$.

1. There exists a $G$-map $f : G/H \to G/K$ if and only if $(H) \leq (K)$, i.e., $H \leq aKa^{-1}$ for some $a \in G$.
2. There exists a $G$-isovariant map $f : G/H \to G/K$ if and only if $(H) = (K)$. In this case, a $G$-isovariant map $f$ is defined by $f(gH) = gaK$, $H = aKa^{-1}$.

2. Isovariant Maps Between Representations

The following result says that isovariant maps between representations are essentially same as those between representation spheres.

**Proposition 2.1.** Let $V$, $W$ be (orthogonal) $G$-representations. The following are equivalent.

1. There exists a $G$-isovariant map $f : V \to W$.
2. There exists a $G$-isovariant map $f : V^G \to W^G$.
3. There exists a $G$-isovariant map $f : S(V^G) \to S(W^G)$.

Here $V^G$ is the orthogonal complement of $V^G$ in $V$. In particular, if $V^G = W^G = 0$, then there exists a $G$-isovariant map $f : V \to W$ if and only if $f : SV \to SW$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Composing the inclusion $i$ and the projection $p$ with $f : V \to W$, we have an isovariant map

$$\overline{f} : V^G \overset{i}{\to} V \overset{f}{\to} W \overset{p}{\to} W^G.$$

Composing the inclusion $j$ and the normalization map with $\overline{f}$, we have an isovariant map

$$\overline{\overline{f}} : S(V^G) \overset{j}{\to} V^G \setminus \{0\} \overset{\overline{f}}{\to} W^G \setminus \{0\} \overset{\text{norm}}{\to} S(W^G).$$

(1) $\Leftarrow$ (2) $\Leftarrow$ (3)
Let $g : S(V^G) \rightarrow S(W^G)$ be an isovariant map. By the radial extension, we have an isovariant map

$$\tilde{g} : V^G \rightarrow W^G.$$ 

By adding the zero map to $\tilde{g}$, we have an isovariant map

$$h := \tilde{g} \oplus 0 : V = V^G \oplus V^G \rightarrow W^G \oplus W^G = W.$$

By further arguments, we also obtain

**Proposition 2.2.** When $V^G = W^G = 0$, there is a one-to-one correspondence

$$[V, W]_G^{isov} \cong [SV, SW]_G^{isov}.$$

We here provide some examples. Let $G = C_n = \langle c \rangle$ be a cyclic group of order $n$, where $c$ is a generator of $C$. Consider the irreducible representations of $C$. Let

$$U_k (= \mathbb{C}) \ (0 \leq k \leq n - 1)$$

denote the irreducible representation with the linear action:

$$c \cdot z = \xi_n^k z \ (z \in U_k), \quad \xi_n = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

Assume $n = pq$, where $p, q$ are distinct primes and $G = C_{pq}$.

**Example 2.3.** If $(k, pq) = (l, pq) = 1$, then there exist a $G$-isovariant map $f : SU_k \rightarrow SU_l$.

In fact, fix $s$ such that $ks \equiv l \mod pq$. We define a map $f$ by

$$f(z) = z^{sl}, \quad z \in SU_k.$$

Then one can check that

1. $f$ is $G$-equivariant,
2. $G$ acts freely on $SU_k$ and $SU_l$.

Hence $f$ is $G$-isovariant.

Further arguments show that the degree of maps classifies isovariant homotopy classes, and we have

$$[U_k, U_l]_{C_{pq}}^{isov} \cong [SU_k, SU_l]_{C_{pq}}^{isov} \cong \mathbb{Z},$$

and the representatives are given by

$$f_m(z) = z^{sl + mpq}, \quad z \in SU_k, \quad m \in \mathbb{Z}.$$

See [3], [4] for the detail.
Example 2.4. There do not exist isovariant maps $f : U_p \to U_q$ and $g : U_1 \to U_q$.

In fact, if $f : X \to Y$ is an isovariant map, then $\text{Iso}(X) \subset \text{Iso}(Y)$. However

$\text{Iso}(U_p) = \{C_p, G\} \not\subset \text{Iso}(U_q) = \{C_q, G\}$

and

$\text{Iso}(U_1) = \{1, G\} \not\subset \text{Iso}(U_q) = \{C_q, G\}$.

Example 2.5. There exists an isovariant map $f : U_1 \to U_p \oplus U_q$.

In fact there are isovariant maps

$f_{\alpha, \beta} : SU_1 \to S(U_p \oplus U_q)$

defined by

$f_{\alpha, \beta}(z) = (z^{(1+\alpha q)p}, z^{(1+\beta p)q}), \quad \alpha, \beta \in \mathbb{Z}, \quad z \in SU_1.$

These are isovariant maps since

$G_{f_{\alpha, \beta}(z)} = G_{z^{(1+\alpha q)p}} \cap G_{z^{(1+\beta p)q}} = 1 \quad (z \in SU_1)$.

In this case, the multidegree classifies isovariant maps and one sees

$[U_1, U_p \oplus U_q]_{C_{pq}}^{\text{isov}} \cong [SU_1, S(U_p \oplus U_q)]_{C_{pq}}^{\text{isov}} \cong \mathbb{Z} \oplus \mathbb{Z}.$

See [3], [4] for the detail.

Example 2.6. There does not exist a $G$-isovariant map $f : U_1 \oplus U_1 \to U_p \oplus U_q$.

If there is an isovariant map, then the isovariant Borsuk-Ulam theorem stated in the next section shows

$\dim U_1 \oplus U_1 - \dim(U_1 \oplus U_1)^{C_p} \leq \dim U_p \oplus U_q - \dim(U_p \oplus U_q)^{C_p}$

$\quad \quad || \quad \quad ||$

$4 - 0 = 4 \quad \quad 4 - 2 = 2.$

This is a contradiction.

Remark. There is a $G$-map $f : S(U_1 \oplus U_1) \to S(U_p \oplus U_q)$. In fact there are $G$-maps $f_i : SU_1 \to SU_i$ defined by $f_i(z) = z^i$ for $i = p$ and $q$. Taking join of $f_p$ and $f_q$, one obtains a $G$-map $f = f_p * f_q : S(U_1 \oplus U_1) \to S(U_p \oplus U_q)$. 
Thus one can finally see

**Proposition 2.7.** Let $G = C_{pq}$, and $V$, $W$ $G$-representations. There exists a $G$-isovariant map $V \to W$ if and only if

$$\begin{cases} 
\dim V - \dim V^H \leq \dim W - \dim W^H \\
\dim V^H - \dim V^G \leq \dim W^H - \dim W^G 
\end{cases}$$

for $H = C_p, C_q$.

See [2] for the detail.

**Question** (unsolved). How about $C_n$ for an arbitrary $n$?

3. **BORSUK-ULAM TYPE THEOREM FOR ISOVARIANT MAPS**

In this section we discuss a Borsuk-Ulam type theorem for isovariant maps, which provides non-existence results on isovariant maps as mentioned in the previous section.

The Borsuk-Ulam theorem due to Borsuk [1] is generalized in various ways (see [6], [7]). The following is one of them. Let $C_p$ be a cyclic group of prime order $p$ and assume that $C_p$ acts freely on spheres $S^m$ and $S^n$.

**Theorem 3.1** (mod $p$ Borsuk-Ulam theorem).

If there exists a $C_p$-map (iff $C_p$-isovariant map) $f : S^m \to S^n$, then $m \leq n$, (or equivalently, if $m > n$, there does not exist a $C_p$-map $f : S^m \to S^n$).

Wasserman first studied the isovariant version of the Borsuk-Ulam theorem and introduced the notion of the Borsuk-Ulam group.

**Definition** (Wasserman). A compact Lie group $G$ is called a *Borsuk-Ulam group (BUG)* if the following statement holds:

For any pair of $G$-representations $V$ and $W$, if there is a $G$-isovariant map $f : V \to W$, then the Borsuk-Ulam inequality:

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

**Proposition 3.2** ([8]). $C_p$ and $S^1$ are BUGs.

The following are fundamental properties of Borsuk-Ulam groups.

**Proposition 3.3** ([8]).

1. If $1 \to H \to G \to K \to 1$ is exact and $H$, $K$ are BUGs, then $G$ is also a BUG.
2. A quotient group of a BUG is also a BUG.
Question (unsolved). Is a subgroup of a BUG also a BUG?

Using this result repeatedly, we have

Corollary 3.4. If

\[ 1 = H_0 < H_1 < H_2 < \cdots < H_r = G \]

and \( H_i/H_{i-1} \) are BUGs \((1 \leq i \leq r)\), then \( G \) is a BUG.

We have the following.

Theorem 3.5 (Isovariant Borsuk-Ulam theorem). Any solvable compact Lie group \( G \) is a BUG.

Proof. As is well-known, \( G \) is solvable if and only if there exists a composition series

\[ 1 = H_0 < H_1 < H_2 < \cdots < H_r = G \]

such that \( H_i/H_{i-1} = C_p \) or \( S^1 \). By Proposition 3.4, \( G \) is a BUG. \( \square \)

So the next question is: how about non-solvable case? Wasserman also found non-solvable examples of BUGs using the prime condition.

Definition (Prime condition (PC)).

(1) We say that a finite simple group \( G \) satisfies the prime condition (PC) if

\[ \sum_{p|o(g)} \frac{1}{p} \leq 1 \]

holds for any \( g \in G \), where \( o(g) \) is the order of \( g \), and the sum is taken over all prime divisors of \( o(g) \).

(2) We say that a finite group \( G \) satisfies (PC) if for a composition series

\[ 1 = H_0 < H_1 < H_2 < \cdots < H_r = G, \]

each simple \( H_i/H_{i-1} \) satisfies (PC) in the sense of (1).

Theorem 3.6 ([8]). If a finite group \( G \) satisfies (PC), then \( G \) is a BUG.

Remark. In the proof of [8], the fact that a cyclic group \( C \) is a BUG is used.

Example 3.7. Alternating groups \( A_5, A_6, \ldots, A_{11} \) satisfy (PC), and hence BUGs. But \( A_n, n \geq 12, \) does not satisfy (PC). In fact \( A_n, n \geq 12, \) has an element of order \( 30 = 2 \cdot 3 \cdot 5 \) and \( 1/2 + 1/3 + 1/5 = 31/30 > 1 \).

Question (unsolved). Is \( A_n \) a BUG for \( n \geq 12 \)?

Example 3.8. \( PSL(2, p) \) satisfies (PC) for \( p: \) prime \( \leq 53; \) hence a BUG. But \( PSL(2, 59), PSL(2, 61) \) do not satisfy (PC). Indeed there are infinitely many primes \( p \) such that \( PSL(2, p) \) does not satisfy (PC).
4. A NEW FAMILY OF BORSUK-ULAM GROUPS

In this section $G$ is a finite group. Let $\mathbb{F}_q$ be a finite field of order $q = p^r$, $p$: prime. Recall

$$PSL(2, q) = SL(2, q)/\{\pm I\}$$
$$= \{A \in M_2(\mathbb{F}_q) \mid \det A = 1\}/\{\pm I\}.$$

Remark. $PSL(2, 2^r) = SL(2, 2^r)$.

Also recall:

(1) If $q = p^r \geq 4$, then $PSL(2, q)$ is simple. On the other hand $PSL(2, 2) \cong S_3$ and $PSL(2, 3) \cong A_4$, which are non-simple.

(2) $|PSL(2, q)| = \begin{cases} q(q-1)(q+1) & p = 2 \\ \frac{1}{2}q(q-1)(q+1) & p : \text{odd prime}. \end{cases}$

We introduce the Möbius condition in [5] and show the following.

Theorem 4.1 ([5]). $PSL(2, q)$ is a BUG for any $q = p^r$.

As a corollary,

Corollary 4.2. $SL(2, q)$, $GL(2, q)$, $PGL(2, q)$ are BUGs.

Proof. These are shown from the following exact sequences.

$$1 \rightarrow \{\pm I\} \rightarrow SL(2, q) \rightarrow PSL(2, q) \rightarrow 1$$
$$1 \rightarrow SL(2, q) \rightarrow GL(2, q) \xrightarrow{\det} \mathbb{F}_q^* \rightarrow 1$$

($F_q^* \cong C_{q-1}$)

$$PGL(2, q) = GL(2, q)/\text{center}$$
($\text{center} = \{aI \mid a \in F_q^*\} \cong F_q^*$).

As seen before, $PSL(2, 59)$, $PSL(2, 61)$ etc. do not satisfy (PC). Our result provides the first example to be a BUG not satisfying (PC).

Finally we announce the following result which will be proved in the forthcoming paper. Let $\text{Syl}_p(G)$ denote a $p$-Sylow subgroup of $G$.

Theorem 4.3 (N-U). If $G$ satisfies one of the following conditions, then $G$ is a BUG.

(1) $\text{Syl}_2(G)$ is a cyclic group $C_{2^r}$ of order $2^r$.
(2) $\text{Syl}_2(G)$ is a dihedral group $D_{2^r}$ of order $2^r$ ($r \geq 2$). As a convention, $D_4 = C_2 \times C_2$.
(3) $\text{Syl}_2(G)$ is a generalized quaternion group $Q_{2^r}$ of order $2^r$ ($r \geq 3$).
(4) $\text{Syl}_2(G)$ is abelian and $\text{Syl}_p(G)$ is cyclic for every odd prime $p$.  

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Example 4.4.

(1) $PSL(2, q)$, $q$: odd, is an example of (2).

(2) $SL(2, q)$, $q$: odd, is an example of (3).

(3) $SL(2, 2^r)$ is an example of (4).

(4) A finite group with periodic cohomology is an example of (1), (3) or (4).

For the proof, we use the fact that $PSL(2, q)$ is a BUG and several deep results of finite group theory.

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