ON BORSUK-ULAM GROUPS

Ikumitsu NAGASAKI (京都府立医科大学 医学部・長崎 生光)
Department of Mathematics
Faculty of Medicine
Kyoto Prefectural University of Medicine

Fumihiro USHITAKI (京都産業大学 理学部・牛瀧 文宏)
Department of Mathematics
Faculty of Science
Kyoto Sangyo University

ABSTRACT. A Borsuk-Ulam group is a group for which the isovariant Borsuk-Ulam theorem holds. A fundamental question is: which groups are Borsuk-Ulam groups? In this article, we shall recall some properties and previous results on a Borsuk-Ulam group. After that, we provide a new family of Borsuk-Ulam groups. We also pose some open questions.

1. NOTATION AND TERMINOLOGY

Let $G$ be a compact Lie group and $V$ an (orthogonal or unitary) representation space of $G$. We denote by $SV$ the unit sphere of $V$, called a $G$-representation sphere. A $G$-equivariant map (or $G$-map for short) $f : X \to Y$ is a continuous map between $G$-spaces satisfying

$$f(gx) = gf(x), \forall x \in X, g \in G.$$ 

It is easy to see that if $f$ is $G$-equivariant, then

1. $f(X^H) \subset Y^H$, so we have the restriction map

$$f^H : X^H \to Y^H.$$ 

2. $G_x \leq G_{f(x)} (\forall x \in X)$.

Definition. A continuous map $f : X \to Y$ is called a $G$-isovariant map if $f$ is a $G$-equivariant map satisfying $G_x = G_{f(x)} (\forall x \in X)$.

It is easy to see that $f : X \to Y$ is $G$-isovariant if and only if $f$ is a $G$-equivariant map such that $f|_{G(x)} : G(x) \to Y$ is injective for any $x \in X$, where $G(x)$ is the orbit of $x$. Similarly we define an isovariant homotopy as follows.

2000 Mathematics Subject Classification. 57S17, 55M20,
The first author was partially supported by JSPS KAKENHI Grant Number 23540101.
**Definition.** Let $f, g$ be $G$-isovariant maps. We call $f$ and $g$ isovariantly $G$-homotopic if there exists a $G$-isovariant map $H : X \times I \to Y$, called a $G$-isovariant homotopy, such that $H(-, 0) = f$ and $H(-, 1) = g$.

Let $[X, Y]^\text{isov} = \text{Iso}_G^G$ denote the set of $G$-isovariant homotopy classes of $G$-isovariant maps.

By the definition of isovariance, we easily see the following.

1. Let $X$ and $Y$ be free $G$-spaces. Then $G$-equivariance is equivalent to $G$-isovariance.
2. If $f : X \to Y$ is an injective $G$-map, then $f$ is $G$-isovariant.
3. If there exists a $G$-isovariant map $f : X \to Y$, then $\text{Iso}(X) \subset \text{Iso}(Y)$, where $\text{Iso}(X)$ is the set of isotropy subgroups of $X$.

**Example 1.1.** Let $X = G/H$ and $Y = G/K$.

1. There exists a $G$-map $f : G/H \to G/K$ if and only if $(H) \leq (K)$, i.e., $H \leq aKa^{-1}$ for some $a \in G$.
2. There exists a $G$-isovariant map $f : G/H \to G/K$ if and only if $(H) = (K)$.
   In this case, a $G$-isovariant map $f$ is defined by $f(gH) = gaK$, $H = aKa^{-1}$.

2. **ISOVARIANT MAPS BETWEEN REPRESENTATIONS**

The following result says that isovariant maps between representations are essentially same as those between representation spheres.

**Proposition 2.1.** Let $V, W$ be (orthogonal) $G$-representations. The following are equivalent.

1. There exists a $G$-isovariant map $f : V \to W$.
2. There exists a $G$-isovariant map $f : V^G \to W^G$. 
3. There exists a $G$-isovariant map $f : S(V^G) \to S(W^G)$.

Here $V^G$ is the orthogonal complement of $V^G$ in $V$. In particular, if $V^G = W^G = 0$, then there exists a $G$-isovariant map $f : V \to W$ if and only if $f : SV \to SW$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Composing the inclusion $i$ and the projection $p$ with $f : V \to W$, we have an isovariant map

$$\overline{f} : V^G \to V \to W \to W^G.$$

Composing the inclusion $j$ and the normalization map with $\overline{f}$, we have an isovariant map

$$\overline{\overline{f}} : S(V^G) \to V^G \to W^G \to S(W^G).$$

(1) $\Leftarrow$ (2) $\Leftarrow$ (3)
Let \( g : S(V^{G\perp}) \rightarrow S(W^{G\perp}) \) be an isovariant map. By the radial extension, we have an isovariant map
\[
\tilde{g} : V^{G\perp} \rightarrow W^{G\perp}.
\]
By adding the zero map to \( \tilde{g} \), we have an isovariant map
\[
h := \tilde{g} \oplus 0 : V = V^{G\perp} \oplus V^{G} \rightarrow W^{G\perp} \oplus W^{G} = W.
\]

By further arguments, we also obtain

**Proposition 2.2.** When \( V^{G} = W^{G} = 0 \), there is a one-to-one correspondence
\[
[V, W]_{G}^{isov} \cong [SV, SW]_{G}^{isov}.
\]

We here provide some examples. Let \( G = C_{n} = \langle c \rangle \) be a cyclic group of order \( n \), where \( c \) is a generator of \( C \). Consider the irreducible representations of \( C \). Let
\[
U_{k} (= \mathbb{C}) \ (0 \leq k \leq n-1)
\]
denote the irreducible representation with the linear action:
\[
c \cdot z = \xi_{n}^{k} z \ (z \in U_{k}), \quad \xi_{n} = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).
\]
Assume \( n = pq \), where \( p, q \) are distinct primes and \( G = C_{pq} \).

**Example 2.3.** If \((k, pq) = (l, pq) = 1\), then there exist a \( G \)-isovariant map \( f : SU_{k} \rightarrow SU_{l} \).

In fact, fix \( s \) such that \( ks \equiv 1 \mod pq \). We define a map \( f \) by
\[
f(z) = z^{sl}, \quad z \in SU_{k}.
\]

Then one can check that

1. \( f \) is \( G \)-equivariant,
2. \( G \) acts freely on \( SU_{k} \) and \( SU_{l} \).

Hence \( f \) is \( G \)-isovariant.

Further arguments show that the degree of maps classifies isovariant homotopy classes, and we have
\[
[U_{k}, U_{l}]_{C_{pq}}^{isov} \cong [SU_{k}, SU_{l}]_{C_{pq}}^{isov} \cong \mathbb{Z},
\]
and the representatives are given by
\[
f_{m}(z) = z^{sl+mpq}, \quad z \in SU_{k}, \quad m \in \mathbb{Z}.
\]
See [3], [4] for the detail.
Example 2.4. There do not exist isovariant maps \( f : U_p \rightarrow U_q \) and \( g : U_1 \rightarrow U_q \).

In fact, if \( f : X \rightarrow Y \) is an isovariant map, then \( \text{Iso}(X) \subseteq \text{Iso}(Y) \). However

\[
\text{Iso}(U_p) = \{C_p, G\} \not\subseteq \text{Iso}(U_q) = \{C_q, G\}
\]

and

\[
\text{Iso}(U_1) = \{1, G\} \not\subseteq \text{Iso}(U_q) = \{C_q, G\}.
\]

Example 2.5. There exists an isovariant map \( f : U_1 \rightarrow U_p \oplus U_q \).

In fact there are isovariant maps

\[
f_{\alpha, \beta} : U_1 \rightarrow S(U_p \oplus U_q)
\]

defined by

\[
f_{\alpha, \beta}(z) = (z^{(1+\alpha q)p}, z^{(1+\beta p)q}), \quad \alpha, \beta \in \mathbb{Z}, \ z \in SU_1.
\]

These are isovariant maps since

\[
G_{f_{\alpha, \beta}(z)} = G_{z^{(1+\alpha q)p}} \cap G_{z^{(1+\beta p)q}} = 1 (\ z \in SU_1).
\]

In this case, the multidegree classifies isovariant maps and one sees

\[
[U_1, U_p \oplus U_q]^{\text{isov}} \cong [SU_1, S(U_p \oplus U_q)]^{\text{isov}} \cong \mathbb{Z} \oplus \mathbb{Z}.
\]

See [3], [4] for the detail.

Example 2.6. There does not exist a \( G \)-isovariant map \( f : U_1 \oplus U_1 \rightarrow U_p \oplus U_q \).

If there is an isovariant map, then the isovariant Borsuk-Ulam theorem stated in the next section shows

\[
\dim U_1 \oplus U_1 - \dim(U_1 \oplus U_1)^{C_p} \leq \dim U_p \oplus U_q - \dim(U_p \oplus U_q)^{C_p}
\]

\[
\| \| \quad \| \| \quad 4 - 0 = 4 \quad 4 - 2 = 2.
\]

This is a contradiction.

Remark. There is a \( G \)-map \( f : S(U_1 \oplus U_1) \rightarrow S(U_p \oplus U_q) \). In fact there are \( G \)-maps

\[
f_i : SU_1 \rightarrow SU_i \text{ defined by } f_i(z) = z^i \text{ for } i = p \text{ and } q.
\]

Taking join of \( f_p \) and \( f_q \), one obtains a \( G \)-map \( f = f_p \ast f_q : S(U_1 \oplus U_1) \rightarrow S(U_p \oplus U_q) \).
Thus one can finally see

**Proposition 2.7.** Let $G = C_{pq}$, and $V, W$ $G$-representations. There exists a $G$-isovariant map $V \to W$ if and only if
\[
\begin{align*}
\dim V - \dim V^H & \leq \dim W - \dim W^H \\
\dim V^H - \dim V^G & \leq \dim W^H - \dim W^G
\end{align*}
\]
for $H = C_p, C_q$.

See [2] for the detail.

**Question** (unsolved). How about $C_n$ for an arbitrary $n$?

3. **Borsuk-Ulam Type theorem for isovariant maps**

In this section we discuss a Borsuk-Ulam type theorem for isovariant maps, which provides non-existence results on isovariant maps as mentioned in the previous section.

The Borsuk-Ulam theorem due to Borsuk [1] is generalized in various ways (see [6]. [7]). The following is one of them. Let $C_p$ be a cyclic group of prime order $p$ and assume that $C_p$ acts freely on spheres $S^m$ and $S^n$.

**Theorem 3.1** (mod $p$ Borsuk-Ulam theorem).

If there exists a $C_p$-map ($\iff$ $C_p$-isovariant map) $f : S^m \to S^n$, then $m \leq n$, (or equivalently, if $m > n$, there does not exist a $C_p$-map $f : S^m \to S^n$).

Wasserman first studied the isovariant version of the Borsuk-Ulam theorem and introduced the notion of the Borsuk-Ulam group.

**Definition** (Wasserman). A compact Lie group $G$ is called a *Borsuk-Ulam group (BUG)* if the following statement holds:

For any pair of $G$-representations $V$ and $W$, if there is a $G$-isovariant map $f : V \to W$, then the Borsuk-Ulam inequality:
\[
\dim V - \dim V^G \leq \dim W - \dim W^G
\]
holds.

**Proposition 3.2** ([8]). $C_p$ and $S^1$ are BUGs.

The following are fundamental properties of Borsuk-Ulam groups.

**Proposition 3.3** ([8]).

(1) If $1 \to H \to G \to K \to 1$ is exact and $H, K$ are BUGs, then $G$ is also a BUG.

(2) A quotient group of a BUG is also a BUG.
Question (unsolved). Is a subgroup of a BUG also a BUG?

Using this result repeatedly, we have

**Corollary 3.4.** If

\[ 1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G \]

and \( H_i/H_{i-1} \) are BUGs (\( 1 \leq i \leq r \)), then \( G \) is a BUG.

We have the following.

**Theorem 3.5** (Isovariant Borsuk-Ulam theorem). *Any solvable compact Lie group \( G \) is a BUG.*

Proof. As is well-known, \( G \) is solvable if and only if there exists a composition series

\[ 1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G \]

such that \( H_i/H_{i-1} = C_p \) or \( S^1 \). By Proposition 3.4, \( G \) is a BUG. \( \square \)

So the next question is: how about non-solvable case? Wasserman also found non-solvable examples of BUGs using the prime condition.

**Definition** (Prime condition (PC)).

1. We say that a finite simple group \( G \) satisfies the prime condition (PC) if

\[ \sum_{p|o(g)} \frac{1}{p} \leq 1 \]

holds for any \( g \in G \), where \( o(g) \) is the order of \( g \), and the sum is taken over all prime divisors of \( o(g) \).

2. We say that a finite group \( G \) satisfies (PC) if for a composition series

\[ 1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G, \]

each simple \( H_i/H_{i-1} \) satisfies (PC) in the sense of (1).

**Theorem 3.6** ([8]). *If a finite group \( G \) satisfies (PC), then \( G \) is a BUG.*

**Remark.** In the proof of [8], the fact that a cyclic group \( C \) is a BUG is used.

**Example 3.7.** Alternating groups \( A_5, A_6, \ldots, A_{11} \) satisfy (PC), and hence BUGs. But \( A_n, n \geq 12, \) does not satisfy (PC). In fact \( A_n, n \geq 12, \) has an element of order \( 30 = 2 \cdot 3 \cdot 5 \) and \( 1/2 + 1/3 + 1/5 = 31/30 > 1. \)

**Question** (unsolved). Is \( A_n \) a BUG for \( n \geq 12? \)

**Example 3.8.** \( PSL(2, p) \) satisfies (PC) for \( p: \) prime \( \leq 53; \) hence a BUG. But \( PSL(2, 59), PSL(2, 61) \) do not satisfy (PC). Indeed there are infinitely many primes \( p \) such that \( PSL(2, p) \) does not satisfy (PC).
4. A NEW FAMILY OF BORSUK-ULAM GROUPS

In this section $G$ is a finite group. Let $\mathbb{F}_q$ be a finite field of order $q = p^r$, $p$: prime. Recall

$$PSL(2, q) = SL(2, q)/\{\pm I\}$$
$$= \{ A \in M_2(\mathbb{F}_q) | \det A = 1 \}/\{\pm I\}.$$

**Remark.** $PSL(2, 2^r) = SL(2, 2^r)$.

Also recall:

1. If $q = p^r \geq 4$, then $PSL(2, q)$ is simple. On the other hand $PSL(2, 2) \cong S_3$ and $PSL(2, 3) \cong A_4$, which are non-simple.

2. $|PSL(2, q)| = \begin{cases} q(q-1)(q+1) & p = 2 \\ \frac{1}{2}q(q-1)(q+1) & p : \text{odd prime}. \end{cases}$

We introduce the Möbius condition in [5] and show the following.

**Theorem 4.1 ([5]).** $PSL(2, q)$ is a BUG for any $q = p^r$.

As a corollary,

**Corollary 4.2.** $SL(2, q), GL(2, q), PGL(2, q)$ are BUGs.

**Proof.** These are shown from the following exact sequences.

$$1 \rightarrow \{ \pm I \} \rightarrow SL(2, q) \rightarrow PSL(2, q) \rightarrow 1$$
$$1 \rightarrow SL(2, q) \rightarrow GL(2, q)^{\det} \rightarrow \mathbb{F}_q^{*} \rightarrow 1$$

$$(F_q^{*} \cong C_{q-1})$$

$$PGL(2, q) = GL(2, q)/\text{center}$$

(centre = $\{ aI | a \in \mathbb{F}_q^{*} \} \cong \mathbb{F}_q^{*}$).

As seen before, $PSL(2, 59), PSL(2, 61)$ etc. do not satisfy (PC). Our result provides the first example to be a BUG not satisfying (PC).

Finally we announce the following result which will be proved in the forthcoming paper. Let $\text{Syl}_p(G)$ denote a $p$-Sylow subgroup of $G$.

**Theorem 4.3 (N-U).** If $G$ satisfies one of the following conditions, then $G$ is a BUG.

1. $\text{Syl}_2(G)$ is a cyclic group $C_{2^r}$ of order $2^r$.
2. $\text{Syl}_2(G)$ is a dihedral group $D_{2^r}$ of order $2^r$ ($r \geq 2$). As a convention, $D_4 = C_2 \times C_2$.
3. $\text{Syl}_2(G)$ is a generalized quaternion group $Q_{2^r}$ of order $2^r$ ($r \geq 3$).
4. $\text{Syl}_2(G)$ is abelian and $\text{Syl}_p(G)$ is cyclic for every odd prime $p$. 
Example 4.4.

(1) $PSL(2, q)$, $q$: odd, is an example of (2).
(2) $SL(2, q)$, $q$: odd, is an example of (3).
(3) $SL(2, 2^r)$ is an example of (4).
(4) A finite group with periodic cohomology is an example of (1), (3) or (4).

For the proof, we use the fact that $PSL(2, q)$ is a BUG and several deep results of finite group theory.

REFERENCES


DEPARTMENT OF MATHEMATICS, KYOTO PREFECTURAL UNIVERSITY OF MEDICINE, 13 NISHITAKATSUKASA-CHO, TAISHOGUN KITA-KU, KYOTO 603-8334, JAPAN
E-mail address: nagasaki@koto.kpu-m.ac.jp (I. Nagasaki)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO SANGYO UNIVERSITY, KAMIGAMO MOTOYAMA, KITA-KU, KYOTO 603-8555, JAPAN
E-mail address: ushitaki@ksuvx0.kyoto-su.ac.jp (F. Ushitaki)