数理解析研究所講究録
第 1816 巻 2012 年 36-43

ON BORSUK-ULAM GROUPS

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ABSTRACT. A Borsuk-Ulam group is a group for which the isovariant Borsuk-Ulam theorem holds. A fundamental question is: which groups are Borsuk-Ulam groups? In this article, we shall recall some properties and previous results on a Borsuk-Ulam group. After that, we provide a new family of Borsuk-Ulam groups. We also pose some open questions.

1. NOTATION AND TERMINOLOGY

Let G be a compact Lie group and V an (orthogonal or unitary) representation space of G. We denote by SV the unit sphere of V, called a G-representation sphere. A G-equivariant map (or G-map for short) $f : X \to Y$ is a continuous map between G-spaces satisfying

$$f(gx) = gf(x),\ \forall x \in X, g \in G.$$

It is easy to see that if f is G-equivariant, then

(1) $f(X^H) \subset Y^H$, so we have the restriction map

$$f^H : X^H \to Y^H.$$

(2) $G_x \leq G_{f(x)}$ $(\forall x \in X)$.

Definition. A continuous map $f : X \to Y$ is called a G-isovariant map if f is a G-equivariant map satisfying $G_x = G_{f(x)}$ $(\forall x \in X)$.

It is easy to see that $f : X \to Y$ is G-isovariant if and only if f is a G-equivariant map such that $f_{|G(x)} : G(x) \to Y$ is injective for any $x \in X$, where $G(x)$ is the orbit of x. Similarly we define an isovariant homotopy as follows.

2000 *Mathematics Subject Classification.* 57S17, 55M20,

The first author was partially supported by JSPS KAKENHI Grant Number 23540101.

Definition. Let f, g be G-isovariant maps. We call f and g isovariantly G-homotopic if there exists a G-isovariant map $H : X \times I \to Y$, called a G-isovariant homotopy, such that $H(-,0) = f$ and $H(-,1) = g$.

Let $[X,Y]_G^{\mathrm{isov}}$ denote the set of G-isovariant homotopy classes of G-isovariant maps.

By the definition of isovariance, we easily see the following.

(1) Let X and Y be free G-spaces. Then G-equivariance is equivalent to G-isovariance.
(2) If $f : X \to Y$ is an injective G-map, then f is G-isovariant.
(3) If there exists a G-isovariant map $f : X \to Y$, then $\mathrm{Iso}\,(X) \subset \mathrm{Iso}\,(Y)$, where $\mathrm{Iso}\,(X)$ is the set of isotropy subgroups of X.

Example 1.1. Let $X = G/H$ and $Y = G/K$.

(1) There exists a G-map $f : G/H \to G/K$ if and only if $(H) \leq (K)$, i.e., $H \leq aKa^{-1}$ for some $a \in G$.
(2) There exists a G-isovariant map $f : G/H \to G/K$ if and only if $(H) = (K)$. In this case, a G-isovariant map f is defined by $f(gH) = gaK$, $H = aKa^{-1}$.

2. Isovariant maps between representations

The following result says that isovariant maps between representations are essentially same as those between representation spheres.

Proposition 2.1. *Let V, W be (orthogonal) G-representations. The following are equivalent.*

(1) *There exists a G-isovariant map $f : V \to W$.*
(2) *There exists a G-isovariant map $f : V^{G\perp} \to W^{G\perp}$.*
(3) *There exists a G-isovariant map $f : S(V^{G\perp}) \to S(W^{G\perp})$.*

Here $V^{G\perp}$ is the orthogonal complement of V^G in V. In particular, if $V^G = W^G = 0$, then there exists a G-isovariant map $f : V \to W$ if and only if $f : SV \to SW$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Composing the inclusion i and the projection p with $f : V \to W$, we have an isovariant map

$$\overline{f} : V^{G\perp} \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{p} W^{G\perp}.$$

Composing the inclusion j and the normalization map with $\overline{f}$, we have an isovariant map

$$\overline{\overline{f}} : S(V^{G\perp}) \xrightarrow{j} V^{G\perp} \setminus \{0\} \xrightarrow{\overline{f}} W^{G\perp} \setminus \{0\} \xrightarrow{\text{norm.}} S(W^{G\perp}).$$

(1) $\Leftarrow$ (2) $\Leftarrow$ (3)

Let $g : S(V^{G^\perp}) \to S(W^{G^\perp})$ be an isovariant map. By the radial extension, we have an isovariant map

$$\tilde{g} : V^{G^\perp} \to W^{G^\perp}.$$

By adding the zero map to $\overline{g}$, we have an isovariant map

$$h := \tilde{g} \oplus 0 : V = V^{G^\perp} \oplus V^G \to W^{G^\perp} \oplus W^G = W.$$

□

By further arguments, we also obtain

Proposition 2.2. *When $V^G = W^G = 0$, there is a one-to-one correspondence*

$$[V, W]_G^{\text{isov}} \cong [SV, SW]_G^{\text{isov}}.$$

We here provide some examples. Let $G = C_n = \langle c \rangle$ be a cyclic group of order n, where c is a generator of C. Consider the irreducible representations of C. Let

$$U_k \ (= \mathbb{C}) \ (0 \leq k \leq n-1)$$

denote the irreducible representation with the linear action:

$$c \cdot z = \xi_n^k z \ (z \in U_k), \quad \xi_n = \exp(\frac{2\pi\sqrt{-1}}{n}).$$

Assume $n = pq$, where p, q are distinct primes and $G = C_{pq}$.

Example 2.3. If $(k, pq) = (l, pq) = 1$, then there exist a G-isovariant map $f : SU_k \to SU_l$.

In fact, fix s such that $ks \equiv 1 \bmod pq$. We define a map f by

$$f(z) = z^{sl}, \quad z \in SU_k.$$

Then one can check that

(1) f is G-equivariant,

(2) G acts freely on SU_k and SU_l.

Hence f is G-isovariant.

Further arguments show that the degree of maps classifies isovariant homotopy classes, and we have

$$[U_k, U_l]_{C_{pq}}^{\text{isov}} \cong [SU_k, SU_l]_{C_{pq}}^{\text{isov}} \cong \mathbb{Z},$$

and the representatives are given by

$$f_m(z) = z^{sl+mpq}, \quad z \in SU_k, \quad m \in \mathbb{Z}.$$

See [3], [4] for the detail.

Example 2.4. There do not exist isovariant maps $f : U_p \to U_q$ and $g : U_1 \to U_q$.

In fact, if $f : X \to Y$ is an isovariant map, then $\mathrm{Iso}\,(X) \subset \mathrm{Iso}\,(Y)$. However

$$\mathrm{Iso}\,(U_p) = \{C_p, G\} \not\subset \mathrm{Iso}\,(U_q) = \{C_q, G\}$$

and

$$\mathrm{Iso}\,(U_1) = \{1, G\} \not\subset \mathrm{Iso}\,(U_q) = \{C_q, G\}.$$

Example 2.5. There exists an isovariant map $f : U_1 \to U_p \oplus U_q$.

In fact there are isovariant maps

$$f_{\alpha,\beta} : SU_1 \to S(U_p \oplus U_q)$$

defined by

$$f_{\alpha,\beta}(z) = (z^{(1+\alpha q)p}, z^{(1+\beta p)q}), \quad \alpha, \beta \in \mathbb{Z},\ z \in SU_1.$$

These are isovariant maps since

$$G_{f_{\alpha,\beta}(z)} = G_{z^{(1+\alpha q)p}} \cap G_{z^{(1+\beta p)q}} = 1\ (z \in SU_1).$$

In this case, the multidegree classifies isovariant maps and one sees

$$[U_1, U_p \oplus U_q]^{\mathrm{isov}}_{C_{pq}} \cong [SU_1, S(U_p \oplus U_q)]^{\mathrm{isov}}_{C_{pq}} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

See [3], [4] for the detail.

Example 2.6. There does not exist a G-isovariant map $f : U_1 \oplus U_1 \to U_p \oplus U_q$.

If there is an isovariant map, then the isovariant Borsuk-Ulam theorem stated in the next section shows

$$\begin{array}{ccc} \dim U_1 \oplus U_1 - \dim(U_1 \oplus U_1)^{C_p} & \leq & \dim U_p \oplus U_q - \dim(U_p \oplus U_q)^{C_p} \\ \| & & \| \\ 4 - 0 = 4 & & 4 - 2 = 2. \end{array}$$

This is a contradiction.

Remark. There is a G-map $f : S(U_1 \oplus U_1) \to S(U_p \oplus U_q)$. In fact there are G-maps $f_i : SU_1 \to SU_i$ defined by $f_i(z) = z^i$ for $i = p$ and q. Taking join of f_p and f_q, one obtains a G-map $f = f_p * f_q : S(U_1 \oplus U_1) \to S(U_p \oplus U_q)$.

Thus one can finally see

Proposition 2.7. *Let $G = C_{pq}$, and V, W G-representations. There exists a G-isovariant map $V \to W$ if and only if*

$$\begin{cases} \dim V - \dim V^H \leq \dim W - \dim W^H \\ \dim V^H - \dim V^G \leq \dim W^H - \dim W^G \end{cases}$$

for $H = C_p$, C_q.

See [2] for the detail.

Question (unsolved). How about C_n for an arbitrary n?

3. Borsuk-Ulam type theorem for isovariant maps

In this section we discuss a Borsuk-Ulam type theorem for isovariant maps, which provides non-existence results on isovariant maps as mentioned in the previous section.

The Borsuk-Ulam theorem due to Borsuk [1] is generalized in various ways (see [6]. [7]). The following is one of them. Let C_p be a cyclic group of prime order p and assume that C_p acts freely on spheres S^m and S^n.

Theorem 3.1 (mod p Borsuk-Ulam theorem).
If there exists a C_p-map ($\Longleftrightarrow$ C_p-isovariant map) $f : S^m \to S^n$, then $m \leq n$, (or equivalently, if $m > n$, there does not exist a C_p-map $f : S^m \to S^n$).

Wasserman first studied the isovariant version of the Borsuk-Ulam theorem and introduced the notion of the Borsuk-Ulam group.

Definition (Wasserman). A compact Lie group G is called a *Borsuk-Ulam group (BUG)* if the following statement holds:

For any pair of G-representations V and W, if there is a G-isovariant map $f : V \to W$, then the Borsuk-Ulam inequality:

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Proposition 3.2 ([8]). *C_p and S^1 are BUGs.*

The following are fundamental properties of Borsuk-Ulam groups.

Proposition 3.3 ([8]).

1. *If $1 \to H \to G \to K \to 1$ is exact and H, K are BUGs, then G is also a BUG.*
2. *A quotient group of a BUG is also a BUG.*

Question (unsolved). Is a subgroup of a BUG also a BUG?

Using this result repeatedly, we have

Corollary 3.4. *If*
$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G$$
and H_i/H_{i-1} *are BUGs* $(1 \leq i \leq r)$, *then* G *is a BUG.*

We have the following.

Theorem 3.5 (Isovariant Borsuk-Ulam theorem). *Any solvable compact Lie group* G *is a BUG.*

Proof. As is well-known, G is solvable if and only if there exists a composition series
$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G$$
such that $H_i/H_{i-1} = C_p$ or S^1. By Proposition 3.4, G is a BUG. □

So the next question is: how about non-solvable case? Wasserman also found non-solvable examples of BUGs using the prime condition.

Definition (Prime condition (PC)). (1) We say that a finite simple group G satisfies the prime condition (PC) if
$$\sum_{p|o(g)} \frac{1}{p} \leq 1$$
holds for any $g \in G$, where $o(g)$ is the order of g, and the sum is taken over all prime divisors of $o(g)$.

(2) We say that a finite group G satisfies (PC) if for a composition series
$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G,$$
each simple H_i/H_{i-1} satisfies (PC) in the sense of (1).

Theorem 3.6 ([8]). *If a finite group* G *satisfies (PC), then* G *is a BUG.*

Remark. In the proof of [8], the fact that a cyclic group C is a BUG is used.

Example 3.7. Alternating groups $A_5, A_6, \ldots, A_{11}$ satisfy (PC), and hence BUGs. But A_n, $n \geq 12$, does not satisfy (PC). In fact A_n, $n \geq 12$, has an element of order $30 = 2 \cdot 3 \cdot 5$ and $1/2 + 1/3 + 1/5 = 31/30 > 1$.

Question (unsolved). Is A_n a BUG for $n \geq 12$?

Example 3.8. $PSL(2,p)$ satisfies (PC) for p: prime ≤ 53; hence a BUG. But $PSL(2,59)$, $PSL(2,61)$ do not satisfy (PC). Indeed there are infinitely many primes p such that $PSL(2,p)$ does not satisfy (PC).

4. A new family of Borsuk-Ulam groups

In this section G is a finite group. Let $\mathbb{F}_q$ be a finite field of order $q = p^r$, p: prime. Recall

$$\begin{aligned} PSL(2,q) &= SL(2,q)/\{\pm I\} \\ &= \{A \in M_2(\mathbb{F}_q) \mid \det A = 1\}/\{\pm I\}. \end{aligned}$$

Remark. $PSL(2,2^r) = SL(2,2^r)$.

Also recall:

(1) If $q = p^r \geq 4$, then $PSL(2,q)$ is simple. On the other hand $PSL(2,2) \cong S_3$ and $PSL(2,3) \cong A_4$, which are non-simple.

(2) $|PSL(2,q)| = \begin{cases} q(q-1)(q+1) & p = 2 \\ \frac{1}{2}q(q-1)(q+1) & p : \text{odd prime.} \end{cases}$

We introduce the Möbius condition in [5] and show the following.

Theorem 4.1 ([5]). *$PSL(2,q)$ is a BUG for any $q = p^r$.*

As a corollary,

Corollary 4.2. *$SL(2,q)$, $GL(2,q)$, $PGL(2,q)$ are BUGs.*

Proof. These are shown from the following exact sequences.

$$1 \to \{\pm I\} \to SL(2,q) \to PSL(2,q) \to 1$$

$$1 \to SL(2,q) \to GL(2,q) \overset{\det}{\to} \mathbb{F}_q^* \to 1$$

$(F_q^* \cong C_{q-1})$

$$PGL(2,q) = GL(2,q)/\text{center}$$

(center $= \{aI \mid a \in \mathbb{F}_q^*\} \cong \mathbb{F}_q^*$). □

As seen before, $PSL(2,59)$, $PSL(2,61)$ etc. do not satisfy (PC). Our result provides the first example to be a BUG not satisfying (PC).

Finally we announce the following result which will be proved in the forthcoming paper. Let $\mathrm{Syl}_p(G)$ denote a p-Sylow subgroup of G.

Theorem 4.3 (N-U). *If G satisfies one of the following conditions, then G is a BUG.*

(1) *$\mathrm{Syl}_2(G)$ is a cyclic group C_{2^r} of order 2^r.*
(2) *$\mathrm{Syl}_2(G)$ is a dihedral group D_{2^r} of oder 2^r ($r \geq 2$). As a convention, $D_4 = C_2 \times C_2$.*
(3) *$\mathrm{Syl}_2(G)$ is a generalized quaternion group Q_{2^r} of order 2^r ($r \geq 3$).*
(4) *$\mathrm{Syl}_2(G)$ is abelian and $\mathrm{Syl}_p(G)$ is cyclic for every odd prime p.*

Example 4.4.

(1) $PSL(2,q)$, q: odd, is an example of (2).
(2) $SL(2,q)$, q: odd, is an example of (3).
(3) $SL(2,2^r)$ is an example of (4).
(4) A finite group with periodic cohomology is an example of (1), (3) or (4).

For the proof, we use the fact that $PSL(2,q)$ is a BUG and several deep results of finite group theory.

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