Simple factor dressing of a minimal surface

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1 Introduction

In this paper, we announce the results in [6]. A simple factor dressing is a transform of a harmonic map from a Riemann surface. A Gauss map of a constant mean curvature surface in $\mathbb{R}^3$ is a harmonic map from a Riemann surface to $S^2$. It is shown that every $\mu$-Draboux transform of a harmonic map $N: M \to S^2$ is given by a simple factor dressing of $N$ ([1], Theorem 6.1). A conformal Gauss map of a Willmore conformal map from a Riemann surface to $S^4$ is a harmonic map from a Riemann surface to $\mathcal{Z} = \{C \in \text{End}(\mathbb{H}) | C^2 = -\text{Id}\}$ (see [7]). It is shown that the Darboux transform of a harmonic sphere congruence $\mathcal{C}$ in $\mathbb{H}$ is a $\mu$-Darboux transform of $\mathcal{C}$ with $\mu \in (\mathbb{R} \setminus \{0\}) \cup S^1$. Moreover, it is a simple factor dressing of $\mathcal{C}$ ([8]).

When we consider these transforms, there is a theory of minimal surfaces in Euclidean space in the intersection of the theory of constant mean curvature surfaces in $\mathbb{R}^3$ and that of Willmore surfaces in $S^4$ (see Table 1). The Gauss map of a minimal surface is a conformal harmonic map and the mean curvature sphere of a minimal surface is a harmonic map. This is an interesting point to consider $\mu$-Darboux transforms and simple factor dressing of a minimal surface.

The definitions and propositions used in this paper are summarized in [7].

2 Minimal surfaces

We recall minimal surfaces in terms of quaternions.

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Table 1: Maps and Gauss maps
2.1 One-forms with values in quaternions

We model $\mathbb{R}^4$ on $\mathbb{H}$. We denote by $S^2$ the two-sphere centered at the origin with radius one. Then

$$\{a \in \text{End}(\mathbb{H}) | a^2 = -1\} = \{a \in \text{Im} \mathbb{H} | |a| = 1\} = S^2.$$  

Hence an element of $S^2$ is a quaternionic linear complex structure of $\mathbb{H}$ and a square root of $-1$.

Let $M$ be a Riemann surface with complex structure $J$. We fix a map $N: M \to S^2$. Then a one-form $\omega$ on $M$ with values in $\mathbb{H}$ is decomposed as

$$\omega = \omega_N + \omega_{-N} = \omega_N^* + \omega_{-N}^*,$$

where $\omega_N := \frac{1}{2}(\omega - N^* \omega), \quad \omega_N^* := \frac{1}{2}(\omega - * \omega N).$

Let $\eta$ be another one-form on $M$ with values in $\mathbb{H}$. Then

$$\omega \wedge \eta = \omega_N \wedge \eta_{-N} + \omega_{-N} \wedge \eta_N.$$

2.2 Minimal surfaces

Let $f: M \to \mathbb{H}$ be a map. Then $f$ is conformal if and only if there exists $N: M \to S^2$ and $R: M \to S^2$ such that $(df)_{-N} = (df)^R = 0$ ([2]). The map $N$ is called the left normal of $f$ and the map $R$ is called the right normal of $f$. A conformal map $f$ with $(df)_{-N} = (df)^R = 0$ is minimal with respect to the induced metric if and only if $(dN)_N = (dN)^{-N} = 0$ and, equivalently, $(dR)_R = (dR)^{-R} = 0$ ([2]). Hence if $f$ is a minimal surface with $(df)_{-N} = (df)^R = 0$, then $N$ and $R$ are conformal maps. In fact, they are holomorphic maps with respect to a standard complex structure of $S^2 \cong \mathbb{C}P^1$. For a minimal surface $f$, there exists locally a map $f^*$ such that $df = \star df$. The map $f^*$ is called a conjugate minimal surface of $f$. We see that $(df^*)_{-N} = (df^*)^R = 0$. For $(p, q) \in \mathbb{H}^2 \setminus \{(0, 0)\}$, $f_{p,q} := fp + f^* q$ and $f^{p,q} := fp + q f^*$ are minimal surfaces.

**Definition 1** ([6]). The family of minimal surfaces $\{f_{p,q}\}_{(p,q) \in \mathbb{H}^2 \setminus \{(0,0)\}}$ is called the right associated family of $f$ and the family of minimal surfaces $\{f^{-p,q}\}_{(p,q) \in \mathbb{H}^2 \setminus \{(0,0)\}}$ is called the left associated family of $f$.

If $f$ is a minimal surface in $\mathbb{R}^3$, then $\{f_{\cos \theta, \sin \theta}\}_{\theta \in \mathbb{R}}$ is the classical associated family. The classical associated family is an isometric deformation of the original minimal surface.

**Theorem 1** ([6]). $f_{p,q} (p, q) \in \mathbb{H}^2 \setminus \{(0, 0)\}$ is isometric to $f$ if and only if $(p, q) = (n \cos \theta, n \sin \theta)$, $n \in S^3 = \{a \in \mathbb{H} | |a| = 1\}$.

3 Holomorphic Gauss maps

We explain transforms of the Gauss map of a minimal surface. These are similar to the transforms of a harmonic map into $S^2$.
3.1 The associated family of a harmonic map into $S^2$

We set $I\phi := \phi i$. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by the complex structure $I$. A map $R: M \to S^2$ is harmonic if and only if $d(R \ast dR) = 0$. We define a family of connections on $\mathbb{H}$ by setting $d_\mu := d + (\mu - 1)Q^{(1,0)} + (\mu^{-1} - 1)Q^{(0,1)}$, where $\mu \in \mathbb{C} \setminus \{0\}$ and $Q = -\frac{1}{2}(\ast dR)_{R}$. 

Lemma 1 ([3]). A map $R: M \to S^2$ is harmonic if and only if $d_\mu$ is flat for any $\mu \in \mathbb{C} \setminus \{0\}$.

3.2 The SFD of a holomorphic Gauss map

Let $r_\lambda: M \to \text{GL}(2, \mathbb{C})$ be a map such that it is meromorphic on $\mathbb{C}P^1$ with respect to $\lambda$ with simple pole away from $\{0, \infty\}$ and $r_1 = \text{Id}$. Set $\hat{d}_\lambda := r_\lambda \circ d_\lambda \circ r_\lambda^{-1}$.

Definition 2. A map $\hat{R}: M \to S^2$ is called a simple factor dressing of $R$ if there exists $r_\lambda$ such that $\hat{d}_\lambda$ is the associated family of $\hat{R}$.

A simple factor dressing is a harmonic map. In fact, it is written as follows.

Theorem 2 ([1]). If $\hat{R}$ is a simple factor dressing of $R$, then $\hat{R} := \hat{T}^{-1}R\hat{T}$, where $\hat{T} := \frac{1}{2}(-R\beta(a - 1)\beta^{-1} + \beta b\beta^{-1})$, $a := \frac{\lambda + \lambda^{-1}}{2}$, $b := \frac{i\lambda^{-1} - \lambda}{2}$, and $d_\lambda \beta = 0$.

We consider the case where $R$ is holomorphic. If $f$ is minimal, then $R$ is holomorphic. The simple factor dressing $\hat{R}$ is harmonic. In fact, we have the following:

Theorem 3 ([6]). Let $f: M \to \mathbb{H}$ be minimal with $(df)^R = 0$. Then a simple factor dressing $\hat{R}$ is a right normal of $f_{p,q}$ for some $(p, q) \in \mathbb{H}^2 \setminus \{(0,0)\}$.

\[ R \leftrightarrow f \]
\[ \hat{R} \leftrightarrow f_{p,q} \]

4 Conformal Gauss maps

We explain transforms of a conformal Gauss map of a minimal surface.

4.1 The SFD of a conformal Gauss map

Let $f: M \to \mathbb{H}$ be a minimal surface with $(df)_{-N} = (df)^R = 0$. It is known that $f$ is Willmore, too. We set the sections $e$ and $\psi$ of $\underline{\mathbb{H}^2}$ as

$e := \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \psi := \begin{pmatrix} f & 1 \end{pmatrix}$.

We define a line bundle $L$ by $L := \psi\mathbb{H}$. Then $L$ is associated with a Willmore conformal map with mean curvature sphere

$S(e \quad \psi) := (e \quad \psi) \begin{pmatrix} N & 0 \\ 0 & -R \end{pmatrix}$.
Let \( d^S_\mu \) be the associated family of \( d \) with \( S \) (see [7]). Set \( a := \frac{\mu + \mu^{-1}}{2}, \ b := i \frac{\mu^{-1} - \mu}{2} \). Then, we have the following:

**Theorem 4 ([6]).** If \( \hat{S} \) is a simple factor dressing of a conformal Gauss map \( S \) of a minimal surface \( f \), then \( \hat{S} \) is the conformal Gauss map of a minimal surface \( \hat{f} = h^{n \frac{b}{a-1} n^{-1},-1}, \ h := -f_{mg_{m^{-1},m \frac{a-1}{2} m^{-1}}} \).

\[
\begin{array}{c}
S \quad \downarrow \quad \hat{S} \\
\downarrow \quad \downarrow \\
\hat{f} \quad \rightarrow \quad f
\end{array}
\]

### 4.2 \( \mu \)-Darboux transform of a conformal Gauss map

Let \( f: M \rightarrow \mathbb{H} \) be a minimal with \( (df)^R = 0 \). Then, the map \( g := f_R - f^* \) is a super-conformal map. A super-conformal map is a conformal map with vanishing Willmore energy.

**Definition 3 ([6]).** The super-conformal map \( g \) is called an associated Willmore surface of \( f \).

The following is a relation between \( \mu \)-Darboux transform, associated family, and associated Willmore surface.

**Theorem 5 ([6]).** Every non-constant \( \mu \)-Darboux transform of a minimal surface \( f \) is an associated Willmore surface of an associated minimal surface \( f^{pa} \).

### References


