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Game Russian option with the finite maturity*

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Abstract

We consider Game Russian options with the finite maturity. Game Russian option is a contract that the seller and the buyer have the rights to cancel and to exercise it at any time, respectively. We discuss the pricing model of Game Russian options when the stock pays dividends continuously. We show that the pricing model can be formulated as a coupled optimal stopping problem which is analyzed as Dynkin game.

1 Introduction

Russian option was introduced by Shepp and Shiryaev [9], [10] and is one of perpetual American lookback options. Russian option with the finite maturity was studied by Duistermaat, Kyprianou and van Schaikb [1], Ekström [2] and Peskir [7]. Duistermaat, Kyprianou and van Schaikb [1] showed the option value can be characterizes as the unique solution to a free boundary problem and gave numerical algorithm for solving the problem. Ekström [2] proved the existence of an optimal stopping boundary. Peskir [7] showed that the optimal stopping boundary can be characterizes as the unique solution of a nonlinear integral equation. It is very interesting to publish these papers at the much same time.

Russian option is the contract that only the buyer has the right to exercise it. On the other hand, Game Russian option is the contract that the seller and the buyer have both the rights to cancel and to exercise it at any time, respectively. Its payoff function depends on the running maximum value of the stock process. This option value is represented as coupled optimal stopping problem for the seller and the buyer. See Cvitanic and Karatzas [3] and Kifer [4]. In the case where there is no dividend and the dividend is positive, Kyprianou [6] and Suzuki and Sawaki [12] derived the value function and its optimal boundaries, respectively. Moreover, Kunita and Seko [5] studied the value function of the game call options and their optimal regions.

In this paper, we study the value function of Game Russian options and their optimal regions. The paper is organized as follows. In Section 2 we introduce a pricing model of Game Russian options with the finite maturity by means of a coupled optimal stopping problem given by Kifer [4]. Section 3 gives the main theorem.

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2 Model

We consider the Black-Scholes model. Let B_t be the riskless asset price at time t defined by $B_t = e^{rt}$, where r is a positive constant. Let S_t be the risky asset price at time t determined by

$$dS_t = S_t(\mu dt + \kappa dW_t), \tag{2.1}$$

where μ and $\kappa > 0$ are constants and W_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We define probability measure $\tilde{P}$ in such a way that its Radon-Nikodym derivative is given by

$$\frac{d\tilde{P}}{dP} = \exp\left(-\frac{\mu + d - r}{\kappa}W_t - \frac{1}{2}\left(\frac{\mu + d - r}{\kappa}\right)^2 t\right),$$

where d is nonnegative constant. Then $\tilde{W}_t \equiv W_t + \frac{\mu+d-r}{\kappa}$ is a standard Brownian motion under the probability measure $\tilde{P}$.

Next we introduce another probability measure $\hat{P}$ by

$$\frac{d\hat{P}}{d\tilde{P}} = \exp\left(\kappa\hat{W}_t - \frac{1}{2}\kappa^2 t\right).$$

Then, $\hat{W}_t \equiv \tilde{W}_t - \kappa t$ is standard Brownian motion with respect to $\hat{P}$ and S_t is represented by

$$S_t = S_0 \exp\left\{\left(r - d + \frac{1}{2}\kappa^2\right)t + \kappa\tilde{W}_t\right\}.$$

We set

$$\Psi_t \equiv \max\left(S_0\psi, \sup_{0\le u\le t} S_u\right)/S_t, \ \psi \ge 1.$$

Let σ be a cancel time for the seller and τ be an exercise time for the buyer. Then the value function $V(x,t)$ with the penalty $\delta > 0$ is given by

$$V(x,t) = \inf_{\sigma\in\mathcal{T}_{t,T}} \sup_{\tau\in\mathcal{T}_{t,T}} \hat{E}[e^{-\alpha(\sigma\wedge\tau-t)}\{(\Psi_\sigma + \delta)1_{\{\sigma<\tau\}} + \Psi_\tau 1_{\{\tau\le\sigma\}}\} \mid \Psi_t = x], \ \alpha > 0, \tag{2.2}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times in the interval $[t,T]$ and the infimum and supremum are taken over all stopping times σ and τ, respectively. We represent the value function as follows.

$$V(x,s) = \inf_{\sigma\in\mathcal{T}_{s,T}} \sup_{\tau\in\mathcal{T}_{s,T}} J_s(\sigma,\tau,x), \tag{2.3}$$

where

$$J_s(\sigma,\tau,x) = \tilde{E}[e^{-\alpha(\sigma\wedge\tau-s)}\{(\Psi_\sigma + \delta)1_{\{\sigma<\tau\}} + \Psi_\tau 1_{\{\tau\le\sigma\}}\}].$$

The value function $V(x,s)$ satisfies the inequalities

$$x \le V(x,s) \le x + \delta.$$

We define the sets A, B and C by

$$\begin{aligned} A &= \{(x,s)\times[0,T) \in \mathbf{R}^+; V(x,s) = x+\delta\}, \\ B &= \{(x,s)\times[0,T) \in \mathbf{R}^+; V(x,s) = x\}. \\ C &= \{(x,s)\times[0,T) \in \mathbf{R}^+; x < V(x,s) < x+\delta\}. \end{aligned}$$

These sets are the subsets of real positive numbers. The set A and B are called the seller's cancellation region and the buyer's exercise region, respectively. C is called the continuation region.

Let σ_A^x and τ_B^x be the first hitting times of the process $\Psi_t(x)$ to the set A and B, respectively, i.e.,

$$\begin{aligned}\sigma_A^x &= \inf\{t>0 \mid \Psi_t(x)\in A\}\wedge T\\ \tau_B^x &= \inf\{t>0 \mid \Psi_t(x)\in B\}\wedge T.\end{aligned}$$

For any $x>0$, $\hat{\sigma}_s^x \equiv \sigma_A^x$ and $\hat{\tau}_s^x \equiv \tau_B^x$ attain the infimum and the supremum. Therefore, we have

$$V(x,s) = J_s(\hat{\sigma}_s^x, \hat{\tau}_s^x, x).$$

When the sets A and B are empty, we understand that $\hat{\sigma}_s^x = T$ and $\hat{\tau}_s^x = T$.

Lemma 1 *The value function is nondecreasing in x for any s. and is Lipschitz continuous in x for any s. And it holds*

$$0 \le \frac{\partial V(x,s)}{\partial x} \le 1. \tag{2.4}$$

Proof. Replacing the optimal stopping times $\hat{\sigma}_s^x$ and $\hat{\tau}_s^y$ from the nonoptimal stopping times $\hat{\sigma}_s^y$ and $\hat{\tau}_s^x$, we have

$$\begin{aligned}V(y,s) &\ge J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, y)\\ V(x,s) &\le J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, x),\end{aligned}$$

respectively. Note that $z_1^+ - z_2^+ \le (z_1 - z_2)^+$. For any $x>y$, we have

$$\begin{aligned}0 \le V(x,s) - V(y,s) &\le J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, x) - J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, y)\\ &= \hat{E}[e^{-\alpha(\hat{\sigma}_s^y\wedge\hat{\tau}_s^x)}(\Psi_{\hat{\sigma}_s^y\wedge\hat{\tau}_s^x}(x) - \Psi_{\hat{\sigma}_s^y\wedge\hat{\tau}_s^x}(y))]\\ &= \hat{E}[e^{-\alpha(\hat{\sigma}_s^y\wedge\hat{\tau}_s^x)}H^{-1}(s, \hat{\sigma}_s^y\wedge\hat{\tau}_s^x)((x - \sup H(s,u))^+ - (y - \sup H(s,u)^+)]\\ &\le (x-y)\hat{E}[e^{-\alpha(\hat{\sigma}_s^y\wedge\hat{\tau}_s^x)}H^{-1}(s, \hat{\sigma}^y\wedge\hat{\tau}^x)],\end{aligned}$$

where

$$H(s,t) = \exp\left\{\left(r - d + \frac{1}{2}\kappa^2\right)(t-s) + \kappa(\tilde{W}_t - \tilde{W}_s)\right\}.$$

Since the above expectation is less than 1, we have

$$0 \le V(x,s) - V(y,s) \le x - y.$$

This means that $V(x,s)$ is Lipschitz continuous in x and it holds (2.4).

Lemma 2 *Let $V_R(x,s)$ be the value function of Russian option with the finite maturity and let $\delta^* = V_R(1,s) - 1$. If $\delta > \delta^*$, the seller never cancels. Therefore Game Russian options are reduced to Russian options with the finite maturity.*

Proof. We set $U(x) = V_R(x,s) - x - \delta$. $h'(x) = V_R'(x,s) - 1 < 0$. Because we know $h(1) = V_R(1,s) - 1 - \delta = \delta^* - \delta < 0$ by the condition $\delta \geq \delta^*$, we have $h(x) < 0$, *i.e.*, $V_R(x,s) < x + \delta$ holds. By using the relation $V(x,s) \leq V_R(x,s)$ we obtain $V(x,s) < x + \delta$, *i.e.*, it is optimal for the seller not to cancel. Therefore the seller never cancels the contract for $\delta \geq \delta^*$.

Remark 1 *Since $\Psi_t(x) \geq \Psi_0(x) = x \geq 1$, it follows that the seller's optimal cancellation region A is a point $\{1\}$.*

3 Main Theorem

In this section, we give the main theorem. In order to prove it, we needs the following lemmas.

Lemma 3 *The value function $V(x,s)$ is convex in x.*

Proof. The function V satisfies

$$\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 V}{\partial x^2} = -\frac{\partial V}{\partial s} - (r-d)x\frac{\partial V}{\partial x} + \alpha V.$$

If $r \leq d$, we get $\frac{\partial^2 V}{\partial x^2} > 0$. Next assume that $r > d$. We consider function $\tilde{V}(x) = V(-x)$ for $x < 0$. Then,

$$\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 \tilde{V}}{\partial x^2} - (r-d)x\frac{\partial \tilde{V}}{\partial x} - \alpha\tilde{V} = \frac{1}{2}\kappa^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r-d)x\frac{\partial V}{\partial x} - \alpha V = 0.$$

Since we find that $\frac{\partial^2 \tilde{V}}{\partial x^2} > 0$ from the above equation, $\tilde{V}$ is a convex function. It follows from this fact that V is a convex function.

Lemma 4 *Suppose $d = 0$. The the first derivative $\frac{\partial V}{\partial x}(x,s)$ is strictly increasing.*

From the above lemmas, we have the following theorems.

Theorem 1 *Let A and B be the seller's cancellation region and the buyer's exercise region, respectively.*

1. *Let β be the infimum of s such that $V(1,s) < \delta$. In this case, it holds $0 \leq s \leq T$ and the cancellation region B is represented by*

$$A = \begin{cases} \{1\}, & if\ s \leq \beta \\ \emptyset, & if\ s > \beta \end{cases} \tag{3.1}$$

2. *(a) If $d = 0$, the buyer's exercise region is empty, i.e., the buyer never exercises.*

   (b) Suppose $d > 0$. Then the buyer's exercise region is

$$B = \{x; b(s) \leq x < \infty\},$$

where $(b(s), s \in [0,T))$ is a nonincreasing function.

Theorem 2 *Let $V(x,s)$ be the value function of Game Russian option with the finite maturity defined by (2.3). Then we have the following.*

1. *The function $V(x,s)$ is convex with respect to x for any s and Lipschitz continuous with respect to x for any s.*

2. (a) *Suppose $d = 0$. If $\delta \geq \delta^*$, the value function $V(x,s) = V_E(x,s)$. When $\delta < \delta^*$, we get $V(x,s) < V_E(x,s)$, where $V_E(x,s)$ is the value function of the European option.*

   (b) *Suppose $d > 0$. If $\delta \geq \delta^*$, we have $V(x,s) = V_R(x,s)$. When $\delta < \delta^*$, we get $V(x,s) < V^*(x,s)$.*

3. *The first derivative $\frac{\partial V}{\partial x}(x,s)$ is increasing and it satisfies*

$$\frac{\partial V}{\partial x}(b(s)-,s) = \frac{\partial V}{\partial x}(b(s)+,s) = 1.$$

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