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Manhattan Product of Digraphs

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Abstract We review the Manhattan product of digraphs from the viewpoint of spectral analysis and obtain some preliminary formulae. As an example, the spectrum of the Manhattan product of the directed path P_n and the directed cycle C_2 is obtained as well as its asymptotic spectral distribution.

1 Introduction

Quantum probabilistic techniques have been developed for (asymptotic) spectral analysis of graphs, see e.g., [10]. One of the main techniques is based on the relation between notions of independence and product structures of graphs. In this note we initiate an attempt to generalize the quantum probabilistic approach to digraphs (directed graphs).

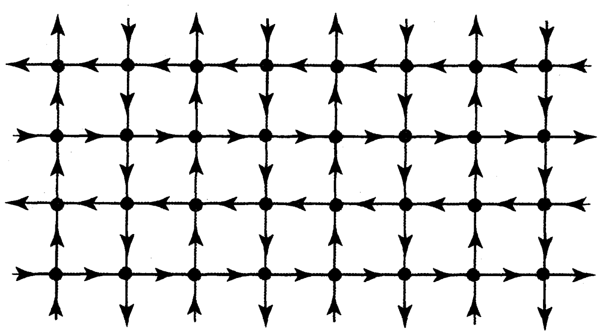

Figure 1.1: Manhattan street network

There is a long history of spectral analysis of digraphs with many relevant topics. From the viewpoint of product structure of digraphs the first non-trivial example we consider would be the Manhattan street network. The spectra of the Manhattan street networks are described by Comellas *et al.* [5, 6]. Their method relies on direct calculation and a more conceptual derivation is desirable. In this line it is natural to formulate the Manhattan street network as a kind of product of digraphs. In fact, in their more recent papers [7, 8] Comellas, Dalfó and Fiol introduce the notion of *Manhattan product* of digraphs and obtain some basic properties. The main purpose of this note is to reformulate the Manhattan product in a slightly more general context and to discuss the spectral properties of simple examples.

Independently of spectral analysis, the Manhattan street network was introduced beforehand by Maxemchuk [12] and Morillo *et al.* [13] for simple and effective structure of communication networks, see also [3, 11, 14]. In some literatures, e.g., [2], the notion of *Manhattan network* appears, however, it is different from the Manhattan street network.

2 Spectrum of a Digraph

A *digraph (directed graph)* is a pair $G = (V, E)$, where V is a non-empty set and E is a subset of $V \times V$. We say that $x \in V$ is a *vertex* and $e = (x, y) \in E$ is an *arc (arrow)* from i to j. In that case we also write $x \to y$. By definition a digraph may have a *loop*, i.e., an arc from a vertex to itself. Throughout this paper, unless otherwise stated, a digraph means a finite digraph, i.e., a digraph with finite number of vertices.

The adjacency matrix of a digraph $G = (V, E)$ is a matrix A with index set $V \times V$ defined by

$$(A)_{xy} = \begin{cases} 1, & \text{if } x \to y, \\ 0, & \text{otherwise.} \end{cases}$$

Then A becomes a $\{0, 1\}$-matrix. Conversely, every $\{0, 1\}$-matrix with index set $V \times V$ defines a digraph with vertex set V. A digraph is called *symmetric* if its adjacency matrix is symmetric. A symmetric digraph with no loops is naturally identified with a graph in the usual sense. In fact, their adjacency matrices are characterized by common conditions.

The set of eigenvalues of a digraph G is denoted by

$$\operatorname{ev} G = \{\lambda_1, \lambda_2, \dots, \lambda_s\},$$

where $\lambda_1, \lambda_2, \dots, \lambda_s$ are distinct eigenvalues of the adjacency matrix A of G. The characteristic polynomial of A, often referred to as the *characteristic polynomial* of G, is factorized as follows:

$$\varphi_G(x) = \det(x - A) = \prod_{i=1}^{s} (x - \lambda_i)^{m_i}, \qquad m_i \geq 1.$$

Then m_i is called the *algebraic multiplicity* of λ_i. While, the dimension l_i of the eigenspace associated with λ_i is called the *geometric multiplicity*. It is obvious that $1 \leq l_i \leq m_i$. Note that $l_i < m_i$ may happen for a general digraph and that $l_i = m_i$ for a symmetric digraph.

The *converse* or *opposite* of a digraph $G = (V, E)$ is a digraph $G^\vee = (V, E^\vee)$, where

$$E^\vee = \{(x, y) \in V \times V;\ (y, x) \in E\}.$$

The adjacency matrix of $G^\vee$ is obtained by transposing that of G. Hence the characteristic polynomials of $G^\vee$ and G coincide, so do their eigenvalues.

The algebraic (resp. geometric) spectrum of a digraph G is the list of its eigenvalues with algebraic (resp. geometric) multiplicities. The spectra of digraphs are characteristic quantities and have many applications. For basic results, in particular on the spectral radius, see the recent survey by Brualdi [1].

Example 2.1 (Cycle) Let $n \geq 2$. We put

$$V = \{0, 1, 2, \dots, n-1\},$$
$$E = \{(0, 1), (1, 2), \dots, (n-2, n-1), (n-1, 0)\}.$$

The digraph (V, E) is called a *cycle* (or more precisely, a *directed cycle*) of degree n and is denoted by C_n. Note that the cycle C_2 is symmetric. From elementary knowledge of linear algebra we know that

$$\varphi_{C_n}(x) = x^n - 1,$$
$$\operatorname{ev} C_n = \{1 = \omega^0, \omega, \omega^2, \dots, \omega^{n-1}\}, \qquad \omega = e^{2\pi i/n}.$$

Moreover, the algebraic multiplicity of each eigenvalue is one, so coincides with the geometric multiplicity.

Example 2.2 (Colliding cycle) Let $n \geq 3$ and $0 \leq k \leq n$. A *colliding cycle* is a digraph $C_{n,k} = (V, E)$, where

$$V = \{0, 1, 2, \dots, n-1\},$$
$$E = \{(0, 1), (1, 2), \dots, (k-1, k)\} \cup \{(k+1, k), \dots, (n-1, n-2), (0, n-1)\}.$$

(Addition is taken by modulo n.) Apparently, $C_n = C_{n,n} = C^\vee_{n,0}$. For a non-trivial colliding cycle $C_{n,k}$ with $k \neq 0, n$, we have

$$\varphi(x) = x^n, \qquad \operatorname{ev} C_{n,k} = \{0\}.$$

The algebraic multiplicity of the eigenvalue 0 is n while the geometric one is 2.

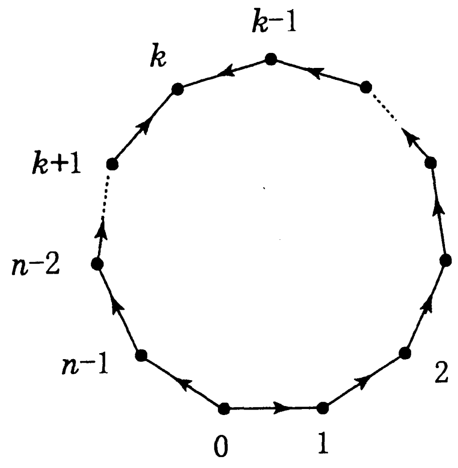

Figure 2.2: Colliding cycle

3 Bipartite Digraphs

A digraph $G = (V, E)$ is called *bipartite* if the vertex set admits a partition

$$V = V^{(0)} \cup V^{(1)} \qquad V^{(0)} \neq \emptyset, \quad V^{(1)} \neq \emptyset, \quad V^{(0)} \cap V^{(1)} = \emptyset$$

such that every arc has its initial vertex in $V^{(0)}$ and final vertex in $V^{(1)}$, or initial vertex in V_1 and final vertex in $V^{(0)}$. By definition a bipartite digraph has no loops. The adjacency matrix of a bipartite digraph is of the form:

$$A = \begin{bmatrix} O & C \\ D & O \end{bmatrix}, \tag{3.1}$$

where C is a $\{0,1\}$-matrix with index set $V^{(0)} \times V^{(1)}$ and D is a $\{0,1\}$-matrix with index set $V^{(1)} \times V^{(0)}$. From elementary knowledge of linear algebra we have the following

Proposition 3.1 *Let G be a bipartite digraph with adjacency matrix* (3.1). *Then the characteristic polynomial is given by*

$$\varphi_G(x) = \det(x - A) = x^{m-n} \det(x^2 - DC),$$

where $m = |V^{(0)}|$ and $n = |V^{(1)}|$ with $m \geq n$.

Given a bipartite digraph $G = (V, E)$ we define the parity function $\pi = \pi_G : V \to \{0, 1\}$ by

$$\pi(x) = \pi_G(x) = \begin{cases} 0, & x \in V^{(0)}, \\ 1, & x \in V^{(1)}. \end{cases}$$

Note that the parity function depends on the partition $V = V^{(0)} \cup V^{(1)}$. For an arc $(x,y) \in E$ we have $\pi(x)+\pi(y) = 1$. We mention some basic properties. The proofs are straightforward so omitted.

Proposition 3.2 *Let $G = (V,E)$ be a bipartite digraph. For any pair of vertices $x,y \in V$, the parity of the length of a path from x to y (whenever exists) is independent of the choice of such a path.*

Proposition 3.3 *A bipartite digraph does not contain a cycle of odd degree. More generally, a bipartite digraph does not contain a colliding cycle of odd degree.*

Proposition 3.4 *A cycle of even degree is bipartite. More generally, so is a colliding cycle of even degree.*

4 Manhattan Product

For $i = 1,2$ let $G_i = (V_i, E_i)$ be a bipartite digraph with parity function $\pi = \pi_i$. Consider the direct product

$$V = V_1 \times V_2 = \{(x,y)\,;\, x \in V_1,\, y \in V_2\}$$

and let E consist of pairs of vertices $((x,y),(x',y'))$ satisfying one of the following two conditions:

(i) $y = y'$, and $(x,x') \in E_1$ or $(x',x) \in E_1$ according as $\pi_2(y) = 0$ or $\pi_2(y) = 1$;

(ii) $x = x'$, and $(y,y') \in E_2$ or $(y',y) \in E_2$ according as $\pi_1(x) = 0$ or $\pi_1(x) = 1$.

The digraph (V,E) is called the *Manhattan product* and is denoted by

$$G = G_1 \# G_2 .$$

Although not explicitly indicated, the Manhattan product depends on the choice of the partitions $V_i = V_i^{(0)} \cup V_i^{(1)}$, or equivalently the choice of the parity functions π_i. The (2-dimensional) Manhattan street network [8] is nothing but the Manhattan product $C_m \# C_n$ with even m, n.

We now observe a simple property of the Manhattan product $G = G_1 \# G_2 = (V,E)$. Take $(x_0, y_0) \in V$. The section $\Sigma = \{(x, y_0)\,;\, x \in V_1\}$ has a digraph structure isomorphic to G_1 or $G_1^\vee$ according as $\pi_2(y_0) = 0$ or $\pi_2(y_0) = 1$. Let

$$(x_0,y_0) \to (x_1,y_0) \to \cdots \to (x_i,y_0) \to \cdots \tag{4.1}$$

be a path in G and consider the sections $\Sigma[x_i] = \{(x_i,y)\,;\, y \in V_2\}$. Then $\Sigma[x_i]$ is isomorphic to G_2 or $G_2^\vee$, and they occur alternately along the path (4.1). This is a typical property of the Manhattan street networks. In order to maintain this property it is natural to take the class of bipartite digraphs for the Manhattan product.

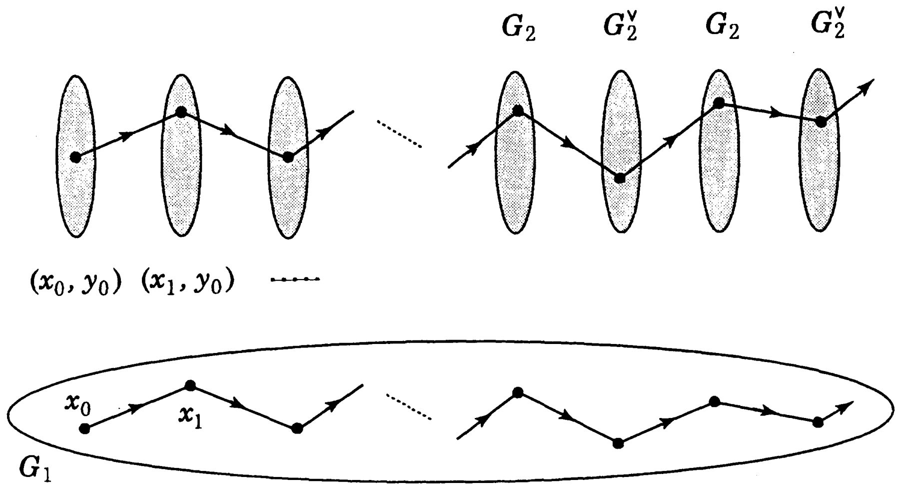

Figure 4.3: Manhattan product

Proposition 4.1 *The Manhattan product of two bipartite digraphs is bipartite.*

Proof. Consider two bipartite digraphs $G_i = (V_i, E_i)$, $i = 1, 2$, with partitions of the vertex sets $V_i = V_i^{(0)} \cup V_i^{(1)}$. Set

$$V^{(0)} = V_1^{(0)} \times V_2^{(0)} \cup V_1^{(1)} \times V_2^{(1)}, \qquad V^{(1)} = V_1^{(0)} \times V_2^{(1)} \cup V_1^{(1)} \times V_2^{(0)}.$$

Then $V = V^{(0)} \cup V^{(1)}$ is a partition of the vertex set V of the Manhattan product $G_1 \# G_2$, where there are no arcs lying in $V^{(0)}$ or $V^{(1)}$. ∎

Proposition 4.2 *Let G_i be a bipartite digraph with the adjacency matrix A_i, $i = 1, 2$. Then the adjacency matrix A of the Manhattan product $G = G_1 \# G_2$ satisfies*

$$(A)_{(x,y)(x',y')} = \delta_{xx'}(t^{\pi_1(x)}(A_2))_{yy'} + (t^{\pi_2(y)}(A_1))_{xx'}\delta_{yy'}, \qquad x, x' \in V_1, \; y, y' \in V_2,$$

where $t(A) = A^T$ stands for the transposition and π_i is the parity function of G_i.

We consider a simple example. Let $G = (V, E)$ be a bipartite digraph and consider the Manhattan product $G \# C_2$. Let B be the adjacency matrix of G. Then the adjacency matrix A is given by

$$A = \begin{bmatrix} B & I \\ I & B^T \end{bmatrix}, \tag{4.2}$$

where I is the identity matrix indexed by $V \times V$.

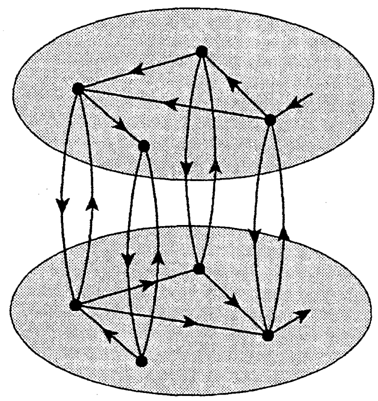

Figure 4.4: $G\# C_2$ (G is not necessarily bipartite.)

Theorem 4.3 *Let $G = (V, E)$ be a bipartite digraph with adjacency matrix B. Then the characteristic polynomial of the Manhattan product $G\# C_2$ is given by*

$$\varphi(x) = \det((x - B)(x - B^T) - I). \tag{4.3}$$

Moreover, if

$$B = \begin{bmatrix} O & C \\ D & O \end{bmatrix},$$

we have

$$\varphi(x) = \det \begin{bmatrix} (x^2 - 1)I + CC^T & -x(C + D^T) \\ -x(C^T + D) & (x^2 - 1)I + DD^T \end{bmatrix}. \tag{4.4}$$

Proof. Let A be the adjacency matrix of the Manhattan product $G\# C_2$. Then the characteristic polynomial is given by

$$\varphi(x) = \det(x - A) = \det \begin{bmatrix} x - B & -I \\ -I & x - B^T \end{bmatrix}.$$

Applying the formula:

$$\det \begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \det(XY - I) = \det(YX - I), \tag{4.5}$$

where X, Y are $n \times n$ matrices and I is the identity matrix, we obtain (4.3). Then (4.4) follows by direct computation. ∎

In fact, $G\#C_2$ may be defined without assuming that G is bipartite, see Fig. 4.4. In that case too, $G\#C_2$ keeps the typical property of the Manhattan street networks and the characteristic polynomial is given by (4.3).

Example 4.4 Let P_n be the directed path with n vertices, i.e., $P_n = (V, E)$ with

$$V = \{1, 2, \dots, n\},$$
$$E = \{(1,2), (2,3), \dots, (n-1, n)\}.$$

The adjacency matrix of P_n is given by

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

We see from Theorem 4.3 that the characteristic polynomial of $P_n \# C_2$ is given by

$$\varphi_n(x) = \det((x - B)(x - B^T) - I).$$

By elementary calculation we obtain

$$\varphi_n(x) = x^2 \varphi_{n-1}(x) - x^2 \varphi_{n-2}(x).$$

Then, recalling the recurrence relation of the Chebyshev polynomial of the second kind, we come to

$$\varphi_n(x) = x^{n-1} \tilde{U}_{n+1}(x),$$

where

$$\tilde{U}_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Consequently,

$$\mathrm{ev}\,(P_n \# C_2) = \left\{ 2\cos\frac{k\pi}{n+2}\,;\ k = 1, 2, \dots, n+1 \right\} \cup \{0\},$$

where every non-zero eigenvalue has algebraic multiplicity one.

The asymptotic spectral distribution as $n \to \infty$ is also interesting.

Theorem 4.5 *The asymptotic (algebraic) spectral distribution of $P_n \# C_2$ is given by*

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho(x)dx,$$

where

$$\rho(x) = \frac{1}{\pi\sqrt{4 - x^2}} \chi_{(-2,2)}(x).$$

Proof. It is sufficient to show that

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{2\cos\frac{k\pi}{n+1}}$$

tends to $\rho(x)dx$ as $n \to \infty$. Let $f(x)$ be a bounded continuous function. Then we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \frac{1}{n}\sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right) \to \int_0^1 f(2\cos\pi t)dt, \quad \text{as } n \to \infty,$$

which follows by the definition of Riemann integral. By change of variable, one gets

$$\int_0^1 f(2\cos\pi t)dt = \int_{-2}^{2} f(x)\frac{dx}{\pi\sqrt{4-x^2}}.$$

Consequently,

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \int_{-2}^{2} f(x)\frac{dx}{\pi\sqrt{4-x^2}} = \int_{-\infty}^{+\infty} f(x)\rho(x)dx,$$

which completes the proof. ∎

Remark 4.6 The probability distribution $\rho(x)dx$ in Theorem 4.5 is called the *arc-sine law* (with mean 0 and variance 2).

Remark 4.7 As another generalization of (4.2) it is interesting to consider

$$A = \begin{bmatrix} B & I & & & & & \\ & B^T & I & & & & \\ & & B & I & & & \\ & & & \ddots & \ddots & & \\ & & & & & B & I \\ I & & & & & & B^T \end{bmatrix}. \tag{4.6}$$

This is a kind of product of G and C_n (with even n), which is considered something between the Manhattan product and the direct product.

References

[1] R. A. Brualdi: *Spectra of digraphs*, Linear Algebra Appl. **432** (2010), 2181–2213.

[2] M. Benkert, A. Wolff, F. Widmann and T. Shirabe: *The minimum Manhattan network problem: Approximations and exact solutions*, Comput. Geom. **35** (2006), 188–208.

[3] F. Comellas and C. Dalfó: *Optimal broadcasting in 2-dimensional Manhattan street networks*, Parallel and Distributed Computing and Networks **246** (2005), 135–140.

[4] F. Comellas, C. Dalfó and M. A. Fiol: *Multidimensional Manhattan street networks*, SIAM J. Discrete Math. **22** (2008), 1428–1447.

[5] F. Comellas, C. Dalfó, M. A. Fiol and M. Mitjana: *A spectral study of the Manhattan networks*, Electronic Notes in Discrete Mathematics **29** (2007) 267–271.

[6] F. Comellas, C. Dalfó, M. A. Fiol and M. Mitjana: *The spectra of Manhattan street networks*, Linear Algebra Appl. **429** (2008), 1823–1839.

[7] F. Comellas, C. Dalfó, M. A. Fiol: *A new operation on digraphs: the Manhattan product*,

[8] F. Comellas, C. Dalfó and M. A. Fiol: *The Manhattan product of digraphs*, preprint, 2009.

[9] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, 1979.

[10] A. Hora and N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer, 2007.

[11] B. Khasnabish: *Topological properties of Manhattan Street networks*, Electron. Lett. **25** (1989), 1388–1389.

[12] N. F. Maxemchuk: *Routing in the Manhattan Street Network*, IEEE Trans. Comm. **35** (1987), 503–512.

[13] P. Morillo, M. A. Fiol, J. Fàbrega: *The diameter of directed graphs associated to plane tessellations*, Ars Comb. **20A** (1985) 17–27.

[14] E. A. Varvarigos: *Optimal communication algorithms for Manhattan street networks*, Network communications broadcasting and gossiping. Discrete Appl. Math. **83** (1998), 303–326.