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Understanding Capacities on a Finite Lattice

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This article summarizes the results obtained by the author [4] who explored a combinatorial approach when capacities are defined over a finite lattice. Let $L$ be a finite lattice with partial ordering $\leq$, and let $\hat{0}$ and $\hat{1}$ denote the minimum and the maximum element of $L$. A monotone function $\varphi$ on $L$ is called a capacity if $\varphi(\hat{0}) = 0$ and $\varphi(\hat{1}) = 1$. Let $\mathcal{L}$ denote the collection of nonempty dual order ideals in $L$, and let $\mathcal{X}$ be an $\mathcal{L}$-valued random variable on some probability space $(\Omega, \mathbb{P})$, distributed as $\mathbb{P}(\mathcal{X} = V) = f(V)$. If $\mathbb{P}(\hat{0} \in \mathcal{X}) = 0$ then

(1) \[ \varphi(x) = \mathbb{P}(x \in \mathcal{X}) \]

gives a capacity, which is viewed as a marginal condition for $\mathcal{X}$. From another viewpoint, the collection of capacities on $L$ is a convex polytope, every element of which can be represented as the convex combination

(2) \[ \varphi(x) = \sum_{V \in \mathcal{L}} f(V) \chi_{V}(x), \quad x \in L, \]

where $\chi_{V}$ denotes an indicator function of $V$. It should be noted, however, that the choice of $f$ is not necessarily unique. In the way of formulating (2), the weight $f(V)$ determines a probability mass function (pmf) for $\mathcal{X}$, in which (2) is deemed to be (1). This probabilistic interpretation of a capacity was first considered by Choquet [1] and independently by Murofushi and Sugeno [6].

For $a_{1}, a_{2}, \ldots \in L$, we define the difference operator $\nabla_{a_{1}}$ by

(3) \[ \nabla_{a_{1}} \varphi(x) = \varphi(x) - \varphi(x \wedge a_{1}), \quad x \in L, \]

and the successive difference operator $\nabla_{a_{1}, \ldots, a_{n}}$ recursively by

(4) \[ \nabla_{a_{1}, \ldots, a_{n}} \varphi = \nabla_{a_{n}}(\nabla_{a_{1}, \ldots, a_{n-1}} \varphi), \quad n = 2, 3, \ldots. \]

The monotonicity of $\varphi$ is characterized by $\nabla_{a} \varphi \geq 0$ for any $a \in L$; furthermore, $\varphi$ is called completely monotone (or monotone of order $\infty$; see [1]) if $\nabla_{a_{1}, \ldots, a_{n}} \varphi \geq 0$ for any $a_{1}, \ldots, a_{n} \in L$ and for any $n \geq 1$. 
Let $X$ be an $L$-valued random variable with pmf $f(x) = \mathbb{P}(X = x)$. If $f(\hat{0}) = 0$ then

$$\varphi(x) = \sum_{y \leq x} f(y), \quad x \in L,$$

gives a capacity, which is viewed as a cumulative distribution function (cdf), also known as a belief function in [2]. The existence of the cdf (5) for a capacity $\varphi$ is necessary and sufficient for the completely monotonicity of $\varphi$. This crucial observation, known as Choquet's theorem, was made by Choquet [1] for the class of compact sets in a topological space, and it has been instrumental in the studies of random sets. See [5] for a comprehensive review on random sets on topological spaces. This result in case of lattices was due to Norberg [7] who studied measures on continuous posets.

The function $f$ in (5) is called the Möbius inverse of $\varphi$, by which the successive difference operators are fully characterized as follows.

**Theorem 1.** The Möbius inverse $f$ of $\varphi$ satisfies

$$\nabla_{a_1, \ldots, a_n} \varphi(x) = \sum \{f(y) : y \leq x, y \not\leq a_i \text{ for all } i = 1, \ldots, n \}.$$

Particularly we can show the Choquet's theorem for a finite lattice via combinatorial techniques.

**Corollary 2.** Assume $\varphi(\hat{0}) \geq 0$. Then the Möbius inverse $f$ of $\varphi$ is nonnegative if and only if $\varphi$ is completely monotone.

The collection $\mathcal{L}$ is itself a distributive lattice when it is equipped with the order relation $U \preceq V$ by $U \supseteq V$. The lattice $L$ is embedded as the subposet $\mathcal{L}_0 := \{\langle a \rangle^* : a \in L\}$ of principal dual order ideals. Here we introduce a completely monotone capacity $\Phi$ on $\mathcal{L}$, and call it a completely monotone extension of $\varphi$ if it satisfies the marginal condition

$$\varphi(x) = \Phi(\langle x \rangle^*), \quad x \in L.$$

The marginal condition (7) is equivalent to (2), in which the weight $f(V)$ determines the Möbius inverse of $\Phi$. By the same token, (1) and (7) are the same when we express $\Phi(U) = \mathbb{P}(\mathcal{X} \preceq U)$ as a cdf for $\mathcal{L}$-valued random variable $\mathcal{X}$.

Kellerer [3] and Rüschendorf [8] investigated the optimal bounds analogous to the classical Fréchet bounds systematically for various marginal problems. Let $R(\mathcal{L})$ be the space of real-valued functions on $\mathcal{L}$. Given $\Phi \in M_{\infty}(\mathcal{L})$ we can formulate the nonnegative linear functional

$$\Phi(g) = \sum_{V \in \mathcal{L}} f(V) g(V), \quad g \in R(\mathcal{L}),$$
where $f$ is the Möbius inverse of $\Phi$. Assuming $\varphi \in M_1(L)$, we can define the Fréchet bound

\begin{equation}
B_{\varphi}(g) = \min\{\Phi(g) : \Pi(\Phi) = \varphi\}
\end{equation}

for any $g \in R(\mathcal{L})$. Duality follows from the relationship between primal and dual problem of linear programming, but it is also viewed as a straightforward application of the Hahn-Banach theorem (cf. Kellerer [3]).

**Theorem 3.** The dual problem

\begin{equation}
S^\varphi(g) = \max \left\{ \sum_{x \in L} r_x \varphi(x) : \sum_{x \in V} r_x \leq g(V), V \in \mathcal{L} \right\}
\end{equation}

satisfies $B_{\varphi}(g) = S^\varphi(g)$ for any $g \in R(\mathcal{L})$.

In particular we formulate the optimal lower bound $\lambda(\varphi; a, b) = B_{\varphi}(\langle a, b \rangle^*)$ at the dual order ideal $\langle a, b \rangle^*$ generated by a pair $\{a, b\}$ of $L$. Then we apply the value $\lambda(\varphi; a, x)$ to replace $\varphi(a \wedge x)$ in (3)–(4), and propose the $\lambda$-difference operator $\Lambda_a$ by

\begin{equation}
\Lambda_a \varphi(x) = \varphi(x) - \lambda(\varphi; a, x), \quad x \in L,
\end{equation}

and the successive $\lambda$-difference operator recursively by

\begin{equation}
\Lambda_{a_1, \ldots, a_n} \varphi = \Lambda_{a_n}(\Lambda_{a_1, \ldots, a_{n-1}} \varphi), \quad n = 2, 3, \ldots.
\end{equation}

Then we consider a stochastic comparison between $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$ and $\psi(y) = \mathbb{P}(Y \leq y)$, and obtain a sufficient condition for $\mathbb{P}(Y \in \mathcal{X}) = 1$.

**Theorem 4.** If

\begin{equation}
\Lambda_{a_1, \ldots, a_k} \varphi(\hat{1}) \leq \nabla_{a_1, \ldots, a_k} \psi(\hat{1}) \quad \text{for every monotone path } (a_1, \ldots, a_k),
\end{equation}

then there exists a joint cdf $\Gamma$ for $(\mathcal{X}, Y)$ satisfying $\mathbb{P}(Y \in \mathcal{X}) = 1$ given the marginal conditions.

**References**


