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Integral representation of monotone functions

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Abstract

Integral representation of monotone functions has been studied by Choquet [1], Murofushi and Sugeno [4], Norberg [5], and many others, but not necessarily been their primal interest due to the lack of uniqueness in their representations. Here we present a brief overview of different approaches and generalizations, and show our own version of integral representation from the ongoing investigation.

1 Choquet theory of integral representation

In his treatise on theory of capacity, Choquet outlined a series of applications for integral representation on the set $\mathcal{E}$ of extreme points of a compact convex Hausdorff space $C$ (Chapter VII of [1]). Let $L$ be a partially ordered set (poset) with a maximum element $e$, and let $C$ be the convex set of nonnegative monotone functions $\varphi$ on $L$ with $\varphi(e) \leq 1$. Assuming the topology of simple (i.e., pointwise) convergence on functions over $L$, we can show that $C$ is compact, and the set $\mathcal{E}$ of extreme points of $C$ consists of indicator functions of the form

\[
\chi(x) = \begin{cases} 
1 & \text{if } x \in U; \\
0 & \text{otherwise.}
\end{cases}
\]

The monotonicity of $\chi$ implies that $y \in U$ whenever $x \in U$ and $x \leq y$, and such subset $U$ is called an upper set. The set $\mathcal{E}$ is compact, and any element $\varphi$ of $C$ is represented in the integral form

\[
\varphi(x) = \int \chi(x) \, d\mu(\chi), \quad x \in L,
\]

with a Radon measure $\mu$ on $\mathcal{E}$ (Section 40 of [1]).

Let $S$ be a compact Hausdorff space, and $\mathcal{K}$ be the class of compact subsets of $S$. Then a nonnegative monotone function $\varphi$ on $\mathcal{K}$ is called a capacity if it is upper semicontinuous (i.e., $\varphi(E) \downarrow \varphi(F)$ whenever $E \downarrow F$) in the exponential (i.e., Vietoris) topology. Here the convex set $C$ of capacities $\varphi$ with $\varphi(S) \leq 1$ is considered similarly; however, the topology of simple convergence is not suitable
for the space $C$. Over the convex cone $Q$ of nonnegative continuous functions on $S$, a capacity $\varphi$ uniquely corresponds to the functional

\[ (3) \quad \varphi(\xi) = \int_0^{\max \xi} \varphi(\{x \in E : \xi(x) \geq r\}) \, dr, \quad \xi \in Q. \]

Then we can introduce the topology of vague convergence on capacities in which a net $\{\varphi_\alpha\}$ converges to $\varphi$ if and only if $\varphi_\alpha(\xi)$ converges to $\varphi(\xi)$ for any $\xi \in Q$. Under this topology the convex set $C$ is compact Hausdorff, and the indicator function $\chi$ in (1) corresponds to a closed upper set $U$ in the exponential topology (Section 48 of [1]).

When $S$ is a locally compact Hausdorff space, it is not necessary for $K$ to contain $S$. Here we can introduce a partial ordering on $K$ by the dual (i.e., the reverse order) of inclusion, and denote the poset by $L$ with the maximum element $\emptyset$. Then we can set the convex set $C^*$ of lower semicontinuous and nonnegative monotone functions $\varphi$ on $L$ with $\varphi(\emptyset) \leq 1$. Observe that a lower semicontinuous and nonnegative monotone functions $\varphi$ on $L$ uniquely corresponds to a bounded capacity $\psi$ on $K$ via

\[ \varphi(E) = \sup_{F \in K} \psi(F) - \psi(E) + \psi(\emptyset), \quad E \in K. \]

The topology of vague convergence is introduced by (3) over the convex cone $Q$ of nonnegative continuous functions with compact support, in which the convex set $C^*$ becomes compact Hausdorff.

### 2. A framework of continuous semilattice

In the application of integral representation for capacities on a locally compact Hausdorff $S$, the Hausdorff assumption seems indispensable in order for $C^*$ to be compact Hausdorff. Then the set $E^*$ of extreme points of $C^*$ is compact and homeomorphic to the family of open upper subsets $U$, and the integral representation (2) of $\varphi \in C^*$ is equivalently formulated as

\[ (4) \quad \varphi(x) = \mu(U_x), \quad x \in L, \]

where $U_x := \{U \in E^* : x \in U\}$ is an open set in $E^*$.

In the framework of continuous posets (cf. Giertz et al. [3]), the compact Hausdorff set $E^*$ is homeomorphic to the family of Scott open subsets of $L$. Here the topology of vague convergence corresponds to the Lawson topology, which comes solely from the fact that $L$ is a continuous semilattice. Norberg [5] showed that it is entirely possible to construct a Borel measure $\mu$ on the family $E^*$ of Scott open subsets satisfying (4) if $L$ is a continuous semilattice and $E^*$ is second countable. Thus, we can choose $S$ to be a locally compact sober and second countable space, which is not necessarily Hausdorff. Note that the Borel measure $\mu$ is a Radon measure when $E^*$ is second countable; see [2].

We claim that $E^*$ is not necessarily second countable, and demonstrate it by a rather straightforward construction of a Radon measure $\mu$ satisfying (4) due to
Murofushi and Sugeno [4]. Let \( \varphi \in C^* \) be fixed, and let \( e \) denote the top element of the continuous semilattice \( L \). Observe that

\[
F(r) = \{ x \in L : \varphi(x) > r \}
\]

maps from \( r \in [0, \varphi(e)] \) to \( E^* \), and \( F \) is Borel-measurable. For a Borel measurable subset \( V \) of \( E^* \) we can define \( \mu(V) := m(F^{-1}(V)) \) with the Lebesgue measure \( m \) on \( [0, \varphi(e)] \). Then we can show that \( \mu \) is a Radon measure, and it satisfies

\[
\mu(U_x) = m([0, \varphi(x)]) = \varphi(x).
\]

It should be noted that Norberg [5] has investigated a Borel measure \( \mu \) on the family \( L^* \) of Scott open filters in \( L \), and proved a bijection between Borel measures on \( L^* \) and lower semicontinuous and completely monotone nonnegative functions on \( L \). The above construction immediately fails for this purpose since (5) does not map into \( L^* \) in general even if \( \varphi \) is completely monotone.

Finally we present our own version of construction without assuming the second countable \( E^* \). Let \( C(E^*) \) be the space of continuous functions on \( E^* \), and let \( \delta_x \) be a point mass probability measure (i.e., Dirac delta) at \( x \in L \). Here we will use the following proposition, but leave the proof for the future publication.

**Proposition 1.** There exists a subspace \( R \) of \( C(E^*) \) such that \( (i) \) each \( g \in R \) is uniquely extended to a signed Radon measure \( R \) on \( L \) so that \( g(U) = R(U) \) for any \( U \in E^* \), and \( (ii) \) for each \( x \in L \) there is an increasing net \( \{ g_\alpha \} \) of \( R \) satisfying \( \sup_\alpha g_\alpha(U) = \delta_x(U) \) for any \( U \in E^* \).

For a fixed \( \varphi \in C^* \), we can introduce a nonnegative homogeneous and superadditive functional on \( C(E^*) \) by

\[
M(g) = \sup \left\{ \int \varphi \, dR : R \leq g, R \in R \right\}, \quad g \in C(E^*).
\]

By applying the Hahn-Banach theorem we obtain a linear functional \( \Phi \) on \( C(E^*) \) satisfying \( (a) \) \( M \leq \Phi \) on \( C(E^*) \), and \( (b) \) \( M = \Phi \) on \( R \). The condition \( (a) \) implies that \( \Phi \) is positive, and that \( \Phi \) uniquely corresponds to a Radon measure \( \mu \) on \( E^* \) via the Riesz representation \( \Phi(g) = \int g \, d\mu \). By applying Proposition 1 together with the condition \( (b) \), we can show that if an increasing net \( \{ R_\alpha \} \) of \( R \) satisfies \( \sup_\alpha R_\alpha(U) = \delta_x(U) \) for \( U \in E^* \) then

\[
\mu(U_x) = \sup_\alpha \Phi(R_\alpha) = \sup_\alpha M(R_\alpha) = \sup_\alpha \int \varphi \, dR_\alpha = \varphi(x),
\]

as desired. A variation of this construction can be used to show the existence of a Radon measure \( \mu \) whose support lies on \( L^* \) when \( \varphi \) is completely monotone (which is a part of the ongoing investigation).
References


