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THE LÉVY-PROKHOROV TOPOLOGY ON NONADDITIVE MEASURES ON METRIC SPACES

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ABSTRACT. We formalize the Lévy–Prokhorov metric and the Fortet–Mourier metric for nonadditive measures on a metric space and show that the Lévy topology on every uniformly equi-autocontinuous set of Radon nonadditive measures can be metrized by such metrics. This result is proved using the uniformity for Lévy convergence on a bounded subset of Lipschitz functions. We describe some applications to stochastic convergence of a sequence of measurable mappings on a nonadditive measure space.

1. INTRODUCTION

This is an announcement of the forthcoming paper [13]. Weak convergence of measures on a topological space not only plays a very important role in probability theory and statistics, but is also interesting from a topological measure theoretic view, since it gives a convergence closely related to the topology of the space on which the measures are defined. Thus, it is possible to study weak convergence of measures on a topological space in association with some topological properties of the space, such as the metrizability, separability and compactness; for comprehensive information on this convergence, readers are referred to Alexandroff [1], Billingsley [2], Dudley [5], Parthasarathy [21], Vakhania *et al.* [23], Varadarajan [24], and references therein.

Nonadditive measures, which are set functions that are monotonic and vanish at the empty set, have been extensively studied [4, 22, 25]. They are closely related to nonadditive probability theory and the theory of capacities and random capacities. Nonadditive measures have been used in expected utility theory, game theory, and some economic topics under Knightian uncertainty [3, 8, 14, 16, 19, 20].

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The notion of weak convergence of nonadditive measures was formulated by Girotto and Holzer in a fairly abstract setting [10]. Some of their fundamental results for weak convergence, such as the portmanteau theorem and the direct and converse Prokhorov theorems, have been extended to the nonadditive case. In particular, the portmanteau theorem allows us to show that weak topology, which is topology generated by weak convergence, coincides with the Lévy topology, which is the topology generated by convergence of measures on a special class of sets. Although the metrizability of a topology is usually one of the main topics in topological theory, there seem to be no reports that focus on the metrizability of the Lévy topology of nonadditive measures. The aims of this paper are (i) to present successful nonadditive analogs of the theory of weak convergence of measures with a particular focus on metrizability, and (ii) to supply weak convergence methods to related fields.

The remainder of the paper is organized as follows. In Section 2 we recall the notion of regularity systems of sets that are crucial for formalizing the portmanteau theorem [10]. We also describe some new properties for later use and some examples of regularity systems. In Section 3 we discuss the possibility of metrizing subsets of nonadditive measures on a metric space. In particular, we show that the Lévy topology on the set of all Radon and autocontinuous nonadditive measures on a separable metric space can be metrized. In Section 4 we introduce a Lévy–Prokhorov metric and a Fortet–Mourier metric on the space of nonadditive measures on a metric space. In Section 5 we show that the Lévy topology on a uniformly equi-autocontinuous set of Radon nonadditive measures can be metrized by the Lévy–Prokhorov and Fortet–Mourier metrics. This result is proved using the uniformity of the Lévy convergence on a bounded subset of Lipschitz functions. In Section 6 we describe some applications to stochastic convergence of a sequence of measurable mappings defined on a nonadditive measure space.

2. Regularity system and the portmanteau theorem

Throughout the paper, unless stated otherwise, X is a Hausdorff space and $\mathcal{B}$ is a field containing all open subsets of X. Let 2^X be the family of all subsets of X. For a set $A \subset X$, A^-, A°, and ∂A denote the closure, interior, and boundary of A, respectively. $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers, respectively. Let χ_A be the characteristic function of a set A. For each $x \in X$, δ_x denotes the unit mass at $x \in X$ defined by $\delta_x(A) := \chi_A(x)$ for all $A \subset X$. We say that a set function $\mu : \mathcal{B} \to [0, \infty)$ is a *nonadditive measure* on

X if $\mu(\emptyset) = 0$ and $\mu(A) \le \mu(B)$ whenever $A, B \in \mathcal{B}$ and $A \subset B$. Let $\mathcal{M}(X)$ be the set of all such measures. As usual, the *conjugate* $\bar{\mu}$ of μ is defined by $\bar{\mu}(B) := \mu(X) - \mu(B^c)$ for all $B \in \mathcal{B}$, where B^c denotes the complement of the set B. For a subset $\mathcal{P} \subset \mathcal{M}(X)$, let $\overline{\mathcal{P}} := \{\bar{\mu} : \mu \in \mathcal{P}\}$; this is called the *conjugate space* of $\mathcal{P}$. Girotto and Holzer formulated a nonadditive extension of the portmanteau theorem for weak convergence of measures using the following regularity systems of sets [10].

Definition 1. Let $\mu \in \mathcal{M}(X)$.

(1) The *outer regularization* μ^* of μ is the nonadditive measure defined by $\mu^*(A) := \inf\{\mu(U) : A \subset U \text{ and } U \text{ is open}\}$ for every subset A of X, and the *inner regularization* μ_* of μ is the nonadditive measure defined by $\mu_*(A) := \sup\{\mu(C) : C \subset A \text{ and } C \text{ is closed}\}$ for every subset A of X. We denote by $\mathcal{R}_\mu$ the family of all $B \in \mathcal{B}$ satisfying $\mu(B) = \mu^*(B) = \mu_*(B)$ and we call this the *μ-regularity system.*

(2) The *strongly outer regularization* $\mu^\natural$ of μ is the nonadditive measure defined by $\mu^\natural(A) := \inf\{\mu(C) : A \subset C \text{ and } C \in \mathcal{R}_\mu \text{ is closed}\}$ for every subset A of X, and the *strongly inner regularization* $\mu_\natural$ of μ is the nonadditive measure defined by $\mu_\natural(A) := \sup\{\mu(U) : U \subset A \text{ and } U \in \mathcal{R}_\mu \text{ is open}\}$ for every subset A of X. We denote by $\mathcal{R}_\mu^\circ$ the family of all $B \in \mathcal{B}$ satisfying $\mu(B) = \mu^\natural(B) = \mu_\natural(B)$ and we call this the *μ-strong regularity system.*

Remark 1. (1) This type of regularity was introduced and discussed by Narukawa and Murofushi for nonadditive measures on locally compact spaces [18].

(2) The regularity notion defined above is different from that in [12, Definition 5] and it is appropriate for our purpose. If μ is autocontinuous, then the former follows from the latter.

For later use, we collect some properties of the μ-regularity and μ-strong regularity systems; see [10, Remark 2.2 and Theorem 2.3] for proofs.

Proposition 1. *Let $\mu \in \mathcal{M}(X)$ and $B \in \mathcal{B}$.*

(1) $\mathcal{R}_\mu^\circ \subset \mathcal{R}_\mu$.

(2) $\emptyset, X \in \mathcal{R}_\mu^\circ$.

(3) $B \in \mathcal{R}_\mu$ *if and only if* $B^c \in \mathcal{R}_{\bar{\mu}}$.

(4) $B \in \mathcal{R}_\mu^\circ$ *if and only if* $B^c \in \mathcal{R}_{\bar{\mu}}^\circ$.

Let $\mu \in \mathcal{M}(X)$ and let f be a real-valued function on X such that $\{f > t\} \in \mathcal{B}$ for all $t \in \mathbb{R}$. The (asymmetric) *Choquet integral* of f with respect to μ is defined by

$$\int_X f d\mu := \int_0^\infty \mu(\{f > t\})dt - \int_{-\infty}^0 \{\mu(X) - \mu(\{f > t\})\}dt$$

whenever the Lebesgue integrals of the right-hand side of the above equation are not both ∞. We say that f is *Choquet integrable* if $\int_X f d\mu < \infty$. Every bounded continuous function and every characteristic function of a set in $\mathcal{B}$ is Choquet integrable. In this paper, the symbol $\int_X f d\mu$ always denotes the Choquet integral.

We denote by $C_b(X)$ the real Banach space of all bounded, continuous real-valued functions on X with norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$. The *weak topology* on $\mathcal{M}(X)$ is the topology such that, for any $\mu \in \mathcal{M}(X)$, the basic neighborhoods of μ are sets of the form:

$$V_{\mu,\varepsilon,f_1,\ldots,f_k} := \left\{ \nu \in \mathcal{M}(X) : \left| \int_X f_i d\nu - \int_X f_i d\mu \right| < \varepsilon \ \ (i = 1, \ldots, k) \right\},$$

where $\varepsilon > 0$, $k \in \mathbb{N}$, and $f_1, \ldots, f_k \in C_b(X)$ [10, Definition 3.1]. Given a net $\{\mu_\alpha\}_{\alpha \in \Gamma}$ in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$, we say that μ_α *weakly converges* to μ and write $\mu_\alpha \xrightarrow{w} \mu$ if μ_α converges to μ with respect to the weak topology. Obviously, $\mu_\alpha \xrightarrow{w} \mu$ if and only if $\int_X f d\mu_\alpha \to \int_X f d\mu$ for every $f \in C_b(X)$.

There are already some convergence notions of nonadditive measures [17, 26]. However, the following proposition shows that weak convergence has the advantage of giving a convergence related to the topology of X; see [24, Theorem II.9] for the proof.

Proposition 2. *Assume that $\mathcal{D}(X) := \{\delta_x : x \in X\}$ is endowed with the relative topology induced by the weak topology on $\mathcal{M}(X)$. Then X is homeomorphic to $\mathcal{D}(X)$ if and only if X is completely regular.*

Now we are ready to introduce a nonadditive extension of the portmanteau theorem, which gives a comprehensive list of conditions equivalent to weak convergence. The following is a special case of [10, Theorem 3.7] and is enough for our purpose in this paper.

Theorem 1 (The portmanteau theorem). *Let X be a normal space and let $\{\mu_\alpha\}_{\alpha \in \Gamma}$ be a net in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. The following conditions are equivalent:*

(i) $\mu_\alpha \xrightarrow{w} \mu$.
(ii) $\bar{\mu}_\alpha \xrightarrow{w} \bar{\mu}$.
(iii) $\limsup_{\alpha \in \Gamma} \mu_\alpha(C) \le \mu(C)$ *and* $\mu(U) \le \liminf_{\alpha \in \Gamma} \mu_\alpha(U)$ *for every closed* $C \in \mathcal{R}_\mu$ *and every open* $U \in \mathcal{R}_\mu$.
(iv) $\mu_\alpha(B) \to \mu(B)$ *for every* $B \in \mathcal{R}_\mu^\circ$.
(v) $\mu_\alpha(U) \to \mu(U)$ *for every open* $U \in \mathcal{R}_\mu^\circ \cap \mathcal{R}_{\bar{\mu}}^\circ$.
(vi) $\mu_\alpha(C) \to \mu(C)$ *for every closed* $C \in \mathcal{R}_\mu^\circ \cap \mathcal{R}_{\bar{\mu}}^\circ$.

Recall that the *Lévy topology* on $\mathcal{M}(X)$ is the topology such that, for any $\mu \in \mathcal{M}(X)$, the basic neighborhoods of μ are the sets of the form:

$$W_{\mu,\varepsilon,B_1,\ldots,B_k} := \left\{ \nu \in \mathcal{M}(X) : \begin{array}{l} |\nu(B_i) - \mu(B_i)| < \varepsilon \\ |\bar{\nu}(B_i) - \bar{\mu}(B_i)| < \varepsilon \end{array} \quad (i = 1, 2, \ldots, k) \right\},$$

where $\varepsilon > 0$, $k \in \mathbb{N}$, and $B_1, \ldots, B_k \in \mathcal{R}_\mu^\circ \cap \mathcal{R}_{\bar{\mu}}^\circ$ [10, Definition 4.1]. Given a net $\{\mu_\alpha\}_{\alpha\in\Gamma}$ in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$, we say that μ_α *Lévy converges* to μ and write $\mu_\alpha \xrightarrow{L} \mu$ if μ_α converges to μ with respect to the Lévy topology. The portmanteau theorem shows that $\mu_\alpha \xrightarrow{L} \mu$ if and only if $\mu_\alpha(B) \to \mu(B)$ for every $B \in \mathcal{R}_\mu^\circ$, and hence the weak topology and the Lévy topology coincide on $\mathcal{M}(X)$.

In the rest of this section, we study regularity systems in more detail. To this end, we introduce the following continuity notion of nonadditive measures.

Definition 2. Let $\mu \in \mathcal{M}(X)$.

(1) μ is said to be *c-continuous* if $\mu(C) = \inf_{n\in\mathbb{N}} \mu(C_n)$ whenever $\{C_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets with $C = \bigcap_{n=1}^\infty C_n$.

(2) μ is said to be *o-continuous* if $\mu(U) = \sup_{n\in\mathbb{N}} \mu(U_n)$ whenever $\{U_n\}_{n\in\mathbb{N}}$ is an increasing sequence of open sets with $U = \bigcup_{n=1}^\infty U_n$.

(3) μ is said to be *co-continuous* if it is c-continuous and o-continuous.

Remark 2. (1) This type of continuity was introduced and discussed by Narukawa and Murofushi for nonadditive measures on locally compact spaces [18].

(2) The generalized sequence versions of c-continuity and o-continuity are called the total c-continuity and total o-continuity, respectively. They have previously been discussed with applications to convergence theorems for Choquet integrals [12].

In general, c-continuity and o-continuity are independent of each other (Examples 2 and 3). We say that X is *perfectly normal* if X is normal and every closed subset of X is a G_δ-set, which is a countable intersection of open sets. Every metric space is perfectly normal [6, Corollary 4.1.13].

Proposition 3. *Let X be perfectly normal and let $\mu \in \mathcal{M}(X)$.*

(1) *If μ is c-continuous, then $\mathcal{R}_\mu$ contains all closed subsets of X.*

(2) *If μ is o-continuous, then $\mathcal{R}_\mu$ contains all open subsets of X.*

If μ is additive, that is, $\mu(A\cup B) = \mu(A)+\mu(B)$ whenever $A, B \in \mathcal{B}$ and $A\cap B = \emptyset$, then the μ-regularity system $\mathcal{R}_\mu$ and the μ-strong regularity system $\mathcal{R}_\mu^\circ$ are fields, and $\mathcal{R}_\mu^\circ$ is the family of all $B \in \mathcal{B}$ with $\partial B \in \mathcal{R}_\mu$ and $\mu(\partial B) = 0$; see [9, Theorems 2.2 and 2.5] and [10, Remark 2.2(ii)], but this is not the case for nonadditive measures (Examples 1 and 2).

Proposition 4. *Let $\mu \in \mathcal{M}(X)$ and $B \in \mathcal{B}$.*

(1) *If $B \in \mathcal{R}_\mu^\circ$, then $\mu(B^-) = \mu(B^\circ)$.*

(2) *Let X be perfectly normal. Assume that μ is co-continuous. Then $B \in \mathcal{R}_\mu^\circ$ if and only if $\mu(B^-) = \mu(B^\circ)$.*

In probability theory, a Borel set B with $\mu(B^-) = \mu(B^\circ)$ (thus $\mu(\partial B) = 0$) is called a μ-continuity set [2, page 15]. Therefore, Proposition 4 gives a sufficient condition for coincidence of the μ-strong regularity system $\mathcal{R}_\mu^\circ$ with the set of all μ-continuity $B \in \mathcal{R}_\mu$. Note that, in general, $\mu(\partial B) = 0$ is not equivalent to $\mu(B^-) = \mu(B^\circ)$ (Examples 1 and 2).

Example 1. Let $X := [0,1]$. We define the nonadditive measure $\mu : 2^X \to [0,1]$ as

$$\mu(A) := \begin{cases} 0 & \text{if } A \neq X \\ 1 & \text{if } A = X \end{cases}$$

for each subset A of X.

(1) μ is co-continuous.

(2) Let $U := (0,1)$. Then $U^c \in \mathcal{R}_\mu^\circ$, but $U \notin \mathcal{R}_\mu^\circ$. Thus, $\mathcal{R}_\mu^\circ$ is not a field.

(3) Let $U := (0,1)$. Then $\partial U \in \mathcal{R}_\mu$ and $\mu(\partial U) = 0$, but $U \notin \mathcal{R}_\mu^\circ$. Furthermore, $\mu(U^-) \neq \mu(U^\circ)$.

Example 2. Let $X := [0,1]$. We define the nonadditive measure $\mu : 2^X \to [0,1]$ as

$$\mu(A) := \begin{cases} 0 & \text{if } A \subset \{0,1\} \\ 1 & \text{if } A \not\subset \{0,1\} \end{cases}$$

for each subset A of X.

(1) μ is o-continuous but is not c-continuous.

(2) Let $U := (0,1)$. Then $U \in \mathcal{R}_\mu$, but $U^c \notin \mathcal{R}_\mu$. Thus, $\mathcal{R}_\mu$ is not a field.

(3) Let $V := (0,1/2)$. Then $V \in \mathcal{R}_\mu^\circ$, but $\mu(\partial V) \neq 0$. Furthermore, $\mu(V^-) = \mu(V^\circ)$.

Example 3. Let $X := (0,1)$. We define the nonadditive measure $\mu : 2^X \to [0,1]$ as

$$\mu(A) := \begin{cases} 0 & \text{if } A \neq X \\ 1 & \text{if } A = X \end{cases}$$

for each subset A of X. Then μ is c-continuous but is not o-continuous.

3. Metrization of subsets of $\mathcal{M}(X)$

In this section, we discuss the possibility of metrizing subsets of $\mathcal{M}(X)$ in the case in which X is a separable metric space. In the remainder of the paper, let (X,d) be a metric space with metric d. Recall that a real-valued function f on X is called

Lipschitz if there is a constant $K > 0$ such that $|f(x) - f(y)| \le Kd(x,y)$ for all $x, y \in X$. The Lipschitz seminorm is defined as $\|f\|_L := \sup\{|f(x) - f(y)|/d(x,y) : x, y \in X, x \neq y\}$. Let $BL(X,d)$ denote the real Banach space of all bounded, real-valued Lipschitz functions f on X with norm $\|f\|_{BL} := \|f\|_L + \|f\|_\infty$. For any subset A of X, let $d(x,A)$ be the distance from x to A, defined as $d(x,A) := \inf\{d(x,a) : a \in A\}$.

Proposition 5. *Let $\mu, \nu \in \mathcal{M}(X)$. Assume that $\int_X f d\mu = \int_X f d\nu$ for every $f \in BL(X,d)$.*

(1) *If μ and ν are c-continuous, then $\mu(C) = \nu(C)$ for every closed $C \subset X$.*

(2) *If μ and ν are o-continuous, then $\mu(U) = \nu(U)$ for every open $U \subset X$.*

Proposition 6. *Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Assume that μ is co-continuous. The following conditions are equivalent:*

(i) $\mu_\alpha \xrightarrow{w} \mu$.

(ii) $\int_X f d\mu_\alpha \to \int_X f d\mu$ *for every* $f \in BL(X,d)$.

Let $\mathcal{M}_{rco}(X)$ denote the space of all $\mu \in \mathcal{M}(X)$ that are co-continuous and satisfy $\mu(B) = \mu^*(B) = \mu_*(B)$ for all $B \in \mathcal{B}$. Then it is readily evident that $\mathcal{M}_{rco}(X)$ coincides with its conjugate space $\overline{\mathcal{M}_{rco}(X)}$. The following theorem can be proved by Theorem 1 and Propositions 2, 5 and 6 in the same way as [21, Theorem II.6.2].

Theorem 2. *The Lévy topology on $\mathcal{M}_{rco}(X)$ is metrizable as a separable metric space if and only if X is a separable metric space.*

Let $\mu \in \mathcal{M}(X)$. We say that μ is *autocontinuous* if $\lim_{n\to\infty} \mu(A \cup B_n) = \lim_{n\to\infty} \mu(A \setminus B_n) = \mu(A)$ whenever $A, B_n \in \mathcal{B}$ for all $n \in \mathrm{N}$ and $\lim_{n\to\infty} \mu(B_n) = 0$. We also say that μ is *Radon* if, for every $B \in \mathcal{B}$, there are an increasing sequence $\{K_n\}_{n\in\mathrm{N}}$ of compact sets and a decreasing sequence $\{U_n\}_{n\in\mathrm{N}}$ of open sets such that $K_n \subset B \subset U_n$ for all $n \in \mathrm{N}$ and $\lim_{n\to\infty} \mu(U_n \setminus K_n) = 0$.

Proposition 7. *Let $\mu \in \mathcal{M}(X)$. If μ is Radon and autocontinuous, then $\mu, \bar{\mu} \in \mathcal{M}_{rco}(X)$.*

By Theorem 2 and Proposition 7, we have the following theorem.

Theorem 3. *Each of the Lévy topologies on the set of all Radon and autocontinuous $\mu \in \mathcal{M}(X)$ and on its conjugate space is metrizable as a separable metric space if and only if X is a separable metric space.*

4. The Lévy–Prokhorov metric and the Fortet–Mourier metric

In this section, we introduce a Lévy–Prokhorov metric and a Fortet–Mourier metric on $\mathcal{M}(X)$. For any subset A of X and $\varepsilon > 0$, let $A^\varepsilon := \{x \in X : d(x, A) < \varepsilon\}$.

Definition 3. For any $\mu, \nu \in \mathcal{M}(X)$, let $\rho(\mu,\nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ for all } B \in \mathcal{B}\}$.

In the same way as for the additive case [5, Theorem 11.3.1], it can be proved that ρ satisfies $\rho(\mu,\nu) \geq 0, \rho(\mu,\mu) = 0$, and $\rho(\mu,\nu) \leq \rho(\mu,\lambda) + \rho(\lambda,\nu)$ for any $\mu,\nu,\lambda \in \mathcal{M}(X)$.

Example 4. The symmetry relation $\rho(\mu,\nu) = \rho(\nu,\mu)$ does not hold in general. Let $X := [0,1]$ and let $\mu := \delta_0$. We define the nonadditive measure $\nu : 2^X \to [0,1]$ as

$$\nu(A) := \begin{cases} 0 & \text{if } A = \emptyset \\ 1/2 & \text{if } A \neq \emptyset \text{ and } 0 \notin A \\ 1 & \text{if } 0 \in A. \end{cases}$$

Then $\rho(\mu,\nu) = 0$, but $\rho(\nu,\mu) = 1/2$.

Proposition 8. *Let $\mu,\nu \in \mathcal{M}(X)$ and assume that $\mu(X) = \nu(X)$. Then $\rho(\mu,\nu) = \rho(\bar{\nu},\bar{\mu})$.*

Example 5. Proposition 8 is no longer true if $\mu(X) \neq \nu(X)$. Let $X := [0,1]$ and let $\mu := \delta_0/2$ and ν be the same as in Example 4. Then $\mu(X) = 1/2 \neq 1 = \nu(X)$. The conjugate $\bar{\nu}$ of ν is

$$\bar{\nu}(A) := \begin{cases} 0 & \text{if } A = \emptyset \text{ or } 0 \notin A \\ 1/2 & \text{if } A \neq X \text{ and } 0 \in A \\ 1 & \text{if } A = X. \end{cases}$$

Then $\rho(\mu,\nu) = 0$, but $\rho(\bar{\nu},\bar{\mu}) = 1/2$.

Definition 4. For any $\mu,\nu \in \mathcal{M}(X)$, let $\pi(\mu,\nu) := \rho(\mu,\nu) + \rho(\nu,\mu) + \rho(\bar{\mu},\bar{\nu}) + \rho(\bar{\nu},\bar{\mu})$; we call this the *Lévy–Prokhorov semimetric.*

The following properties can be proved by the standard argument in the additive case.

Proposition 9. *Let $\mu,\nu \in \mathcal{M}(X)$.*

(1) *π is a semimetric on $\mathcal{M}(X)$. Furthermore, $\pi(\mu,\nu) = \pi(\bar{\mu},\bar{\nu})$.*

(2) *Assume that μ and ν are c-continuous and $\mu(B) = \mu_*(B)$ and $\nu(B) = \nu_*(B)$ for all $B \in \mathcal{B}$. Then $\mu = \nu$ whenever $\rho(\mu,\nu) = \rho(\nu,\mu) = 0$.*

(3) *Assume that μ and ν are o-continuous and $\mu(B) = \mu^*(B)$ and $\nu(B) = \nu^*(B)$ for all $B \in \mathcal{B}$. Then $\mu = \nu$ whenever $\rho(\bar{\mu},\bar{\nu}) = \rho(\bar{\nu},\bar{\mu}) = 0$.*

Thus, by Propositions 7 and 9, we have the following.

Proposition 10. *The Lévy–Prokhorov semimetric π is a metric on $\mathcal{M}_{rco}(X)$, and hence it is also a metric on the set of all Radon and autocontinuous $\mu \in \mathcal{M}(X)$ and on its conjugate space.*

Next we define the Fortet–Mourier metric on $\mathcal{M}(X)$ that is closely linked to weak convergence [7].

Definition 5. For any $\mu, \nu \in \mathcal{M}(X)$, let

$$\kappa(\mu,\nu) := \sup\left\{\left|\int_X f d\mu - \int_X f d\nu\right| : f \in BL(X,d), \|f\|_{BL} \le 1\right\}.$$

We call this the *Fortet–Mourier semimetric.*

Proposition 11. *Let $\mu, \nu \in \mathcal{M}(X)$.*

(1) *κ is a semimetric on $\mathcal{M}(X)$. Furthermore, $\kappa(\mu,\nu) = \kappa(\bar{\mu},\bar{\nu})$.*

(2) *Assume that μ and ν are c-continuous and $\mu(B) = \mu_*(B)$ and $\nu(B) = \nu_*(B)$ for all $B \in \mathcal{B}$. Then $\mu = \nu$ whenever $\kappa(\mu,\nu) = 0$.*

(3) *Assume that μ and ν are o-continuous and $\mu(B) = \mu^*(B)$ and $\nu(B) = \nu^*(B)$ for all $B \in \mathcal{B}$. Then $\mu = \nu$ whenever $\kappa(\mu,\nu) = 0$.*

(4) *The semimetric κ is a metric on $\mathcal{M}_{rco}(X)$, and hence it is also a metric on the set of all Radon and autocontinuous $\mu \in \mathcal{M}(X)$ and on its conjugate space.*

Proposition 12. *For any $\mu, \nu \in \mathcal{M}(X)$, we have $\rho(\mu,\nu) \le \kappa(\mu,\nu) + \kappa(\mu,\nu)^{1/2}$.*

5. Metrization by the Lévy–Prokhorov and Fortet–Mourier metrics

In this section, we show that the Lévy topology on certain sets of nonadditive measures can be metrized by the Lévy–Prokhorov metric and the Fortet–Mourier metric. To this end, we introduce the notion of uniform equi-autocontinuity of nonadditive measures.

Definition 6. Let $\mathcal{P}$ be a subset of $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$.

(1) μ is said to be *uniformly autocontinuous* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for any $A, B \in \mathcal{B}$, $\mu(B) < \delta$ implies $\mu(A \cup B) - \varepsilon \le \mu(A) \le \mu(A \setminus B) + \varepsilon$.

(2) $\mathcal{P}$ is said to be *uniformly equi-autocontinuous* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for any $\mu \in \mathcal{P}$ and $A, B \in \mathcal{B}$, $\mu(B) < \delta$ implies $\mu(A \cup B) - \varepsilon \le \mu(A) \le \mu(A \setminus B) + \varepsilon$.

Let $\mu \in \mathcal{M}(X)$. We say that μ is *subadditive* if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for every $A, B \in \mathcal{B}$. Obviously, every subadditive measure is uniformly autocontinuous, and every subset of the subadditive measures is uniformly equi-autocontinuous. Recall that μ is *tight* if, for every $\varepsilon > 0$, there is a compact subset K of X such that $\mu(X \setminus K) < \varepsilon$. It is obvious that every Radon $\mu \in \mathcal{M}(X)$ is tight.

Unlike the Lebesgue integral, the Choquet integral is generally nonlinear with respect to its integrand owing to the nonadditivity of μ. That is, we may have $\int_X (f+g)d\mu \neq \int_X f d\mu + \int_X g d\mu$ for some functions f and g. This is one of the reasons why we should assume the uniform equi-autocontinuity of a set of nonadditive measures in the following theorems. The following theorem shows the uniformity for the Lévy convergence on a bounded subset of Lipschitz functions and it plays an essential part in the proof of Theorem 5.

Theorem 4. *Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Assume that $\{\mu_\alpha\}_{\alpha\in\Gamma}$ is uniformly equi-autocontinuous and μ is uniformly autocontinuous. Furthermore, assume that μ is tight. The following conditions are equivalent:*

(i) $\int_X f d\mu_\alpha \to \int_X f d\mu$ *for every* $f \in BL(X, d)$.
(ii) $\kappa(\mu_\alpha, \mu) \to 0$.

Theorem 5. *Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Assume that $\{\mu_\alpha\}_{\alpha\in\Gamma}$ is uniformly equi-autocontinuous and μ is uniformly autocontinuous. Furthermore, assume that μ is tight and co-continuous. The following conditions are equivalent:*

(i) $\mu_\alpha \xrightarrow{L} \mu$.
(ii) $\mu_\alpha \xrightarrow{w} \mu$.
(iii) $\int_X f d\mu_\alpha \to \int_X f d\mu$ *for every* $f \in BL(X, d)$.
(iv) $\kappa(\mu_\alpha, \mu) \to 0$.
(v) $\pi(\mu_\alpha, \mu) \to 0$.

Remark 3. In Theorem 5, the co-continuity of μ is needed only in the proof of the implication (v) $\Rightarrow$ (i).

Since $\pi(\mu, \nu) = \pi(\bar{\mu}, \bar{\nu})$ and $\kappa(\mu, \nu) = \kappa(\bar{\mu}, \bar{\nu})$ for any $\mu, \nu \in \mathcal{M}(X)$, the following proposition easily follows from Theorem 1.

Proposition 13. *Let $\mathcal{P} \subset \mathcal{M}(X)$. The Lévy topology on $\mathcal{P}$ is metrizable with respect to π (or κ) if and only if the Lévy topology on $\overline{\mathcal{P}}$ is metrizable with respect to π (or κ).*

By Theorem 5 and Propositions 13, we have the following corollaries.

Corollary 1. *Let $\mathcal{P} \subset \mathcal{M}(X)$ be uniformly equi-autocontinuous. Assume that every $\mu \in \mathcal{P}$ is Radon. Each Lévy topology on $\mathcal{P}$ and on its conjugate space $\overline{\mathcal{P}}$ is metrizable with respect to π and κ.*

Corollary 2. *Each Lévy topology on the set of all Radon and subadditive $\mu \in \mathcal{M}(X)$ and on its conjugate space is metrizable with respect to π and κ.*

Let $\mu \in \mathcal{M}(X)$. We say that μ is *submodular* if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ for every $A, B \in \mathcal{B}$ and it is *supermodular* if the reverse inequality holds. We also say that μ is *continuous* if it is continuous from above, that is, $\mu(A_n) \downarrow \mu(A)$ whenever $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ and $A \in \mathcal{B}$ satisfy $A_n \downarrow A$, and it is continuous from below, that is, $\mu(A_n) \uparrow \mu(A)$, whenever $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ and $A \in \mathcal{B}$ satisfy $A_n \uparrow A$. We denote by $\mathcal{CSUBM}(X)$ the set of all continuous and submodular $\mu \in \mathcal{M}(X)$ and by $\mathcal{CSUPM}(X)$ the set of all continuous and supermodular $\mu \in \mathcal{M}(X)$. Let $\mathcal{B}(X)$ be the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X.

Theorem 6. *Let X be a complete or locally compact, separable metric space. Let $\mathcal{B} = \mathcal{B}(X)$.*

(1) $\overline{\mathcal{CSUBM}(X)} = \mathcal{CSUPM}(X)$.

(2) *The Lévy topology on $\mathcal{CSUBM}(X)$ is metrizable with respect to π and κ.*

(3) *The Lévy topology on $\mathcal{CSUPM}(X)$ is metrizable with respect to π and κ.*

Let $\mu \in \mathcal{M}(X)$ and let $\lambda \in \mathbb{R}$. We say that μ satisfies the λ-*rule* if $\mu(A \cup B) = \mu(A) + \mu(B) + \lambda \cdot \mu(A) \cdot \mu(B)$ whenever $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$ [25, Definition 4.3]. Every nonadditive measure satisfying the λ-rule is subadditive when $\lambda < 0$; it is superadditive when $\lambda > 0$; and it is additive when $\lambda = 0$. The following example gives a uniformly equi-autocontinuous set of Radon nonadditive measures that are both subadditive and superadditive.

Example 6. Let $\mathcal{Q}$ be a set of Radon finitely additive $\nu \in \mathcal{M}(X)$ with $\sup_{\nu \in \mathcal{Q}} \nu(X) < \infty$. Let λ_1 and λ_2 be real constants with $\lambda_1 < 0 < \lambda_2$. For each $\lambda \in \mathbb{R}$, we define the function $\varphi_\lambda : [0, \infty) \to [0, \infty)$ as

$$\varphi_\lambda(t) := \begin{cases} \dfrac{e^{\lambda t} - 1}{\lambda} & (\lambda \neq 0) \\ t & (\lambda = 0) \end{cases}$$

and we define the Radon nonadditive measure $\varphi_\lambda \circ \nu : \mathcal{B} \to [0, \infty)$ as $(\varphi_\lambda \circ \nu)(B) := \varphi_\lambda(\nu(B))$ for all $B \in \mathcal{B}$. Then it is routine to show that each $\varphi_\lambda \circ \nu$ satisfies the λ-rule and the set $\mathcal{P} := \{\varphi_\lambda \circ \nu : \lambda_1 \leq \lambda \leq \lambda_2, \nu \in \mathcal{Q}\}$ is uniformly equi-autocontinuous.

Consequently, by Corollary 1, the Lévy topology on $\mathcal{P}$ is metrizable with respect to π and κ.

A set $M \subset \mathcal{M}(X)$ is said to be *uniformly tight* if, for every $\varepsilon > 0$, there is a compact subset K of X such that $\mu(X \setminus K) < \varepsilon$ for all $\mu \in M$. The uniform equi-autocontinuity can also be utilized in the proof of a nonadditive version of the well-known LeCam theorem concerning the uniform tightness of a weakly convergent sequence of measures [5, Theorem 11.5.3].

Theorem 7. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ be a sequence and let $\mu \in \mathcal{M}(X)$. Assume that $\{\mu_n\}$ is uniformly equi-autocontinuous and each μ_n is Radon. Furthermore, assume that μ is tight and c-continuous. If $\mu_n \xrightarrow{L} \mu$, then $\{\mu_n\}$ is uniformly tight.*

6. Applications

In this section, we give some applications to stochastic convergence of sequences of metric space-valued measurable mappings on a nonadditive measure space. Let $(\Omega, \mathcal{A})$ be a measurable space, that is, Ω is a non-empty set and $\mathcal{A}$ is a σ-field of subsets of Ω. Throughout this section, let (X, d) be a metric space and let $\mathcal{B} = \mathcal{B}(X)$. We denote by $\mathcal{F}(\Omega, X)$ the set of all mappings $\xi : \Omega \to X$ that are measurable with respect to the σ-fields $\mathcal{A}$ and $\mathcal{B}(X)$.

Let $\xi \in \mathcal{F}(\Omega, X)$. Let $P : \mathcal{A} \to [0, \infty)$ be a nonadditive measure. The nonadditive measure $P \circ \xi^{-1} \in \mathcal{M}(X)$ defined by $P \circ \xi^{-1}(E) := P(\xi^{-1}(E))$ for every $E \in \mathcal{B}(X)$ is called the *distribution* of ξ. Let $\xi, \eta \in \mathcal{F}(\Omega, X)$, one of which is separably valued, that is, the range space is a separable subset of X. Then the function $\omega \in \Omega \mapsto d(\xi(\omega), \eta(\omega))$ is measurable with respect to the σ-fields $\mathcal{A}$ and $\mathcal{B}(\mathbb{R})$ [23, Proposition I.1.9]. Thus, we say that a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Omega, X)$ *converges in measure* to a separably valued $\xi \in \mathcal{F}(\Omega, X)$ and write $\xi_n \xrightarrow{P} \xi$ if $\lim_{n\to\infty} P(\{\omega \in \Omega : d(\xi_n(\omega), \xi(\omega)) \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. We also say that ξ_n *converges in distribution* to ξ if $P \circ \xi_n^{-1} \xrightarrow{L} P \circ \xi^{-1}$.

Theorem 8. *Let $P : \mathcal{A} \to [0, \infty)$ be an autocontinuous nonadditive measure. Let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Omega, X)$ and $\xi \in \mathcal{F}(\Omega, X)$. Assume that ξ is separably valued and $P \circ \xi^{-1}$ is co-continuous. If $\xi_n \xrightarrow{P} \xi$, then $P \circ \xi_n^{-1} \xrightarrow{L} P \circ \xi^{-1}$.*

Remark 4. (1) By the same proof as for [15, Theorem 3.3], we can show that the autocontinuity of P is a necessary and sufficient condition for $P \circ \xi_n^{-1} \xrightarrow{L} P \circ \xi^{-1}$ for any sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Omega, \mathbb{R})$ and $\xi \in \mathcal{F}(\Omega, \mathbb{R})$ satisfying $\xi_n \xrightarrow{P} \xi$ and $P \circ \xi^{-1}$ is co-continuous.

(2) Let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Omega,\mathbb{R})$ and $\xi \in \mathcal{F}(\Omega,\mathbb{R})$. Denneberg [4, page 97] and Murofushi and co-workers [15] defined the notion of convergence in distribution as $P(\xi_n > t) \to P(\xi > t)$ except at most countably many values t. It is easy to show that the Lévy convergence $P\circ\xi_n^{-1} \xrightarrow{L} P\circ\xi^{-1}$ and the co-continuity of $P\circ\xi^{-1}$ imply convergence in distribution in the sense of Denneberg. However, the converse is not true.

Example 7. For each $n \in \mathbb{N}$, we define the nonadditive measure $\mu_n : \mathcal{B}(\mathbb{R}) \to [0,1]$ as

$$\mu_n(E) := \begin{cases} \delta_{-1}(E) & \text{if } (t,\infty) \subset E \text{ for some } t \in \mathbb{R} \\ \delta_{-1}(E)/n & \text{if } (t,\infty) \not\subset E \text{ for any } t \in \mathbb{R} \end{cases}$$

for every $E \in \mathcal{B}(\mathbb{R})$. Then $\mu_n((t,\infty)) \to \mu((t,\infty))$ for every $t \in \mathbb{R}$, but μ_n does not Lévy converge to μ.

Theorem 9. *Let $P : \mathcal{A} \to [0,\infty)$ be a uniformly autocontinuous nonadditive measure. Let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Omega,X)$ and $\xi \in \mathcal{F}(\Omega,X)$. Assume that each $P\circ\xi_n^{-1}$ is Radon and $P\circ\xi^{-1}$ is co-continuous. Furthermore, assume that $P\circ\xi_n^{-1} \xrightarrow{L} P\circ\xi^{-1}$. The following conditions are equivalent:*

(i) *$P\circ\xi^{-1}$ is tight.*

(ii) *$\{P\circ\xi_n^{-1}\}_{n\in\mathbb{N}}$ is uniformly tight.*

7. Conclusion

We formalized the Lévy–Prokhorov metric and the Fortet–Mourier metric for nonadditive measures on a metric space and showed that the Lévy topology on every uniformly equi-autocontinuous set of Radon nonadditive measures can be metrized by such metrics. This result was proved using the uniformity for the Lévy convergence on a bounded subset of Lipschitz functions. In applications to nonadditive measure theory, we showed that convergence in measure implies convergence in distribution for a sequence of measurable mappings on an autocontinuous measure space. We also showed the uniform tightness of a Lévy convergent sequence of measurable mappings on a uniformly autocontinuous measure space. Our results could supply weak convergence methods for related fields, such as Choquet expected utility theory, game theory, and some economic topics under Knightian uncertainty.

An open problem is the fact that the Lévy topology can be metrized by the Lévy–Prokhorov metric and/or the Fortet–Mourier metric on the set of all Radon autocontinuous nonadditive measures, which is not necessarily uniformly equi-autocontinuous.

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