A representation of unital completely positive maps

Marie Choda

Abstract

Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices, and let $\Phi$ be a unital completely positive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. With the notion of the von Neumann entropy for a state in mind, we give a model of $r$-tupple $\{v_j\}_{j=1}^r$ so that $\Phi(x) = v_1^*xv_1 + \cdots + v_r^*xv_r$, $(x \in M_n(\mathbb{C}))$. The $r$ is uniquely determined for $\Phi$ and the $r$-tupple is also unique up to a $r \times r$ unitary matrix.

1 Introduction

In the framework of the theory of operator algebras, the notion of entropy for automorphisms was introduced by Connes-Stømer in [8], Connes-Narnhofer-Thirring in [9] and Voiculescu in [14] (which is extended by Brown [4]). The Connes-Stømer entropy $H(\theta)$ is defined for a *-automorphism $\theta$ of finite von Neumann algebra $M$ with $\tau = \tau \circ \theta$, where $\tau$ is a fixed given finite trace of $M$. After then, the Connes-Narnhofer-Thirring entropy $h_\phi(\theta)$ is given as an extended version of $H(\theta)$ for a *-automorphism $\theta$ of a $C^*$-algebra $A$ by replacing the trace $\tau$ to a state $\phi$ of $A$, and if $A$ is a finite von Neumann algebra then $h_\tau(\theta) = H(\theta)$. Voiculescu's topological entropy $ht(\theta)$ is defined as an independent version of any state of $A$.

We studied these entropies in [6] and [7] for not only *-automorphisms but also *-endomorphisms like so called canonical shifts. As one of interesting such *-endomorphisms, we picked up the Cuntz canonical endomorphism $\Phi_n$ on the Cuntz algebra $O_n$ which has a strong connection to Longo's canonical shift (cf. [6]). The $O_n$ is the $C^*$-algebra generated by isometries $\{S_1, \cdots, S_n\}$ such that $S_1S_1^* + \cdots + S_nS_n^* = 1$, and the $\Phi_n$ is defined as

$$\Phi_n(x) = S_1xS_1^* + \cdots + S_nxS_n^*, \quad (x \in O_n).$$ (1.1)
Such maps given by the form as the right hand side of (1.1) are unital completely positive maps, and the above notions $H(\cdot), h_\phi(\cdot)$ and $ht(\cdot)$ are available for unital completely positive maps too.

Conditional expectations are the most typical examples of unital completely positive maps, ans states of the matrix algebras $M_n(\mathbb{C})$ are considered as the most elementary example of conditional expectations. However, for a conditional expectation $E$, by their definitions it holds always that $H(E) = h_\phi(E) = ht(E) = 0$. On the other hand, in the case of the von Neumann entropy $S(\phi)$ for a state $\phi$ of $M_n(\mathbb{C})$, it is possible that $S(\phi) \neq 0$.

In order to define "entropy", we need the notion of "finite partition of unity" (see for example, [11]). The most generalized one of "finite partition of unity" was introduced by Lindblad ([10]), and it is called the "finite operational partition of unity".

With these facts in mind, here we give a method to induce the finite operational partition of unity for a given unital completely positive map. That is, let $A$ and $B$ be unital $C^*$-algebras and let $\Phi$ be a unital completely positive map of $A$ to $B$. We give a method to get a model of $r$-tuple $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ such that

$$\Phi(x) = v_1^* xv_1 + \cdots + v_r^* xv_r, \quad (x \in A). \quad (1.2)$$

When $A$ and $B$ are matrix algebras, such a representation is called Kraus representation (cf. Appendix in [13]), or obtained as a straightforward application of Stinespring's theorem (see for example, [1, 3]). We note that this representation is not unique.

Our main purpose in this note is to show, for a given completely positive map $\Phi$, a unique $r$-tuple $v(\Phi) = \{v_1, \cdots, v_r\}$ which is suitable to extend the notion of von Neumann entropy $S(\phi)$ for a state $\phi$ of matrix algebras to the entropy $S(\Phi)$ for a unital completely positive map $\Phi$.

First, for a given completely positive map $\Phi$ from $B(H)$ to $B(K)$ of finite dimensional Hilbert spaces $H, K$, we construct the Hilbert spaces $H \otimes_{\Phi} K$. Let $r = \dim(H \otimes_{\Phi} K)$. Next, we give a $r$-tuple $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ which satisfy that $\Phi(x) = v_1^* xv_1 + \cdots + v_r^* xv_r$. The $r$-tuple is unique up to unitaries and induces $S(\Phi)$. After that, we apply these to the non-commutative Bernoulli shift $\beta$ and we define the entropy $S_\Phi(\Phi)$ with respect to a state $\varphi$ with $\varphi = \varphi \cdot \beta$. These results in this note are in [5].
2 Preliminaries

Here, we denote some notions and terminologies which we use later. We denote by $M_n(C)$ the algebra of $n \times n$ complex matrices, and by $\text{Tr}_n$ the standard trace, that is, the sum of all diagonal components. A matrix $D \in M_n(C)$ is called a density matrix if $D$ is a positive operator with $\text{Tr}_n(D) = 1$ (cf. [11] [12]).

The notation $\eta$ is called the entropy function in usual, and it is the function defined by

$$\eta(t) = \begin{cases} -t \log t, & (0 < t \leq 1) \\ 0, & t = 0 \end{cases}$$

2.1 Finite partitions

The notion of "a finite partition of unity" is the starting point of our study.

2.1.1 Finite partitions of 1

The first one is discussed in the real numbers $\mathbb{R}$. Let

$$\lambda = \{\lambda_1, \cdots, \lambda_n\}$$

be the set of real numbers $\lambda_i \geq 0$ with $\sum \lambda_i = 1$. We say that the $n$-tupple $\lambda \subset \mathbb{R}$ is a finite partition of 1.

2.1.2 Finite operational partition of unity

The terminology, a finite operational partition of unity, was first given by Lindblad ([10]) and after then it is used by Alicki-Fannes([2]).

Let $A$ be a unital $C^*$-algebra. Let $x = \{x_1, \ldots, x_k\} \subset A$. Then $x$ is said to be a finite operational partition of unity of size $k$ if

$$\sum_{i}^{k} x_i^* x_i = 1_A. \quad (2.1)$$

Such a finite operational partition of unity $x = \{x_1, \ldots, x_k\}$ in $A$ induces an $A$-coefficient in $M_k(A)$, whose $(i, j)$ coefficient $x(j, i)$ is given by the following:

$$x(j, i) = x_i^* x_j, \quad (1 \leq i, j \leq k). \quad (2.2)$$

We denote this matrix by $[x]$. Then $[x]$ is an $A$-coefficient density matrix in $M_k(A)$, that is, $[x]$ is a positive operator with $Tr([x]) = \sum_{i=1}^{k} x(i, i) = 1_A$. 

2.2 Entropy for finite partitions of unity

2.2.1 Entropy for finite partitions of 1

Let a $n$-tuple $\lambda \subset \mathbb{R}$ be a finite partition of 1. Let

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n).$$  \hspace{1cm} (2.3)

Then $H(\lambda)$ is called the entropy for the finite partition $\lambda$ of 1.

2.2.2 Entropy for Finite operational partition of unity

Let $x = \{x_1, \ldots, x_k\}$ be an operational partition of unity in a unital $C^*$-algebra $A$, and let $\varphi$ be a state of $A$. The $\rho[x]$ is the $k \times k$ matrix whose $(i,j)$-component is defined by

$$\rho[x](i,j) = \varphi(x_j^* x_i), \hspace{1cm} (i,j = 1, \cdots, k).$$  \hspace{1cm} (2.4)

Then $\rho[x]$ is a density matrix. We call $\rho[x]$ the density matrix associate with $x$ and $\varphi$. If $\varphi$ is a unique tracial state, we denote $\rho[x]$ by $\rho[x]$ simply.

Let $\lambda(\rho[x]) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ be the eigenvalues of the matrix $\rho[x]$. Then $\lambda(\rho[x])$ is a finite partition of 1 because $\rho[x]$ is a density matrix. Hence we have the entropy $H(\lambda(\rho[x]))$.

Let $S(\rho[x])$ be the von Neumann entropy (cf. [11, 12]) for the density matrix $\rho[x]$. Then $S(\rho[x])$ is nothing else but $H(\lambda(\rho[x]))$, that is,

$$S(\rho[x]) = \text{Tr}_k(\eta(\rho[x])) = H(\lambda(\rho[x])) = \sum_i \eta(\lambda_i).$$  \hspace{1cm} (2.5)

3 Representation of completely posive maps

Let $\Phi$ be a completely positive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. Put $A = M_n(\mathbb{C})$. We give a method to get a "finite" family $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ for $\Phi$ which satisfies that $\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r$ for all $x \in A$. We remark that if $\Phi$ is unital, then $v(\Phi)$ is a finite operational partition:

$$\sum_{j=1}^r v_j^* v_j = \Phi(1_A) = 1_{B(K)}.$$  \hspace{1cm} (3.1)
3.1 Hilbert space $H \otimes_{\Phi} K$

Let $H$ be an $n$-dimensional Hilbert space, and let $\Phi: A \to B(K)$ be a completely positive linear map. Let $\{e_1, \cdots, e_m\}$ be the set of mutually orthogonal minimal projections in $B(H)$ with $\Phi(e_i) \neq 0$ for all $i$. Let $\xi_i \in e_i(H)$ be a vector with $\|\xi_i\| = 1$ for $i = 1, \cdots, m$ and we extend $\{\xi_1, \cdots, \xi_m\}$ to an orthonormal basis of $H$ as $\{\xi_i\}$. Let $\{e_{ij}; i, j = 1, \cdots, n\}$ be a matrix units of $A$ with $e_{ij} \xi_j = \xi_i$ so that $e_{ii} = e_i$ for $i = 1, \cdots, m$. Then each $\xi \in H \otimes K$ (the algebraic tensor product $H \otimes K$ of $H$ and $K$) is written by

$$\zeta = \sum_{i=1}^{n} \xi_i \otimes \mu_i, \quad \text{for some } \mu_i \in K. \quad (3.2)$$

**Definition 3.1.1.** We define a sesquilinear form $\langle \cdot, \cdot \rangle_{\Phi}$ on the space $H \otimes K$ by

$$\langle \sum_{i=1}^{n} \xi_i \otimes \mu_i; \sum_{j=1}^{n} \xi_j \otimes \nu_j \rangle_{\Phi} = \sum_{i,j} \Phi(e_{ji}) \mu_i \nu_j, \quad (3.3)$$

where $\langle \cdot, \cdot \rangle_K$ means the inner product of the Hilbert space $K$.

Since $\Phi$ is completely positive, this form $\langle \cdot, \cdot \rangle_{\Phi}$ turns out positive semidefinite. The value $\langle \cdot, \cdot \rangle_{\Phi}$ depends on the choice of the orthonormal basis of $H$. However the kernel of this sesquilinear form $\langle \cdot, \cdot \rangle_{\Phi}$ is unique up to unitaries on $H \otimes K$ as follows:

**Proposition 3.1.2.** Let $\{\xi_1, \cdots, \xi_n\}$ (resp., $\{\xi'_1, \cdots, \xi'_n\}$) be an orthonormal basis of $H$, and let $u$ be a unitary on $H \otimes K$ with $\xi_i = u \xi_i$ for all $i = 1, \cdots, n$. Let $\text{Ker}(\Phi)$ (resp., $\text{Ker}'(\Phi)$) be the kernel of this form via $\{\xi_i\}$ (resp., $\{\xi'_i\}$) Then

$$\text{Ker}'(\Phi) = (\overline{u}u^* \otimes 1) \text{Ker}(\Phi)$$

where $\overline{u}$ is the unitary matrix on $H$ whose $(i, j)$-entry is the conjugate complex number of $u(i, j)$ for all $i, j = 1, \cdots, n$.

**Definition 3.1.3.** Now taking the quotient by the space $\text{Ker}(\Phi)$, we have a preHilbert space and complete to get the Hilbert space $H \otimes_{\Phi} K$.

We denote by $(\sum_{i=1}^{n} \xi_i \otimes \mu_i)_{\Phi}$ the element in $H \otimes_{\Phi} K$ corresponding to $\sum_{i=1}^{n} \xi_i \otimes \mu_i \in H \otimes K$. 

If $K$ is finite dimensional, of course the $H \otimes_{\Phi} K$ is finite dimensional. We can extend this method to get the Hilbert space $H \otimes_{\Phi} K$ to (for an example) non-commutative Bernoulli shifts, and it induces finite dimensional $H \otimes_{\Phi} K$ even if $H$ and $K$ are infinite dimensional. By Proposition 3.1.2, we have the following:

**Proposition 3.1.4.** The dimension of $H \otimes_{\Phi} K$ does not depend on the choice of orthonormal basis of $H$.

**Example 3.1.5.**

1. If $\phi$ is a state of $M_n(\mathbb{C})$, then
   \[
   \dim(\mathbb{C}^n \otimes_{\phi} \mathbb{C}) = \text{rank of } \phi, \text{ (i.e., the rank of the density matrix of } \phi).\]

2. If $E$ is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra $A$ of $M_n(\mathbb{C})$, then
   \[
   \dim(\mathbb{C}^n \otimes_{E} \mathbb{C}^n) = n.
   \]
   Here, we remark that $A$ is isomorphic to the diagonal subalgebra $D_n(\mathbb{C})$.

3. Let $B$ be a subfactor of $M_n(\mathbb{C})$. If $E$ is the conditional expectation $M_n(\mathbb{C})$ to $B$, then
   \[
   \dim(\mathbb{C}^n \otimes_{E} \mathbb{C}^m) = \frac{n}{m}
   \]
   Here, we remark that $B$ is isomorphic to $M_m(\mathbb{C})$ for some $m$ by which $n$ can be divided.

4. If $\alpha$ is an automorphism of $M_n(\mathbb{C})$, then
   \[
   \dim(\mathbb{C}^n \otimes_{\alpha} \mathbb{C}^n) = 1.
   \]

The following shows that $\dim(H \otimes_{\Phi} K)$ can be finite, even if $H$ and $K$ are infinite dimensional.

**Example 3.1.6.** Let $\beta$ be the non-commutative Bernoulli shift of $A = \otimes_{i=1}^{\infty} M_n(\mathbb{C})$. That is, let $A_i = M_n(\mathbb{C})$ for all $i = 1, 2, \ldots$, and for each $m \in \mathbb{N}$, let
   \[
   A(m) = A_1 \otimes \cdots \otimes A_m \otimes 1 \otimes \cdots \subset A, \tag{3.4}
   \]
   where $1$ is the unit of $M_n(\mathbb{C})$. 


The $\beta$ is given as the shift as the followings:

$$\beta(x) = 1 \otimes x \otimes 1 \otimes \cdots, \text{ for all } m, \ x \in A(m). \quad (3.5)$$

Let $H_i = \mathbb{C}^n$ for all $i = 1, 2, \cdots$, and for each $m \in \mathbb{N}$. Fix an vector $\Omega \in \mathbb{C}$ with $||\Omega|| = 1$, and let

$$H(m) = H_1 \otimes \cdots \otimes H_m \otimes \Omega \otimes \cdots \subset H = \bigotimes_{i=1}^{\infty} H_i, \quad (3.6)$$

The $\beta$ is a unital completely positive map from $A \subset B(H)$ to $B(H)$, and the restriction $\beta|_{A(m)}$ of $\beta$ to $A(m)$ is a unital completely positive map from $A(m) \subset B(H(m))$ to $B(H(m+1))$. Apply the above method to $\beta|_{A(m)}$, we have always that

$$\dim(H(m) \bigotimes_{\beta|_{A(m)}} H(m+1)) = n, \text{ for all } m$$

As a result, we have that

$$\dim(H \bigotimes_{\beta} H) = n = \dim(H \bigotimes_{E} H),$$

where $E : A \rightarrow \beta(A)$ is the conditional expectation.

Now, we call the dimension of $H \otimes_{\Phi} K$ the rank of $\Phi$.

**A phenomenon** As an example, we show a phenomenon of the above discussion in the case of a state of $M_2(\mathbb{C})$ which indicates how the dimension of $H \otimes_{\Phi} K$ coincides with the rank of a state $\phi$ of the usual sense.

**Example 3.1.7.** Let $\{\xi_1, \xi_2\}$ be an orthonormal basis of $\mathbb{C}^2$ and let $\{e_{ij}; i, j = 1, 2\}$ be a matrix units of $\mathbb{C}^2$ with $e_{ij}\xi_j = \xi_i$. We give a vector representation for each $\xi \in \mathbb{C}^2$ relative to this $\{\xi_i; i = 1, 2\}$ and a matrix representation for each $x \in M_2(\mathbb{C})$ relative to this $\{e_{ij}; i, j = 1, 2\}$:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.7)$$

and

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.8)$$
Assume that $\phi$ is a state given by
\[
\phi(x) = \phi\left(\begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array}\right) = \frac{1}{2} \sum_{i,j=1}^{2} x_{ij}.
\] (3.9)

Then $\phi(e_i) = 1/2$ for $i = 1, 2$ and the discussion with respect to $\{\xi_1, \xi_2\}$ and $\{e_{ij}; i, j = 1, 2\}$ is as follows:

Let $\Omega$ be a fixed unit vector in $\mathbb{C}$ and let
\[
\zeta_i = \sqrt{2}\xi_i \otimes \Omega \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}.
\] (3.10)

Then
\[
<\zeta_i, \zeta_j>_{\phi} = 2\phi(e_{ij}) = 1, \quad \text{for all} \quad i, j = 1, 2.
\] (3.11)
This implies that $\zeta_i \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}$ has norm 1 for $i = 1, 2$ and
\[
\zeta_1 = \sqrt{2}\xi_1 \otimes \Omega = \sqrt{2}\xi_2 \otimes \Omega = \zeta_2.
\] (3.12)

Next we choose another family of minimal projections with $\phi(p) \neq 0$ and orthonormal basis of $\mathbb{C}^2$. Let
\[
e_{11}' = \frac{1}{2}\begin{array}{ll} 1 & 1 \\ 1 & 1 \end{array},
\] (3.13)
then $\phi(e_{11}') = 1$ so that the set containing $e_{11}'$ of minimal projections with $\phi(\cdot) \neq 0$ is the one point set $\{e_{11}'\}$. The corresponding orthonormal basis of $\mathbb{C}^2$ and the corresponding matrix units are as follows:
\[
\xi_1' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_2' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\] (3.14)
and
\[
e_{11}'', e_{12}' = \frac{1}{2}\begin{array}{ll} 1 & -1 \\ 1 & -1 \end{array}, \quad e_{21}' = \frac{1}{2}\begin{array}{ll} 1 & 1 \\ -1 & -1 \end{array}, \quad e_{22}' = \frac{1}{2}\begin{array}{ll} 1 & -1 \\ -1 & 1 \end{array}
\] (3.15)

Let
\[
\zeta_i' = \frac{1}{\sqrt{2}}\xi_i' \otimes \Omega \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}, \quad (i = 1, 2).
\] (3.16)
Then
\[
<\zeta_1', \zeta_1'>_{\phi} = \frac{1}{2}\phi(e_{11}') = 1,
\] (3.17)
and
\[ < \zeta'_2, \zeta'_2 >_{\phi} = \frac{1}{2} \phi(e_{22}') = 0 \]  
(3.18)

This means that
\[ \mathbb{C}^2 \otimes_{\phi} \mathbb{C} = \mathbb{C} \zeta'_i = \xi'_i \otimes \mathbb{C}. \]

We remark that \( \{\xi_1, \xi_2\} \) and \( \{\xi'_1, \xi'_2\} \) are combined as \( u \xi_i = \xi'_i \) (\( i = 1, 2 \)) by the unitary \( u \):
\[ u = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]. \]  
(3.19)

**Some relation to the Choi matrix.** For a completely positive map \( \Phi \) of \( M_n(\mathbb{C}) \), it is given the so called the Choi matrix \( C_{\Phi} \).

In the case of \( \Phi \) is a state \( \phi \) of \( M_n(\mathbb{C}) \), we have the following relation between the sesquilinear form \( < \cdot, \cdot >_{\phi} \) and the coefficient of \( C_{\phi} \):
\[ < \xi_i \otimes \Omega, \xi_j \otimes \Omega >_{\phi} = C_{\phi}(j,i) \]  
(3.20)

where \( \{\xi_1, \cdots, \xi_n\} \) is a orthonormal basis of \( \mathbb{C}^n \) and \( \Omega \) is 1 considered as the vector in \( \mathbb{C} \).

### 3.2 Operators \( \{v_j; j = 1, \cdots, r\} \)

As the above section, let \( \Phi : B(H) \rightarrow B(K) \) be a completely positive linear map. Here, we assume that \( H \) and \( K \) be finite dimensional, and let \( r = \dim(H \otimes_{\phi} K) \).

**Definition 3.2.1.** Let \( \{\xi_i; i = 1, \cdots, n\} \) be an orthonormal basis of \( H \), and let \( \{\zeta_j; j = 1, \cdots, r\} \) be an orthonormal basis of \( H \otimes_{\phi} K \). Define \( v_j : K \rightarrow H \) by
\[ v_j(\mu) = \sum_{i=1}^{n} < \xi_i \otimes \mu, \zeta_j >_{\phi} \xi_i, \quad (\mu \in K) \]  
(3.21)

**Proposition 3.2.2.** Let \( \{v_1, v_2, \cdots, v_r\} \) be the tuple obtained by (3.17). Then
(i) They satisfies the desired following property:
\[ \Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r, \quad (x \in B(H)) \]  
(3.22)

(ii) \( ||v_i|| \leq 1 \), for all \( i = 1, \cdots, r \);
(iii) \( \{v_1, v_2, \cdots, v_r\} \) are linearly independent.

**Proposition 3.2.3.** The tuple \( \{v_1, v_2, \cdots, v_r\} \) for unital completely positive map \( \Phi \) satisfy the following convenient properties to compute the von Neumann type entropy \( S(\Phi) \).

1. If \( E \) is a conditional expectation of \( M_n(\mathbb{C}) \) onto \( D_n(\mathbb{C}) \) then \( \{v_j : 1 \leq j \leq r\} \) are mutually orthogonal minimal projections.

2. If \( \Phi \) is a *-homomorphism, then \( \{v_j : 1 \leq j \leq r\} \) are isometries with \( v_i v_j^* = \delta_{ij}1 \). (3.23)

We remark that the following results are well known (see for example [1, 13]) so that our tuple \( \{v_1, v_2, \cdots, v_r\} \) for unital completely positive map \( \Phi \) is unique up to a unitary matrix:

**Proposition 3.2.4.** Let \( H \) and \( K \) be finite dimensional Hilbert spaces. Assume that \( v = \{v_1, v_2, \cdots, v_r\} \) and \( w = \{w_1, \cdots, w_r\} \) are two families of operators in \( B(K,H) \). Then

\[
v_1^* x v_1 + \cdots + v_r^* x v_r = w_1^* x w_1 + \cdots + w_r^* x w_r, \quad \text{for all} \quad x \in B(H)
\]

if and only if there is a unitary matrix \( [u(i,j)] \in M_r(\mathbb{C}) \) such that

\[
v_i = \sum_{j=1}^{r} u(i,j) w_j, \quad i = 1, \cdots, r.
\]

**Example 3.2.5.** Let \( \beta \) be the non-commutative Bernoulli shift on \( A = \bigotimes_{i=1}^{\infty} M_n(\mathbb{C}) \). The \( n \)-tupple \( \{v_j\}_j \) of \( \beta \) are as follows. We use the same notation as Example 3.1.6. Let

\[
W_m = \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m), \quad \alpha_j \in \{1, 2, \cdots, n\}\}. \quad (3.24)
\]

and let

\[
\xi_\alpha = \xi_{\alpha_1} \otimes \xi_{\alpha_2} \otimes \cdots \otimes \xi_{\alpha_m} \otimes \Omega \otimes \cdots \in H(m) \quad (3.25)
\]

Then \( \{\xi_\alpha; \alpha \in W_m\} \) is an orthonormal basis of \( H(m) \), and

\[
\{\xi_\alpha \otimes_\beta (\xi_i \otimes_\beta \xi_\alpha); i = 1, 2, \cdots, n\} \quad (3.26)
\]

is an orthonormal basis of \( H(m) \otimes_\beta H(m+1) \). Then our tuple for \( \beta \) is given by

\[
v_j(\xi_i \otimes_\beta \xi_\beta) = \delta_{ij} \xi_\beta, \quad \text{for all} \quad m, \beta \in W_m. \quad (3.27)
\]

We remark that \( \{v_j\}_j \) are isometries satisfying the Cuntz relation.
3.3 Relation to Stinespring’s theorem

Let $\Phi : A \subset B(H) \rightarrow B(K)$ be a completely positive map, where $H$ and $K$ are finite dimensional. Let $\{v_j : K \rightarrow H\}_{j=1}^r$ be the tuple by (3.17). We denote by $L$ the Hilbert space of $r$-direct sum of $H$, i.e., $H \oplus \cdots \oplus H$. Let

$$V(\xi) = (v_1\xi, \cdots, v_r\xi) \in L, \quad \xi \in K$$

where $H$ and $K$ are finite dimensional. Let

$$\pi(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ : & x & : & 0 \\ : & : & : & : \\ 0 & 0 & \cdots & x \end{bmatrix}, \quad x \in A.$$

Then we have the followings:

1. The $\pi$ is a representation of the $C^*$-algebra $A$ on the Hilbert space $L$.

2. The property that $\sum_{i=1}^r v_i^*v_i = 1_K$ means that the operator $V$ defined by (3.20) is an isometry from $K$ to $L$.

3. The property $\Phi(x) = \sum_{j=1}^r v_j^*xv_j$ is written as

$$\Phi(x) = (v_1^*, \cdots, v_r^*) \begin{bmatrix} x & 0 & \cdots & 0 \\ : & x & : & 0 \\ : & : & : & : \\ 0 & 0 & \cdots & x \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} = V^*\pi(x)V$$

4. These imply that $(\pi, V)$ can be considered as the pair obtained by the Stinespring’s representation.

5. The property that $\{v_j\}_j$ are linearly independent satisfies that $(\pi, V)$ is the minimal pair in the sense of Arveson [1].

4 Entropy for unital completely positive maps

In this section, we denote an application to the notion of entropy. Let $\Phi$ be a unital completely positive map of a $C^*$-algebra $A \subset B(H)$ to $B \subset B(K)$ and let $v(\Phi) = \{v_1, v_2, \cdots, v_r\}$ be the tuple for $\Phi$ obtained by (3.21). Then the tuple $v(\Phi)$ is a finite operational partition of unity in $B(K)$. Hence we can apply the discussion in the section 2.2.2 to unital completely positive map.
4.1 Case of a state $\phi$ of $M_n(\mathbb{C})$

First, we consider the case of a state $\phi$ of $M_n(\mathbb{C})$. Let $\phi$ be a state of $M_n(\mathbb{C})$, and let $v(\phi) = \{v_j : 1 \leq j \leq r\}$ be the tuple associated with $\phi$ defined by (3.20).

Let $\tau$ be the unique tracial state of $M_n(\mathbb{C})$, that is $\tau(x) = Tr(x)/n$ for all $x \in M_n(\mathbb{C})$. The density matrix $\tau[v(\phi)]$ is given by (2.4). Let $\{e_i\}_{i=1}^{m}$ be the mutually orthogonal minimal projections in $M_n(\mathbb{C})$ such that $\phi(e_i) \neq 0$. Then we see that

$$\tau[v(\phi)](i,j) = \delta_{ij} \sqrt{\phi(e_i)\phi(e_j)}. \quad (4.1)$$

It is clear that $\tau[v(\phi)]$ is a diagonal matrix, and the entropy $S(\tau[v(\phi)])$ in the section 2.2.2 is nothing else but the von neumann entropy $S(\phi)$ of $\phi$:

$$S(\tau[v(\phi)]) = \sum_{j=1}^{r} \eta(\phi(e_j)) = -\sum_{j=1}^{r} \phi(e_j) \log \phi(e_j) = S(\phi). \quad (4.2)$$

4.2 Entropy for unital completely positive maps

On the basis of the fact in the above section 4.1, we denote the $S(\rho[v(\Phi)])$ for a unital completely positive map $\Phi : A \rightarrow B$ by $S(\rho(\Phi))$, and in the case of the tracial state $\rho$ we use the same notation $S(\Phi)$ simply.

Here, we show the case of $A$ which has a unique tracial state $\tau$ and we number the values of typical examples of von Neumann type entropy $S(\Phi)$ for unital completely positive maps $\Phi$.

1. If $\phi$ is a state of $M_n(\mathbb{C})$, then

$$S(\phi) = \sum_{j=1}^{r} \eta(\lambda_j) \quad (4.3)$$

where $\{\lambda_j\}$ are eigenvalues of $\phi$.

2. If $E$ is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra $B$ then

$$S(E) = \log n. \quad (4.4)$$

Compare this fact to that $H(E) = ht(E) = 0.$
3. If $E$ is the conditional expectation of $M_n(\mathbb{C})$ to a subfactor $B$ then

$$S(E) = \log \frac{n}{k}. \quad (4.5)$$

Here we remark that a subfactor $B$ of $M_n(\mathbb{C})$ is isomorphic to $M_k(\mathbb{C})$ some $k$, and that $n$ is divisible by $k$. Compare this fact to that $H(E) = \text{ht}(E) = 0$.

4. If $\alpha$ is an automorphism of $M_n(\mathbb{C})$, then

$$S(\alpha) = 0 \quad (4.6)$$

and this coinsides with the fact that $H(\alpha) = \text{ht}(\alpha) = 0$.

5. If $\beta$ is the non-commutative Bernoulli shift on $\bigotimes_{i=1}^{\infty}M_n(\mathbb{C})$, then

$$S(\beta) = \log n \quad (4.7)$$

and this coinsides with the fact that $H(\beta) = \text{ht}(\beta) = \log n$.

6. If $\Phi_n$ is the Cuntz's canonical shift on $O_n$, then

$$S_\Psi(\Phi_n) = \log n \quad (4.8)$$

and this coinsides with the fact that $h_\Psi(\Phi_n) = \text{ht}(\Phi_n) = \log n$. Here $\Psi$ is the state of $O_n$ which is given by the left inverse of $\Phi$.

References


marie@cc.osaka-kyoiku.ac.jp