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Nonlinear Operators and Convergence Theorems in Optimization

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Abstract. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}\alpha(1+\gamma)\|Ux-Uy\|^2 &+ (1-\alpha(1+\gamma))\|x-Uy\|^2 \\ &\le (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\ &\quad - (\alpha-\beta)\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$. In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is called generalized hybrid [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \le \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. Kocourek, Takahashi and Yao [11] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. Notice that the class of the mappings above covers several classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping T is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$\|Tx-Ty\| \le \|x-y\|, \quad \forall x, y \in C.$$

It is also nonspreading [12, 13] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx-Ty\|^2 \le \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x, y \in C.$$

Furthermore, it is hybrid [28] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx-Ty\|^2 \le \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x, y \in C.$$

The classes of nonexpansive mappings, nonspreading mappings and hybrid mappings are deduced from the equilibrium problem in optimization; see [6] and [28]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1-\alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1-\beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an (α, β)-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Recently, Hojo, Takahashi and Yao [8] defined the following class of nonlinear mappings which contains the class of generalized hybrid mappings. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1+\gamma)\|Ux - Uy\|^2 &+ (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. We note that an extended hybrid mapping is not quasi-nonexpansive generally.

In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

2 Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [27], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

From [18], a Hilbert space H satisfies Opial's condition, i.e., for a sequence $\{x_n\}$ of H such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2.3}$$

Let C be a nonempty closed convex subset of H and let $T \colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points of T. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. Since a generalized hybrid mapping T defined in Introduction is quasi-nonexpansive, $F(T)$ is closed and convex.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [24]. Using Banach limits, Kocourek, Takahashi and Yao [11] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 2.1 ([11]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \ge 0$$

for all $x \in H$ and $u \in C$; see [27] for more details. We also know the following lemma.

Lemma 2.2 ([30]). *Let F be a nonempty closed convex subset of a Hilbert space H, let P be the metric projection of H onto F and let $\{x_n\}$ be a sequence in H such that $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly.*

3 New Class of Extended Hybrid Mappings

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $U : C \to H$ is called extended hybrid [8] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha(1+\gamma)\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ &\quad \le (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{3.1}$$

for all $x, y \in C$ and such a mapping U is called (α, β, γ)-extended hybrid. In [8], the authors derived a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space.

Theorem 3.1 ([8]). *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \ne -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an (α, β)-generalized hybrid mapping if and only if $U : C \to H$ is an (α, β, γ)-extended hybrid mapping. In this case, $F(T) = F(U)$.*

A mapping $U : C \to H$ is called a widely strict pseudo-contraction if there exists a real number $k \in \mathbb{R}$ with $k < 1$ such that

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called a widely k-strict pseudo-contraction. A widely k-strict pseudo-contraction [5] is a strict pseudo-contraction if $0 \leq k < 1$. It is also nonexpansive if $k = 0$. Conversely, if $T : C \to H$ is a nonexpansive mapping, then for any $n \in \mathbb{N}$,

$$U = \frac{1}{1+n}T + \frac{n}{1+n}I$$

is a widely $(-n)$-strict pseudo-contraction. The following result is in [32]:

Proposition 3.2 ([32]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha\, U + (1 - 2\alpha)I$. Then, the following are equivalent:*

(a) A is an α-inverse-strongly monotone mapping, i.e.,

$$\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C;$$

(b) U is a widely $(1 - 2\alpha)$-strict pseudo-contraction, i.e.,

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C;$$

(c) U is a $(1, 0, 2\alpha - 1)$-extended hybrid mapping, i.e.,

$$\begin{aligned} &2\alpha\|Ux - Uy\|^2 + (1 - 2\alpha)\|x - Uy\|^2 \\ &\quad\leq (2\alpha - 1)\|Ux - y\|^2 + 2(1 - \alpha)\|x - y\|^2 \\ &\quad\quad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C; \end{aligned}$$

(d) T is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$, where $Z(A) = \{u \in C : Au = 0\}$.

Let $\alpha > 0$ and let $A\colon C \to H$ be α-inverse-strongly monotone. Then for any $\beta \in \mathbb{R}$ with $0 < \beta \leq 2\alpha$, A is $\frac{\beta}{2}$-inverse-strongly monotone. Thus

$$T = I - \beta A = I - \beta(I - U) = \beta U + (1 - \beta)I$$

is nonexpansive. Using Proposition 3.2, we can get the following result:

Proposition 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $k < 1$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = (1 - k)U + kI$. Then, the following are equivalent:*

(a) A is a $\frac{1-k}{2}$-inverse-strongly monotone mapping;
(b) U is a widely k-strict pseudo-contraction;
(c) U is a $(1, 0, -k)$-extended hybrid mapping;
(d) T is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$.

Let $k < 1$ and let U be a widely k-strict pseudo-contraction. Then for any $t \in \mathbb{R}$ with $k \leq t < 1$, U is a widely t-strict pseudo-contraction. Thus

$$T = (1 - t)U + tI$$

is nonexpansive. We also have the following important result [31] for extended hybrid mappings in a Hilbert space.

Theorem 3.4 ([32]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β, γ be real numbers and let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. Then, $I - U$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.*

Using Theorem 3.5, we have the following result for k-strict pseudo-contractions obtained by Marino and Xu [15]; see also [1].

Corollary 3.5 (Marino and Xu [15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. Then, $I - U$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.*

4 Weak Convergence Theorems

Motivated by Propositions 3.2 and 3.3, we are interested in weak and strong convergence theorems for extended hybrid mappings in a Hilbert space. In this section, we first state the following weak convergence theorem of Baillon's type [2] by using Lemma 2.2.

Theorem 4.1 ([8]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and γ be real numbers with $0 \leq -\gamma < 1$. Let $S : C \to C$ be an (α, β, γ)-extend hybrid mapping with $F(S) \neq \emptyset$ and let P be the mertic projection of H onto $F(S)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} ((1+\gamma)S - \gamma I)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^n x$ and $T = (1+\gamma)S - \gamma I$.

The following weak convergence theorem was proved by Takahashi, Wong and Yao [31].

Theorem 4.2 ([31]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let α, β and γ be real numbers. Let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping such that $1 + \gamma > 0$ and $F(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\{\alpha_n x_n + (1 - \alpha_n)((1+\gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n\to\infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

As direct consequences of Theorem 4.2, we obtain the following results.

Corollary 4.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let γ be a real number with $1 + \gamma > 0$ and let $U : C \to H$ be an $(2, 1, \gamma)$-extended hybrid mapping, i.e.,*

$$\begin{aligned} 2(1+\gamma)\|Ux - Uy\|^2 &- (1+2\gamma)\|x - Uy\|^2 \\ &\leq (1+2\gamma)\|Ux - y\|^2 - 2\gamma\|x - y\|^2 \\ &\quad - \gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Suppose that* $\{x_n\}$ *is the sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = P_C\{\alpha_n x_n + (1-\alpha_n)((1+\gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges weakly to an element* v *of* $F(U)$, *where* $v = \lim_{n\to\infty} P_{F(U)}x_n$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Corollary 4.4. *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* P_C *be the metric projection of* H *onto* C. *Let* γ *be a real number with* $1+\gamma > 0$ *and let* $U : C \to H$ *be an* $(\frac{3}{2}, \frac{1}{2}, \gamma)$*-extended hybrid mapping, i.e.,*

$$\begin{aligned} 3(1+\gamma)\|Ux - Uy\|^2 - (1+3\gamma))\|x - Uy\|^2 \\ \leq (1+3\gamma)\|Ux - y\|^2 + (1-3\gamma))\|x-y\|^2 \\ - 2\gamma\|x - Ux\|^2 - 2\gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Suppose that* $\{x_n\}$ *is the sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha_n)((1+\gamma)Ux_n - \gamma x_n)\big), \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges weakly to an element* v *of* $F(U)$, *where* $v = \lim_{n\to\infty} P_{F(U)}x_n$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Taking $\gamma = -\frac{1}{2}$ in Corollaries 4.3 and 4.4, we obtain two mappings such that

$$2\|Ux - Uy\|^2 \leq 2\|x-y\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2$$

and

$$\begin{aligned} 3\|Ux - Uy\|^2 + \|x - Uy\|^2 + \|y - Ux\|^2 \\ \leq 5\|x-y\|^2 + 2\|x - Ux\|^2 + 2\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$, respectively. We can apply Corollaries 4.3 and 4.4 for such mappings and then obtain weak convergence theorems in a Hilbert space.

5 Strong Convergence Theorems

Using an idea of mean convergence, we can prove the following strong convergence theorem [31] of Halpern's type for extended hybrid mappings in a Hilbert space.

Theorem 5.1 ([31]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* α, β *and* k *be real numbers. Let* $U : C \to C$ *be an* $(\alpha, \beta, -k)$*-extended hybrid mapping such that* $0 \leq k < 1$ *and* $F(U) \neq \emptyset$ *and let* P *be the metric projection of* H *onto* $F(U)$. *Suppose that* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$, $u \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{m=1}^{n}((1-k)U + kI)^m x_n \end{cases}$$

for all $n = 1, 2, \ldots$, *where* $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Pu.

Using the hybrid method by Nakajo and Takahashi [17], we can prove the following strong convergence theorem for extended hybrid non-self mappings in a Hilbert space. The method of the proof is due to Nakajo and Takahashi [17] and Marino and Xu [15].

Theorem 5.2 ([31]). *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* α, β *and* k *be real numbers and let* $U : C \to H$ *be an* $(\alpha, \beta, -k)$*-extended hybrid mapping such that* $k < 1$ *and* $F(U) \neq \emptyset$. *Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$, *and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Ux_n + kx_n\}, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ *is the metric projection of* H *onto* $C_n \cap Q_n$ *and* $\{\alpha_n\} \subset (-\infty, 1)$. *Then,* $\{x_n\}$ *converges strongly to* $z_0 = R_{F(U)}x$, *where* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Using Theorem 5.2, we can prove the following theorem obtained by Marino and Xu [15].

Theorem 5.3 (Marino and Xu [15]). *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* k *be a real number with* $0 \le k < 1$ *and let* $U : C \to C$ *be a* k*-strict pseudo contraction such that* $F(U) \neq \emptyset$. *Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Ux_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ *is the metric projection of* H *onto* $C_n \cap Q_n$ *and* $\{\beta_n\} \subset (-\infty, 1)$. *Then,* $\{x_n\}$ *converges strongly to* $z_0 = R_{F(U)}x$, *where* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Proof. We first know that a (1,0,-k)-extended hybrid mapping with $0 \le k < 1$ is a k-strict pseudo contraction. We also have that for any $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)Ux_n \\ &= \frac{\beta_n - k}{1-k} x_n + \Big(1 - \frac{\beta_n - k}{1-k}\Big)\{(1-k)Ux_n + kx_n\}. \end{aligned}$$

Putting $\alpha_n = \frac{\beta_n - k}{1-k}$, we have from $1 > \beta_n$ that $1 - k > \beta_n - k$ and hence $1 > \frac{\beta_n - k}{1-k} = \alpha_n$. Furthermore, we have that for any $n \in \mathbb{N}$ and $z \in C$,

$$\begin{aligned} &\|y_n - z\|^2 \le \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2 \\ &\iff \|y_n - z\|^2 \le \|x_n - z\|^2 - (1-k)\alpha_n(1-k)(1-\alpha_n)\|x_n - Ux_n\|^2 \\ &\iff \|y_n - z\|^2 \le \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2. \end{aligned}$$

From Theorem 5.2, we have the desired result. □

Next, we prove a strong convergence theorem by the shrinking projection method [29] for extended hybrid non-self mappings in a Hilbert space.

Theorem 5.4 ([31]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and k be real numbers and let $U : C \to H$ be an $(\alpha, \beta, -k)$-extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Ux_n + kx_n\}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|Ux_n - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

References

[1] G. L. Acedo and H.-K. Xu, *Iterative methods for strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal. **67** (2007), 2258–2271.

[2] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511–1514.

[3] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[4] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54** (1965), 1041–1044.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[7] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[8] M. Hojo, W. Takahashi and J.-C. Yao, *Weak and strong mean convergence theorems for super hybrid mappings in Hilbert spaces*, Fixed Point Theory **12** (2011), 113–126.

[9] S. Iemoto and W. Takahashi, *Approximating fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.

[10] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[11] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[12] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM. J. Optim. **19** (2008), 824–835.

[13] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[14] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[15] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strich pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[16] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal. **9** (2008), 37–43.

[17] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-378.

[18] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[19] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506–515.

[20] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[21] S. Takahashi, W. Takahashi and M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl. **147** (2010), 27–41.

[22] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[23] W. Takahashi, *Iterative methods for approximation of fixed points and their applications*, J. Oper. Res. Soc. Japan **43** (2000), 87–108.

[24] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[25] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[26] W. Takahashi, *Viscosity approximation methods for resolvents of accretive operators in Banach spaces*, J. Fixed Point Theory Appl. **1** (2007), 135–147.

[27] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[28] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[29] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[30] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[31] W. Takahashi, N.-C. Wong and J.-C. Yao, *Weak and strong convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.

[32] W. Takahashi, N.-C. Wong and J.-C. Yao, *Two generalized strong convergence theorems of Halpern's type in Hilbert spaces and applications*, Taiwanese J. Math., to appear.

[33] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

[34] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.