Nonlinear Operators and Convergence Theorems in Optimization

東京工業大学，慶應義塾大学，東京理科大学
高橋 沙 (Wataru Takahashi)
Tokyo Institute of Technology, Keio University
and Tokyo University of Science, Japan

Abstract. Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\[
\alpha(1+\gamma)\|Ux-Uy\|^2 + (1-\alpha(1+\gamma))\|x-Uy\|^2 \\
\leq (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\
- (\alpha-\beta)\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2
\]
for all $x, y \in C$. In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. A mapping $T : C \to H$ is called generalized hybrid [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that
\[
\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2
\] (1.1)
for all $x, y \in C$. We call such a mapping an $(\alpha, \beta)$-generalized hybrid mapping. Kocourek, Takahashi and Yao [11] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. Notice that the class of the mappings above covers several classes of well-known mappings. For example, an $(\alpha, \beta)$-generalized hybrid mapping $T$ is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,
\[
\|Tx-Ty\| \leq \|x-y\|, \forall x, y \in C.
\]
It is also nonspreading [12, 13] for $\alpha = 2$ and $\beta = 1$, i.e.,
\[
2\|Tx-Ty\|^2 \leq \|Tx-y\|^2 + \|Ty-x\|^2, \forall x, y \in C.
\]
Furthermore, it is hybrid [28] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,
\[
3\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2, \forall x, y \in C.
\]
The classes of nonexpansive mappings, nonspreading mappings and hybrid mappings are deduced from the equilibrium problem in optimization; see [6] and [28]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Recently, Hojo, Takahashi and Yao [8] defined the following class of nonlinear mappings which contains the class of generalized hybrid mappings. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha(1+\gamma)\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2$$

$$\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2$$

$$- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2$$

for all $x, y \in C$. We note that an extended hybrid mapping is not quasi-nonexpansive generally.

In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

2 Preliminaries

Let $H$ be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. From [27], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$  

(2.1)

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$  

(2.2)

From [18], a Hilbert space $H$ satisfies Opial’s condition, i.e., for a sequence $\{x_n\}$ of $H$ such that $x_n \to x$ and $x \neq y$,

$$\lim_{n \to \infty}\inf\|x_n - x\| < \lim_{n \to \infty}\inf\|x_n - y\|.$$  

(2.3)

Let $C$ be a nonempty closed convex subset of $H$ and let $T : C \to H$ be a mapping. We denote by $F(T)$ the set of fixed points of $T$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex; see Ito and Takahashi [10]. Since a generalized hybrid mapping $T$ defined in Introduction is quasi-nonexpansive, $F(T)$ is closed and convex.

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then, we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach
limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$. If $\mu$ is a Banach limit on $l^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,  
\[ \liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n. \]
In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$.
For the proof of existence of a Banach limit and its other elementary properties, see [24]. Using Banach limits, Kocourek, Takahashi and Yao [11] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 2.1** ([11]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T : C \to C$ be a generalized hybrid mapping. Then $T$ has a fixed point in $C$ if and only if \{T^n z\} is bounded for some $z \in C$.

Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive and  
\[ \langle x - P_C x, P_C x - u \rangle \geq 0 \]
for all $x \in H$ and $u \in C$; see [27] for more details. We also know the following lemma.

**Lemma 2.2** ([30]). Let $F$ be a nonempty closed convex subset of a Hilbert space $H$, let $P$ be the metric projection of $H$ onto $F$ and let \{x_n\} be a sequence in $H$ such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then \{Px_n\} converges strongly.

### 3 New Class of Extended Hybrid Mappings

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. A mapping $U : C \to H$ is called extended hybrid [8] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that  
\[ \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \]
\[ \leq (\beta + \alpha \gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha \gamma))\|x - y\|^2 \]
\[ - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \]
for all $x, y \in C$ and such a mapping $U$ is called $(\alpha, \beta, \gamma)$-extended hybrid. In [8], the authors derived a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space.

**Theorem 3.1** ([8]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha, \beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $T$ and $U$ be mappings of $C$ into $H$ such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta, \gamma)$-extended hybrid mapping. In this case, $F(T) = F(U)$.

A mapping $U : C \to H$ is called a widely strict pseudo-contraction if there exists a real number $k \in \mathbb{R}$ with $k < 1$ such that  
\[ \|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C. \]
Such a mapping $U$ is called a widely $k$-strict pseudo-contraction. A widely $k$-strict pseudo-contraction [5] is a strict pseudo-contraction if $0 \leq k < 1$. It is also nonexpansive if $k = 0$. Conversely, if $T : C \to H$ is a nonexpansive mapping, then for any $n \in \mathbb{N}$,

$$\begin{align*}
U &= \frac{1}{1+n}T + \frac{n}{1+n}I
\end{align*}$$

is a widely $(-n)$-strict pseudo-contraction. The following result is in [32]:

**Proposition 3.2 ([32]).** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha > 0$ and let $A, U$ and $T$ be mappings of $C$ into $H$ such that $U = I - A$ and $T = 2\alpha U + (1 - 2\alpha)I$. Then, the following are equivalent:

(a) $A$ is an $\alpha$-inverse-strongly monotone mapping, i.e.,

$$\alpha \|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle, \quad \forall x, y \in C;$$

(b) $U$ is a widely $(1 - 2\alpha)$-strict pseudo-contraction, i.e.,

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1 - 2\alpha)\| (I - U)x - (I - U)y \|^2, \quad \forall x, y \in C;$$

(c) $U$ is a $(1, 0, -k)$-extended hybrid mapping, i.e.,

$$2\alpha \|Ux - Uy\|^2 + (1 - 2\alpha)\|x - Uy\|^2$$

$$\leq (2\alpha - 1)\|x - y\|^2 + 2(1 - \alpha)\|x - y\|^2$$

$$- (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C;$$

(d) $T$ is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$, where $Z(A) = \{u \in C : Au = 0\}$.

Let $\alpha > 0$ and let $A : C \to H$ be $\alpha$-inverse-strongly monotone. Then for any $\beta \in \mathbb{R}$ with $0 < \beta \leq 2\alpha$, $A$ is $\frac{\beta}{2}$-inverse-strongly monotone. Thus

$$T = I - \beta A = I - \beta(I - U) = \beta U + (1 - \beta)I$$

is nonexpansive. Using Proposition 3.2, we can get the following result:

**Proposition 3.3.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $k < 1$ and let $A, U$ and $T$ be mappings of $C$ into $H$ such that $U = I - A$ and $T = (1 - k)U + kI$. Then, the following are equivalent:

(a) $A$ is a $\frac{1-k}{2}$-inverse-strongly monotone mapping;

(b) $U$ is a widely $k$-strict pseudo-contraction;

(c) $U$ is a $(1, 0, -k)$-extended hybrid mapping;

(d) $T$ is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$.

Let $k < 1$ and let $U$ be a widely $k$-strict pseudo-contraction. Then for any $t \in \mathbb{R}$ with $k \leq t < 1$, $U$ is a widely $t$-strict pseudo-contraction. Thus

$$T = (1 - t)U + tI$$

is nonexpansive. We also have the following important result [31] for extended hybrid mappings in a Hilbert space.
Theorem 3.4 ([32]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha, \beta, \gamma$ be real numbers and let $U : C \rightarrow H$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping with $1 + \gamma > 0$. Then, $I - U$ is demiclosed, i.e., $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$ imply $z \in F(U)$.

Using Theorem 3.5, we have the following result for $k$-strict pseudo-contractions obtained by Marino and Xu [15]; see also [1].

Corollary 3.5 (Marino and Xu [15]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 \leq k < 1$ and $U : C \rightarrow H$ be a $k$-strict pseudo-contraction. Then, $I - U$ is demiclosed, i.e., $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$ imply $z \in F(U)$.

4 Weak Convergence Theorems

Motivated by Propositions 3.2 and 3.3, we are interested in weak and strong convergence theorems for extended hybrid mappings in a Hilbert space. In this section, we first state the following weak convergence theorem of Baillon’s type [2] by using Lemma 2.2.

Theorem 4.1 ([8]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha, \beta$ and $\gamma$ be real numbers with $0 \leq -\gamma < 1$. Let $S : C \rightarrow C$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping with $F(S) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(S)$. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} ((1 + \gamma)S - \gamma I)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n \rightarrow \infty} P T^n x$ and $T = (1 + \gamma)S - \gamma I$.

The following weak convergence theorem was proved by Takahashi, Wong and Yao [31].

Theorem 4.2 ([31]). Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\alpha, \beta$ and $\gamma$ be real numbers. Let $U : C \rightarrow H$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping such that $1 + \gamma > 0$ and $F(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\lim \inf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \rightarrow \infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

As direct consequences of Theorem 4.2, we obtain the following results.

Corollary 4.3. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\gamma$ be a real number with $1 + \gamma > 0$ and let $U : C \rightarrow H$ be an $(2, 1, \gamma)$-extended hybrid mapping, i.e.,

$$2(1 + \gamma)\|Ux - Uy\|^2 - (1 + 2\gamma)\|x - Uy\|^2$$

$$\leq (1 + 2\gamma)\|Ux - y\|^2 - 2\gamma\|x - y\|^2$$

$$- \gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2$$
for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \to \infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

**Corollary 4.4.** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\gamma$ be a real number with $1 + \gamma > 0$ and let $U : C \to H$ be an $(\frac{3}{2}, \frac{1}{2}, \gamma)$-extended hybrid mapping, i.e.,

$$3(1 + \gamma)\|Ux - Uy\|^2 - (1 + 3\gamma)\|x - y\|^2 \leq (1 + 3\gamma)\|Ux - y\|^2 + (1 - 3\gamma)\|x - y\|^2 - 2\gamma\|x - Ux\|^2 - 2\gamma\|y - Uy\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \to \infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

Taking $\gamma = -\frac{1}{2}$ in Corollaries 4.3 and 4.4, we obtain two mappings such that

$$2\|Ux - Uy\|^2 \leq 2\|x - y\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2$$

and

$$3\|Ux - Uy\|^2 + \|x - Uy\|^2 + \|y - Ux\|^2 \leq 5\|x - y\|^2 + 2\|x - Ux\|^2 + 2\|y - Uy\|^2$$

for all $x, y \in C$, respectively. We can apply Corollaries 4.3 and 4.4 for such mappings and then obtain weak convergence theorems in a Hilbert space.

## 5 Strong Convergence Theorems

Using an idea of mean convergence, we can prove the following strong convergence theorem [31] of Halpern’s type for extended hybrid mappings in a Hilbert space.

**Theorem 5.1 ([31]).** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $\alpha, \beta$ and $k$ be real numbers. Let $U : C \to C$ be an $(\alpha, \beta, -k)$-extended hybrid mapping such that $0 \leq k < 1$ and $F(U) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(U)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$\begin{align*}
&x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\
z_n = \frac{1}{n} \sum_{m=1}^{n} ((1 - k)U + kI)^m x_n
\end{align*}$$

for all \( n = 1, 2, \ldots \), where \( 0 \leq \alpha_n \leq 1 \), \( \alpha_n \to 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then \( \{x_n\} \) converges strongly to \( Pu \).

Using the hybrid method by Nakajo and Takahashi [17], we can prove the following strong convergence theorem for extended hybrid non-self mappings in a Hilbert space. The method of the proof is due to Nakajo and Takahashi [17] and Marino and Xu [15].

**Theorem 5.2** ([31]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha, \beta \) and \( k \) be real numbers and let \( U : C \to H \) be an \((\alpha, \beta, -k)\)-extended hybrid mapping such that \( k < 1 \) and \( F(U) \neq \emptyset \). Let \( \{x_n\} \subset C \) be a sequence generated by \( x_1 = x \in C \) and

\[
\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)(1-k)Ux_n + kx_n, \\
C_n &= \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2 \alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N},
\end{align*}
\]

where \( P_{C_n \cap Q_n} \) is the metric projection of \( H \) onto \( C_n \cap Q_n \) and \( \{\alpha_n\} \subset (-\infty, 1) \). Then, \( \{x_n\} \) converges strongly to \( z_0 = R_{F(U)}x \), where \( P_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).

Using Theorem 5.2, we can prove the following theorem obtained by Marino and Xu [15].

**Theorem 5.3** (Marino and Xu [15]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( k \) be a real number with \( 0 \leq k < 1 \) and let \( U : C \to C \) be a \((-k)\)-strict pseudo contraction such that \( F(U) \neq \emptyset \). Let \( \{x_n\} \subset C \) be a sequence generated by \( x_1 = x \in C \) and

\[
\begin{align*}
y_n &= \beta_n x_n + (1 - \beta_n)Ux_n, \\
C_n &= \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N},
\end{align*}
\]

where \( P_{C_n \cap Q_n} \) is the metric projection of \( H \) onto \( C_n \cap Q_n \) and \( \{\beta_n\} \subset (-\infty, 1) \). Then, \( \{x_n\} \) converges strongly to \( z_0 = R_{F(U)}x \), where \( P_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).

**Proof.** We first know that a \((1,0,-k)\)-extended hybrid mapping with \( 0 \leq k < 1 \) is a \(-k\)-strict pseudo contraction. We also have that for any \( n \in \mathbb{N} \),

\[
y_n = \beta_n x_n + (1 - \beta_n)Ux_n \\
= \frac{\beta_n - k}{1-k}x_n + \frac{1 - \beta_n - k}{1-k}(1-k)Ux_n + kx_n.
\]

Putting \( \alpha_n = \frac{\beta_n - k}{1-k} \), we have from \( 1 > \beta_n \) that \( 1 - k > \beta_n - k \) and hence \( 1 > \frac{\beta_n - k}{1-k} = \alpha_n \). Furthermore, we have that for any \( n \in \mathbb{N} \) and \( z \in C \),

\[
\|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2
\]

\[
\Leftrightarrow \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)\alpha_n(1-k)(1-\alpha_n)\|x_n - Ux_n\|^2
\]

\[
\Leftrightarrow \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2.
\]

From Theorem 5.2, we have the desired result. \( \square \)

Next, we prove a strong convergence theorem by the shrinking projection method [29] for extended hybrid non-self mappings in a Hilbert space.
Theorem 5.4 ([31]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha$, $\beta$ and $k$ be real numbers and let $U : C \to H$ be an $(\alpha, \beta, -k)$-extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$
\begin{array}{l}
y_n = \alpha_n x_n + (1 - \alpha_n) \{(1-k)Ux_n + kx_n\}, \\
C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2 \alpha_n(1 - \alpha_n) \|Ux_n - x_n\|^2\}, \\
x_{n+1} = P_{C_{n+1}}x, \ \forall n \in \mathbb{N},
\end{array}
$$

where $P_{C_{n+1}}$ is the metric projection of $H$ onto $C_{n+1}$, and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}x$, where $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

References