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<th>WEAK AND STRONG CONVERGENCE THEOREMS FOR UNIFORMLY ASYMPTOTICALLY REGULAR NONEXPANSIVE SEMIGROUPS (Nonlinear Analysis and Convex Analysis)</th>
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1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $C$ be a nonempty closed convex subset of $H$. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. We know iteration procedures for finding a fixed point of a mapping $T$: Let $x$ be an element of $C$ and for each $t$ with $0 < t < 1$, let $x_t$ be a unique element of $C$ satisfying $x_t = tx + (1 - t)Tx_t$. In 1967, Browder [7] proved the following strong convergence theorem.

**Theorem 1.1.** Let $H$ be a Hilbert space, let $C$ be a nonempty bounded closed convex subset of $H$ and let $T$ be a nonexpansive mapping of $C$ into itself. Let $x$ be an element of $C$ and for each $t$ with $0 < t < 1$, let $x_t$ be a unique element of $C$ satisfying

$$x_t = tx + (1 - t)Tx_t.$$ 

Then, $\{x_t\}$ converges strongly to the element of $F(T)$ nearest to $x$ as $t \downarrow 0$.


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On the other hand, Xu and Ori [25] studied the following implicit iterative process for finite nonexpansive mappings $T_1, T_2, \ldots, T_r$ in a Hilbert space: $x_0 = x \in C$ and

$$x_n = \alpha_n + x_{n-1} + (1 - \alpha_n)T_nx_n$$

for every $n = 1, 2, \ldots$, where $\alpha_n$ is a sequence in $(0, 1)$ and $T_n = T_{n+r}$. And they proved the weak convergence of the iterative process defined by (1) in a Hilbert space. Motivated by [25], author and Takahashi [6] introduced an implicit iterative process for a nonexpansive semigroup and then prove a weak convergence theorem for the nonexpansive semigroup by using the idea of mean (see also [2, 3, 4]).

In this paper, we study the implicit iterations (1) for one-parameter nonexpansive semigroups and prove a weak convergence theorem for a uniformly asymptotically regular one-parameter nonexpansive semigroup in a Hilbert space. We also prove a weak convergence theorem for a uniformly asymptotically regular nonexpansive semigroup (see also [22, 23]). Further, we study Browder’s type iterations for nonexpansive semigroups. Then, we prove strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces by using the idea of [1, 7, 9, 22, 23]. And we give a strong convergence theorem for the nonexpansive semigroup by the viscosity approximation method.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $C$ be a nonempty closed convex subset of $H$. Then, for every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_Cx$, such that

$$\|x - P_Cx\| \leq \|x - y\|$$

for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. It is characterized by

$$\langle P_Cx - y, x - P_Cx \rangle \geq 0$$

for all $y \in C$. See [23] for more details. The following result is well-known; see also [23].

Lemma 2.1. Let $C$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself. Then, $F(T) \neq \emptyset$.

We write $x_n \to x$ (or $\lim_{n \to \infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in $H$ converges strongly to $x$. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n \to \infty} x_n = x$) to indicate that
the sequence \( \{x_n\} \) of vectors in \( H \) converges weakly to \( x \). In a Hilbert space, it is well known that \( x_n \to x \) and \( \|x_n\| \to \|x\| \) imply \( x_n \to x \).

Let \( S \) be a semitopological semigroup. A semitopological semigroup \( S \) is called right (resp. left) reversible if any two closed left (resp. right) ideals of \( S \) have nonvoid intersection. If \( S \) is right reversible, \( (S, \leq) \) is a directed system when the binary relation "\( \leq \)" on \( S \) is defined by \( s \leq t \) if and only if \( \{s\} \cup \overline{S_s} \supset \{t\} \cup \overline{S+t} \), \( s, t \in S \), where \( \overline{A} \) is the closure of \( A \). A commutative semigroup \( S \) is a directed system when the binary relation is defined by \( s \leq t \) if and only if \( \{s\} \cup (S + s) \supset \{t\} \cup (S + t) \).

Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). A family \( S = \{T(t) : t \in \mathbb{R}^+\} \) of mappings of \( C \) into itself is said to be a nonexpansive semigroup on \( C \) if it satisfies the following conditions:

(i) For each \( t \in S \), \( T(t) \) is nonexpansive;
(ii) \( T(ts) = T(t)T(s) \) for each \( t, s \in S \).

We denote by \( F(S) \) the set of common fixed points of \( S \), i.e., \( F(S) = \bigcap_{t \in S} F(T(t)) \).

We say that a Banach space \( E \) satisfies Opial’s condition [12] if for each sequence \( \{x_n\} \) in \( E \) which converges weakly to \( x \),

\[
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\| \tag{2}
\]

for each \( y \in E \) with \( y \neq x \). In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [10]. It is well known that this condition is equivalent to the analogous condition of \( \lim \) (see [5]). It is well known that Hilbert spaces satisfy Opial’s condition (see [12, 23]).

**Proposition 2.2** ([12]). Let \( H \) be a Hilbert space. Let \( \{x_n\} \) be a sequence in \( H \) converging weakly to \( x \in H \). Then,

\[
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\| \tag{3}
\]

for each \( y \in E \) with \( y \neq x \).

3. **Convergence Theorems for One-Parameter Nonexpansive Semigroups**

In this section, we prove a weak convergence theorem for an asymptotically regular one-parameter nonexpansive semigroup by using the idea of [1, 9, 22, 23, 25]. Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). A family \( S = \{T(t) : t \in \mathbb{R}^+\} \) of mappings of \( C \) into itself satisfying the following conditions is said to be one-parameter nonexpansive semigroup on \( C \):

(i) for each \( t \in \mathbb{R}^+ \), \( T(t) \) is nonexpansive;
(ii) \( T(0) = I \);
(iii) $T(t + s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

We say that one-parameter nonexpansive semigroup $S = \{T(t) : t \in \mathbb{R}^+\}$ is asymptotically regular if

$$\lim_{s \to \infty} \|T(h + s)x - T(s)x\| = 0$$

for all $h \in \mathbb{R}^+$ and $x \in C$ (see also [22, 23]). The following lemma proved by Acedo and Suzuki ([1]).

**Lemma 3.1** ([1]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S = \{T(s) : s \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on $C$. Assume that $S = \{T(s) : s \in \mathbb{R}^+\}$ is asymptotically regular, that is,

$$\lim_{t \to \infty} \|T(h + t)x - T(t)x\| = 0$$

for all $h \in \mathbb{R}^+$ and $x \in C$. Then,

$$F(T(h)) = F(S)$$

for each $h \in \mathbb{R}^+$.

We say that one-parameter nonexpansive semigroup $S = \{T(t) : t \in \mathbb{R}^+\}$ is uniformly asymptotically regular if for every $h \in \mathbb{R}^+$ and for every bounded subset $K$ of $C$,

$$\lim \sup_{s \in \mathbb{R}^+, x \in K} \|T(h + s)x - T(s)x\| = 0.$$

holds.

We prove a weak convergence theorem for a uniformly asymptotically regular one-parameter nonexpansive semigroup (see [1, 9]).

**Theorem 3.2.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S = \{T(s) : s \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parametr nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)(T(m_n))x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common fixed point of $S$. 
4. **Weak convergence theorems for nonexpansive semigroups**

In this section, we prove a weak convergence theorem for an asymptotically regular nonexpansive semigroup by using the idea of [1, 9, 22, 23, 25]. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $S$ be a commutative semigroup and let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$. We say that nonexpansive semigroup $S = \{T(t) : t \in S\}$ is asymptotically regular if

$$\lim_{s \in S} \|T(h)T(s)x - T(s)x\| = 0$$

for all $h \in S$ and $x \in C$ (see also [22, 23]). The following lemma plays an important role in the proof of main theorem (see [1]).

**Lemma 4.1.** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $S$ be a commutative semigroup. Let $S = \{T(t) : t \in S\}$ be a nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Assume that $S = \{T(t) : t \in S\}$ is asymptotically regular, that is,

$$\lim_{t \in S} \|T(h)T(t)x - T(t)x\| = 0$$

for all $h \in S$ and $x \in C$. Then,

$$F(T(h)) = F(S)$$

for each $h \in S$.

We say that nonexpansive semigroup $S = \{T(t) : t \in S\}$ is uniformly asymptotically regular if for every $h \in S$ and for every bounded subset $K$ of $C$,

$$\lim_{s \in S} \sup_{x \in K} \|T(h)T(s)x - T(s)x\| = 0$$

holds.

We prove a weak convergence theorem for a uniformly asymptotically regular nonexpansive semigroup (see also [1, 9]).

**Theorem 4.2.** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $S$ be a commutative semigroup. Let $S = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)(T(t))^{m_n}x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common fixed point of $S$. 


5. STRONG CONVERGENCE THEOREMS

Motivated by [1, 7, 9], we study Browder's type strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups (see also [22, 23]).

**Theorem 5.1.** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $S$ be a commutative semigroup. Let $S = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n}x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $Pu$, where $P$ is the metric projection from $C$ onto $F(S)$.

We know that $f : C \to C$ is said to be a contraction on $C$ if there exists $r \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq r\|x - y\|$$

for each $x, y \in C$. Using [21] and Theorem 5.1, we obtain the following strong convergence theorem by the viscosity approximation methods (see also [11]).

**Theorem 5.2.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $S$ be a commutative semigroup and let $S = \{T(t) : T \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Let $f$ be a contraction on $C$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)(T(t))^{m_n}x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $Pu$, where $P$ is the metric projection from $C$ onto $F(S)$.

**REFERENCES**


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