数理解析研究所講究録
第 1821 巻 2013 年 107-114

A GENERAL ITERATIVE METHOD UNDER SOME CONTROL CONDITIONS FOR k-STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, we introduce a general iterative method for a k-strictly pseudo-contractive mapping related to an operator F which is κ-Lipschitzian and η-strongly monotone and then prove that under certain different control conditions, the sequence generated by the proposed iterative method converges strongly to a fixed point of the mapping, which solves a variational inequality related to the operator F. Additional results of main results are also obtained. Our results substantially improve and develop the corresponding ones announced by many authors recently.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C. We denote by $F(S)$ the set of fixed points of S.

We recall that a mapping $T : C \to H$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in [0,1)$such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad \forall x,\ y \in C.$$

The mapping T is pseudo-contractive if and only if

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2, \quad \forall x,\ y \in C.$$

T is strongly pseudo-contractive if and only if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx-Ty, x-y\rangle \leq (1-\lambda)\|x-y\|^2, \quad \forall x,\ y \in C.$$

Note that the class of k-strictly pseudo-contractive mappings includes the class of nonexpansive mappings T on C (that is, $\|Tx-Ty\| \leq \|x-y\|$, $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0,1)$ such that $T-\lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [3, 4, 5]). The class of pseudo-contractive mappings is one of the most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudo-contractions, see, for example, [1, 6, 8, 11] and the references therein.

In 2010, Jung [8] introduced the following composite iterative scheme for a k-strictly pseudo-contractive mapping T: $x_0 = x \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)P_C S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$; $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$; $f : C \to C$ is a contractive mapping with constant $\alpha \in (0,1)$ (i.e., there exists a constant $\alpha \in (0,1)$ such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$, $\forall x,\ y \in C$); $A : H \to H$ is a strongly positive bounded linear operator

2000 *Mathematics Subject Classification.* 47H09, 47H10, 47J20, 47J25, 49M05.

Key words and phrases. k-strictly pseudo-contractive mapping; Nonexpansive mapping; Fixed points; Contraction; κ-Lipschitzian and η-strongly monotone operator; Hilbert space; Variational inequality.

The results presented in this lecture are collected mainly from the work [20] by the author of this report.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0003901).

(i.e., there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad x \in H$); and and P_C is the metric projection of H onto C. Under suitable control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, he showed that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality related to the linear operator A:

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0, \quad \forall p \in F(T).$$

By removing the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, the result improves the corresponding results of Cho et al. [6] as well as Marino and Xu [10].

On the other hand, in 2010, by combining Yamada's method [18] with the Marino and Xu's method [10], Tian [14] considered the following general iterative method for a nonexpansive mapping S: $x_0 = x \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)Sx_n, \quad \forall n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset (0, 1)$; $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$ (i.e., there exist positive constants κ and $\eta > 0$ such that $\|Fx - Fy\| \leq \kappa\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \ \forall x, y \in H$); $f : H \to H$ be a contraction with the contractive constant $\alpha \in (0, 1)$; $0 < \mu < \frac{2\eta}{\kappa^2}$; and $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$. By using well-known control conditions on $\{\alpha_n\}$, he proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point $\widetilde{x}$ of S, which is the unique solution of the following variational inequality related to the operator F:

$$\langle \mu F\widetilde{x} - \gamma f(\widetilde{x}), \widetilde{x} - z\rangle \leq 0, \quad \forall z \in F(S). \tag{1.3}$$

In this paper, motivated by the above-mentioned results, we introduce a new general iterative scheme for finding an element of $F(T)$, where $T : C \to H$ is a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$. Under different control conditions, we establish the strong convergence of the sequences generated by the proposed scheme to a point in $F(T)$, which is a solution of a certain variational inequality related to the operator F. The main results improve, develop and complement the corresponding results of Tian [14] as well as Cho et al. [6], Jung [8] and Marino and Xu [10]. Our results also improve the corresponding results of Halpern [7], Moudafi [12], Wittmann [15] and Xu [17].

2. Preliminaries and Lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive.

In a Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([19]). *Let H be a Hilbert space, C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 ([19]). *Let H be a Hilbert space and C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.3 ([19]). *Let H be a Hilbert space, C be a closed convex subset of H, and $T : C \to H$ be a k-strictly pseudo-contractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

The following Lemma 2.4 and 2.5 can be obtained from the Proposition 2.6 of Acedo and Xu [1].

Lemma 2.4. *Let H be a Hilbert space and C be a closed convex subset of H. For any $N \geq 1$, assume that for each $1 \leq i \leq N$, $T_i : C \to H$ is a k_i-strictly pseudo-contractive mapping for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a nonself-k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 2.5. *Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 2.4. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.6 ([9, 16]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n \delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^\infty \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^\infty \lambda_n \delta_n < \infty$,

(iii) $r_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^\infty r_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 ([13]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.8. *In a Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.9. *Let C be a nonempty closed convex subset of a Hilbert space H such that $C \pm C \subset C$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < t < \rho < 1$. Then $S := \rho I - t\mu F : C \to C$ is a contraction with contractive constant $\rho - t\tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2) < 1$ with $t < \frac{1}{\tau}$.*

Proof. From (1.3), (1.4) and (2.1), we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|\rho(x-y) - t\mu(Fx - Fy)\|^2 \\
&= \rho^2\|x-y\|^2 + t^2\mu^2\|Fx - Fy\|^2 - 2t\rho\mu\langle Fx - Fy, x-y\rangle \\
&\leq \rho^2\|x-y\|^2 + t^2\mu^2\kappa^2\|x-y\| - 2t\rho\mu\eta\|x-y\|^2 \\
&< \rho^2\|x-y\|^2 + t\rho\mu^2\kappa^2\|x-y\| - 2t\rho\mu\eta\|x-y\|^2 \\
&= (\rho^2 - t\rho\mu(2\eta - \mu\kappa^2))\|x-y\|^2 \\
&< (\rho - t\tau)^2\|x-y\|^2,
\end{aligned}$$

where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2)$, and so

$$\|Sx - Sy\| < (\rho - t\tau)\|x-y\|.$$

Hence S is a contraction with contractive constant $\rho - t\tau$. $\square$

3. Main results

We need the following result for the existence of solutions of a certain variational inequality, which is slightly an improvement of Theorem 3.1 of Tian [14].

Theorem T. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let x_t be a fixed point of a contraction $S_t \ni x \mapsto t\gamma f(x) + (I - t\mu F)Tx$ for $t \in (0,1)$ and $t < \frac{1}{\tau}$. Then $\{x_t\}$ converges strongly to a fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following variational inequality:*

$$\langle \mu F\widetilde{x} - \gamma f(\widetilde{x}), \widetilde{x} - p\rangle \leq 0, \quad \forall p \in F(T).$$

Equivalently, we have $P_{F(T)}(I - \mu F + \gamma f)\widetilde{x} = \widetilde{x}$.

Now, we study the strong convergence result for a new general iterative scheme.

Theorem 3.1. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (B) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n, \quad \forall n \geq 0, \tag{IS}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C. Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the following variational inequality :

$$\langle \mu Fq - \gamma f(q), q - p\rangle \leq 0, \quad \forall p \in F(T).$$

Proof. First, from the condition (C1), without loss of generality, we assume that $\alpha_n \tau < 1$, $\frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n \alpha\gamma} < 1$ and $\alpha_n < (1 - \beta_n)$ for $n \geq 0$.

We divides the proof several steps:

Step 1. We show that $\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\right\}$ for all $n \geq 0$ and all $p \in F(T) = F(S)$. Indeed, let $p \in F(T)$. Then from Lemma 2.9, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - \mu Fp) + \beta_n(x_n - p) \\
&\quad + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n - ((1 - \beta_n)I - \alpha_n \mu F)P_C Sp\| \\
&\leq (1 - \beta_n - \alpha_n \tau)\|x_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|\gamma f(x_n) - \mu Fp\| \\
&\leq (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - \mu Fp\|) \\
&\leq (1 - (\tau - \gamma\alpha)\alpha_n)\|x_n - p\| + (\tau - \gamma\alpha)\alpha_n \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\right\}.
\end{aligned}$$

Using an induction, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\}$. Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{P_C S x_n\}$ and $\{F P_C S x_n\}$.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To this show, define

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad \text{for all } n \geq 0.$$

Observe that from the definition of z_n,

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}\mu F)P_C S x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n \mu F)P_C S x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}\gamma f(x_n) \\ &\quad + P_C S x_{n+1} - P_C S x_n + \frac{\alpha_n}{1-\beta_n}\mu F P_C S x_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\mu F P_C S x_{n+1} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - \mu F P_C S x_{n+1}) \\ &\quad + \frac{\alpha_n}{1-\beta_n}(\mu F P_C S x_n - \gamma f(x_n)) + P_C S x_{n+1} - P_C S x_n. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(x_{n+1})\| + \mu\|F P_C S x_{n+1}\|) \\ + \frac{\alpha_n}{1-\beta_n}(\mu\|F P_C S x_n\| + \gamma\|f(x_n)\|). \end{aligned}$$

From the condition (C1) and (B), it follows that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 2.7, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - P_C S x_n\| = 0$. Indeed, since

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)P_C S x_n,$$

we have

$$\begin{aligned} \|x_n - P_C S x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - P_C S x_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \mu F P_C S x_n\| + \beta_n\|x_n - P_C S x_n\|, \end{aligned}$$

that is,

$$\|x_n - P_C S x_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - \mu F P_C S x_n\|.$$

So, from the conditions (C1) and (B) and Step 2, it follows that

$$\lim_{n\to\infty}\|x_n - P_C S x_n\| = 0.$$

Step 4. We show that

$$\limsup_{n\to\infty}\langle \gamma f(q) - \mu F q, x_n - q\rangle \le 0,$$

where $q = \lim_{t\to 0} x_t$ being $x_t = t\gamma f(x_t) + (I - t\mu F)P_C S x_t$ for $0 < t < 1$ and $t < \frac{1}{\tau}$. We note that from Lemmas 2.2 and 2.3 and Theorem T2, $q \in F(T) = F(S)$ and q is a solution of a variational inequality

$$\langle \mu F q - \gamma f(q), q - p\rangle \le 0, \quad p \in F(T). \tag{3.1}$$

To show this, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty}\langle \gamma f(q) - \mu F q, x_{n_j} - q\rangle = \limsup_{n\to\infty}\langle \gamma f(q) - \mu F q, x_n - q\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$. Since $\|x_n - P_C S x_n\| \to 0$ by Step 3, we obtain $w = P_C S w$. In fact, if $w \neq P_C S w$, then, by Opial condition,

$$\begin{aligned}\liminf_{j\to\infty} \|x_{n_j} - w\| &< \liminf_{j\to\infty} \|x_{n_j} - P_C S w\| \\ &\le \liminf_{j\to\infty} (\|x_{n_j} - P_C S x_{n_j}\| + \|P_C S x_{n_j} - P_C S w\|) \\ &\le \liminf_{j\to\infty} \|x_{n_j} - w\|,\end{aligned}$$

which is a contradiction. Hence $w = P_C S w$. Since $F(P_C S) = F(S)$, from Lemma 2.3, we have $w \in F(T)$. Therefore, from (3.1), we conclude that

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - \mu F q, x_n - q\rangle &= \lim_{j\to\infty} \langle \gamma f(q) - \mu F q, x_{n_j} - q\rangle \\ &= \langle \gamma f(q) - \mu F q, w - q\rangle \le 0.\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$, where $q = \lim_{t\to 0} x_t$ being $x_t = t\gamma f(x_t) + (I - t\mu F) P_C S x_t$ for $0 < t < 1$ and $t < \frac{1}{\tau}$, and q is a solution of a variational inequality

$$\langle \mu F q - \gamma f(q), q - p\rangle \le 0, \quad p \in F(T).$$

Indeed, from (IS), we have

$$\begin{aligned}x_{n+1} - q = &\ \alpha_n(\gamma f(x_n) - \mu F q) + \beta_n(x_n - q) \\ &+ ((1-\beta_n)I - \alpha_n\mu F) P_C S x_n - ((1-\beta_n)I - \alpha_n \mu F) q.\end{aligned}$$

Applying Lemma 2.8 and Lemma 2.9, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 \le &\ \|\beta_n(x_n - q) + ((1-\beta_n)I - \alpha_n\mu F)P_C S x_n - ((1-\beta_n)I - \alpha_n\mu F)P_C S q\|^2 \\ &+ 2\alpha_n\langle \gamma f(x_n) - \mu F q, x_{n+1} - q\rangle \\ \le &\ ((1-\beta_n - \alpha_n\tau)\|x_n - q\| + \beta_n\|x_n - q\|)^2 \\ &+ 2\alpha_n\gamma\langle f(x_n) - f(q), x_{n+1} - q\rangle + 2\alpha_n\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle \\ \le &\ (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma\alpha\|x_n - q\|\|x_{n+1} - q\| \\ &+ 2\alpha_n\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle \\ \le &\ (1-\tau\alpha_n)^2\|x_n - q\|^2 + \alpha_n\gamma\alpha(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &+ 2\alpha_n\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+1} - q\|^2 \le &\ \frac{1 - 2\tau\alpha_n + \tau^2\alpha_n^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - q\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle \\ = &\ \left(1 - \frac{2(\tau - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right)\|x_n - q\|^2 + \frac{\tau^2\alpha_n^2}{1 - \alpha_n\gamma\alpha}\|x_n - q\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle \\ \le &\ \left(1 - \frac{2(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\alpha_n\right)\|x_n - q\|^2 \\ &+ \frac{2(\tau - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\left(\frac{\tau^2\alpha_n}{2(\tau - \gamma\alpha)}M + \frac{1}{\tau - \gamma\alpha}\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle\right) \\ = &\ (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n\delta_n,\end{aligned}$$

where $M = \sup\{\|x_n - q\|^2 : n \ge 0\}$, $\lambda_n = \frac{2(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\alpha_n$ and

$$\delta_n = \frac{\tau^2\alpha_n}{2(\tau - \gamma\alpha)}M + \frac{1}{\tau - \gamma\alpha}\langle \gamma f(q) - \mu F q, x_{n+1} - q\rangle.$$

From the conditions (C1) and (C2) and Step 4, it is easy to see that $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Hence, by Lemma 2.7, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 3.1. (1) Theorem 3.1 extends and develops Theorem 3.2 of Tian [14] from a nonexpansive mapping to a strictly pseudo-contractive mapping together with removing the condition (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(2) Theorem 3.1 also generalizes Theorem 2.1 of Jung [8] as well as Theorem 2.1 of Cho et al. [6] and Theorem 3.4 of Marino and Xu [10] from a strongly positive bounded linear operator A to a κ-Lipschitzian and η-strongly monotone operator F (In fact, from the definitions, it follows that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and $\overline{\gamma}$-strongly monotone operator).

(3) Theorem 3.1 also improves the corresponding results of Halpern [7], Moudafi [12], Wittmann [15] and Xu [17] as some special cases.

Theorem 3.2. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping for some $0 \le k_i < 1$ and $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (B) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F) P_C S x_n, \quad \forall n \ge 0,$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ with $k = \max\{k_i : 1 \le i \le N\}$ and $\{\eta_i\}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$ and P_C is the metric projection of H onto C. Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the following variational inequality :

$$\langle \mu F q - \gamma f(q), q - p \rangle \le 0, \quad \forall p \in \bigcap_{i=1}^{N} F(T_i).$$

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{i=1}^{N} \eta_i T_i x$. By Lemmas 2.4 and 2.5, we conclude that $T : C \to H$ is a k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \le i \le N\}$ and $F(T) = F(\sum_{i=1}^{N} \eta_i T_i) = \bigcap_{i=1}^{N} F(T_i)$. Then the result follows from Theorem 3.1 immediately. □

As a direct consequence of Theorem 3.2, we have the following result for nonexpansive mappings (that is, 0-strictly pseudo-contractive mappings).

Theorem 3.3. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, $\{T_i\}_{i=1}^{N} : C \to H$ be a finite family of nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0,1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (B) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F) P_C \sum_{i=1}^{N} \eta_i T_i x_n, \quad \forall n \ge 0,$$

where $\{\eta_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$ and P_C is the metric projection of H onto C. Then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_i\}_{i=1}^{N}$, which solves the *following variational inequality :*

$$\langle \mu Fq - \gamma f(q), q - p \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{N} F(T_i).$$

Remark 3.2. (1) Theorem 3.2 and Theorem 3.3 also generalize Theorem 2.2 and Theorem 2.4 of Jung [8] from a strongly positive bounded linear operator A to a κ-Lipschitzian and η-strongly monotone operator F.

(2) Theorem 3.2 and Theorem 3.3 also improve and complement the corresponding results of Cho et al. [6] by removing the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ together with using a κ-Lipschitzian and η-strongly monotone operator F.

(3) As in [2], we also can establish the result for a countable family $\{T_i\}$ of k_i-strict pseudo-contractive mappings with $0 \leq k_i < 1$.

References

[1] G. L. Acedo and H. K. Xu, Iterative methods for strictly pseudo-contractions in Hilbert space, *Nonlinear Anal.* **67** (2007) 2258–2271.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.* **67** (2007) 2350–2360.

[3] F. E. Browder, Fixed point theorems for noncompact mappings, *Proc. Natl. Acad. Sci. USA* **53** (1965) 1272–1276.

[4] F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Ration. Mech. Anal.* **24** (1967) 82–90.

[5] F. E. Browder and W. V. Petryshn, Construction of fixed points of nonlinear mappings Hilbert space, *J. Math. Anal. Appl.* **20** (1967) 197–228.

[6] Y. J. Cho, S. M. Kang and X. Qin, Some results on k-strictly pseudo-contractive mappings in Hilbert spaces, *Nonlinear Anal.* **70** (2009) 1956–1964.

[7] B. Halpern, Fixed points of nonexpansive maps, *Bull. Amer. Math. Soc.* **73** (1967) 957–961.

[8] J. S. Jung, Strong convergence of iterative methods for k-strictly pseudo-contractive mappings in Hilbert spaces, *Applied Math. Comput.* **215** (2010) 3746-3753.

[9] L. S. Liu, Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* **194** (1995) 114-125.

[10] G. Marino and H. X. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **318** (2006) 43–52.

[11] C. H. Morales and J. S. Jung, Convergence of paths for pseudo-contractive mappings in Banach spaces, *Proc. Amer. math. Soc.* **128** (2000) 3411–3419.

[12] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* **241** (2000) 46–55.

[13] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integral, *J. Math. Anal. Appl.* **305** (2005), 227–239.

[14] M. Tian, A general itewrative algorithm for nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.* **73** (2010) 689–694.

[15] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **58** (1992) 486–491.

[16] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* **66** (2002) 240–256.

[17] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* **298** (2004) 279–291.

[18] I. Yamada, The hybrid steepest descent for the variational inequality problems over the intersection of fixed points sets of nonexpansive mappings, in: D. Butnariu, Y. Censor, S. Reich (Eds), Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, Elservier, New York, 2001, pp. 473–504.

[19] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.* 69 (2008) 456–462.

[20] J. S. Jung, Some results on a general iterative method for k-strictly pseudo-contractive mappings, Fixed Point Theory Appl. 2011 (2011) 24 doi:10.1186/1687-1812-2011-24.

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