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Demiclosedness Principles for Total Asymptotically Nonexpansive mappings

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Abstract

In this paper, we first are looking over the demiclosedness principles for non-linear mappings. Next, we give the demiclosedness principle of a continuous non-Lipschitzian mapping which is called totally asymptotically nonexpansive by Alber et al. [Fixed Point Theory and Appl., 2006 (2006), article ID 10673, 20 pages]. This paper is a just survey for demiclosedness principles for nonlinear mappings.

Keywords: totally asymptotically nonexpansive mappings, demiclosedness principle

2000 Mathematics Subject Classification. Primary 47H09; Secondary 65J15.

1 Introduction

Let $X$ be a real Banach space with norm $\| \cdot \|$ and let $X^*$ be the dual of $X$. Denote by $\langle \cdot, \cdot \rangle$ the duality product. Let $\{x_n\}$ be a sequence in $X$, $x \in X$. We denote by $x_n \rightharpoonup x$ the strong convergence of $\{x_n\}$ to $x$ and by $x_n \rightharpoonup x$ the weak convergence of $\{x_n\}$ to $x$. Also, we denote by $\omega_w(x_n)$ the weak $\omega$-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : \exists x_{n_k} \rightharpoonup x\}.$$

Let $C$ be a nonempty closed convex subset of $X$ and let $T : C \to C$ be a mapping. Now let $Fix(T)$ be the fixed point set of $T$; namely,

$$Fix(T) := \{x \in C : Tx = x\}.$$

Recall that $T$ is a Lipschitzian mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^nx - T^ny\| \leq k_n\|x - y\|$$ (1.1)
for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping $T$ is called uniformly $k$-Lipschitz if $k_n = k$ for all $n \geq 1$, nonexpansive if $k_n = 1$ for all $n \geq 1$, and asymptotically nonexpansive if $\lim_{n \to \infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [15] as a generalization of the class of nonexpansive mappings. They proved that if $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that $T$ is a mapping of asymptotically nonexpansive type [18] if for each $x \in C$, 

$$\limsup_{n \to \infty} \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

and $T^N$ is continuous for some $N \geq 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [5]. They say that $T$ is asymptotically nonexpansive in the intermediate sense if $T$ is (uniformly) continuous and 

$$\limsup_{n \to \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

In this case, observe that if we define

$$\delta_n := \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

(here $a \vee b := \max\{a, b\}$), then $\delta_n \geq 0$ for all $n \geq 1$, $\delta_n \to 0$ as $n \to \infty$, and thus (1.3) immediately reduces to 

$$\|T^n x - T^n y\| \leq \|x - y\| + \delta_n$$

for all $x, y \in C$ and $n \geq 1$.

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [9]. They say that a mapping $T : C \to C$ is said to be total asymptotically nonexpansive (TAN, in brief) [1] (or [9]) if there exists two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with $c_n, d_n \to 0$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that 

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n,$$

for all $x, y \in K$ and $n \geq 1$, where $\mathbb{R}^+ := [0, \infty)$ and

$$\phi \in \Gamma(\mathbb{R}^+) \Leftrightarrow \phi \text{ is strictly increasing, continuous on } \mathbb{R}^+ \text{ and } \phi(0) = 0.$$

**Remark 1.1.** If $\varphi(t) = t$, then (1.6) reduces to 

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \|x - y\| + d_n$$

for all $x, y \in C$ and $n \geq 1$. In addition, if $d_n = 0, k_n = 1 + c_n$ for all $n \geq 1$, then the class of total asymptotically nonexpansive mappings coincides with the class of asymptotically nonexpansive mappings. If $c_n = 0$ and $d_n = 0$ for all $n \geq 1$, then (1.6) reduces to the class of nonexpansive mappings. Also, if we take $c_n = 0$ and $d_n = \delta_n$ as in (1.4), then (1.6) reduces to (1.5); see [9] for more details.
Let $C$ be a nonempty closed convex subset of a real Banach space $X$, and let $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Recall that the following Mann [21] iterative method is extensively used for solving a fixed point equation of the form $Tx = x$:

$$x_{n+1} = (1 - \alpha_{n})x_{n} + \alpha_{n}Tx_{n}, \quad n \geq 0,$$

(1.7)

where $\{\alpha_{n}\}$ is a sequence in $[0,1]$ and $x_{0} \in C$ is arbitrarily chosen. In infinite-dimensional spaces, Mann’s algorithm has generally only weak convergence. In fact, it is known [29] that if the sequence $\{\alpha_{n}\}$ is such that $\sum_{n=1}^{\infty} \alpha_{n}(1 - \alpha_{n}) = \infty$, then Mann’s algorithm (1.7) converges weakly to a fixed point of $T$ provided the underlying space is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial’s property. Furthermore, Mann’s algorithm (1.7) also converges weakly to a fixed point of $T$ if $X$ is a uniformly convex Banach space such that its dual $X^{*}$ enjoys the Kadec-Klee property (KK-property, in brief), i.e., $x_{n} \rightharpoonup x$ and $\|x_{n}\| \rightharpoonup \|x\| \Rightarrow x_{n} \rightarrow x$. It is well known [12] that the duals of reflexive Banach spaces with a Fréchet differentiable norms have the KK-property. There exists uniformly convex spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the KK-property; see Example 3.1 of [14].

In this paper, we first are looking over the demiclosedness principles for nonlinear mappings. Next, we give the demiclosedness principle of continuous TAN mappings.

## 2 Geometrical properties of $X$

Let $X$ be a real Banach space with norm $\|\cdot\|$ and let $X^{*}$ be the dual of $X$. Denote by $\langle \cdot, \cdot \rangle$ the duality product. When $\{x_{n}\}$ is a sequence in $X$, we denote the strong convergence of $\{x_{n}\}$ to $x \in X$ by $x_{n} \rightarrow x$ and the weak convergence by $x_{n} \rightharpoonup x$. We also denote the weak $\omega$-limit set of $\{x_{n}\}$ by $\omega_{w}(x_{n}) = \{x : \exists x_{n_{j}} \rightharpoonup x\}$. The normalized duality mapping $J$ from $X$ to $X^{*}$ is defined by

$$J(x) = \{x^{*} \in X^{*} : \langle x, x^{*} \rangle = \|x\|^{2} = \|x^{*}\|^{2}\}$$

for $x \in X$.

Now we summarize some well known properties of the duality mapping $J$ for our further argument.

**Proposition 2.1.** [10, 30, 34]

1. for each $x \in X$, $Jx$ is nonempty, bounded, closed and convex (hence weakly compact).

2. $J(0) = 0$.

3. $J(\lambda x) = \lambda J(x)$ for all $x \in X$ and real $\lambda$.

4. $J$ is monotone, that is, $\langle x - y, j(x) - j(y) \rangle \geq 0$ for all $x, y \in X$, $j(x) \in J(x)$ and $j(y) \in J(y)$. 


(5) \( \|x\|^2 - \|y\|^2 \geq 2(x - y, j(y)) \) for all \( x, y \in X \) and \( j(x) \in J(y) \); equivalently,

\[
\|x + y\|^2 \leq \|x\|^2 + 2(y, j(x + y))
\]

for all \( x, y \in X \) and \( j(x + y) \in J(x + y) \).

**Remark 2.2.** Note that (5) in Proposition 2.1 can be quickly computed by the well known Cauchy-Schwartz inequality:

\[
2(x, j) \leq 2\|x\|\|y\| \leq \|x\|^2 + \|y\|^2.
\]

Recall that a Banach space \( X \) is said to be **strictly convex** (SC) [7] if any non-identically zero continuous linear functional takes maximum value on the closed unit ball at most at one point. It is also said to be **uniformly convex** if \( \|x_n - y_n\| \to 0 \) for any two sequences \( \{x_n\}, \{y_n\} \) in \( X \) such that \( \|x_n\| = \|y_n\| = 1 \) and \( \|(x_n + y_n)/2\| \to 1 \).

We introduce some equivalent properties of strict convexity of \( X \); see Proposition 2.13 in [7] for the detailed proof.

**Proposition 2.3.** ([7]) A linear normed space \( X \) is strictly convex if and only if one of the following equivalent properties holds:

(a) if \( \|x + y\| = \|x\| + \|y\| \) and \( x \neq 0 \), \( y = tx \) for some \( t \geq 0 \);

(b) if \( \|x\| = \|y\| = 1 \) and \( x \neq y \), then \( \|\lambda x + (1 - \lambda)y\| < 1 \) for all \( \lambda \in (0, 1) \), namely, the unit sphere (or any sphere) contains no line segment;

(c) if \( \|x\| = \|y\| = 1 \) and \( x \neq y \), then \( \|(x + y)/2\| < 1 \);

(d) the function \( x \to \|x\|^2, x \in X \), is strictly convex.

**Remark 2.4.** From (b), note that any three points \( x, y, z \) satisfying \( \|x - z\| + \|y - z\| = \|x - y\| \) must lie on a line; specially, if \( \|x - z\| = r_1 \), \( \|y - z\| = r_2 \), and \( \|x - y\| = r = r_1 + r_2 \), then \( z = \frac{r_1}{r} x + \frac{r_2}{r} y \); see [15] for more details. Indeed, taking \( u := \frac{z - x}{r_1}, v := \frac{z - y}{r_2} \), we see \( \|u\| = \|v\| = 1 \) and

\[
\|\lambda u + (1 - \lambda)v\| = \|(x - y)/r\| = 1
\]

for some \( \lambda = \frac{r_1}{r} \in (0, 1) \). Therefore, by (b), it must be \( u = v \iff z = \frac{r_1}{r} x + \frac{r_2}{r} y \).

Let \( S(X) := \{x \in X : \|x\| = 1\} \) be the unit sphere of \( X \). Then the Banach space \( X \) is said to be **smooth** provided

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = (2.1)
\]

exists for each \( x, y \in S(X) \). In this case, the norm of \( X \) is said to be **Gâteaux differentiable**. The space \( X \) is said to be a **uniformly Gâteaux differentiable norm** if for each \( y \in S(X) \), the limit (2.1) is attained uniformly for \( x \in S(X) \). The norm of \( X \) is said to be **Fréchet differentiable** if for each \( x \in S(X) \), the limit (2.1) is attained uniformly for \( y \in S(X) \). The norm of \( X \) said to be **uniformly Fréchet differentiable**
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(or $X$ is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for $x, y \in S(X)$.

A Banach space $X$ is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of $X$ satisfying that $x_n \to x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if $X$ is uniformly convex, then $X$ has the Kadec-Klee property; see [10, 34] for more details.

Again, we introduce some well known properties of the duality mapping $J$ relating to geometrical properties of $X$.

Proposition 2.5. ([10, 30, 34])

1. $X$ is smooth if and only if $J$ is single valued. In this case, $J$ is norm-to-$\omega$ uniform continuous;

2. if $X$ is strictly convex, then $J$ is one to one (or injective), i.e.,
   $$x \neq y \implies Jx \cap Jy = \emptyset.$$ 

3. $X$ is strictly convex if and only if $J$ is a strictly monotone operator, i.e.,
   $$x \neq y, j_x \in Jx, j_y \in Jy \implies \langle x - y, j_x - j_y \rangle > 0.$$ 

4. if $X$ is reflexive, then $J$ is a mapping of $X$ onto $X^*$.

5. if $X^*$ is strictly convex (resp., smooth), then $X$ is smooth (resp., strictly convex). Further, the converse is satisfied if $X$ is reflexive.

6. if $X$ has a Fréchet differentiable norm, then $J$ is norm-to-norm continuous.

7. if $X$ has a uniformly Gâteaux differentiable norm, then $J$ is norm-to-$\omega$ uniformly continuous on each bounded subset of $X$.

8. if $X$ is uniformly smooth, then $J$ is norm-to-norm uniformly continuous on each bounded subset of $X$.

Finally, we shall add the well-known properties between $X$ and its dual $X^*$.

9. $X$ is uniformly convex if and only if $X^*$ is uniformly smooth.

10. $X$ is reflexive, strictly convex, and has the Kadec-Klee property if and only if $X^*$ has a Fréchet differentiable norm.

3 Demiclosedness for nonlinear mappings

Recall that a Banach space $X$ is said to satisfy Opial's condition [25] if whenever a sequence $\{x_n\}$ in $X$ converges weakly to $x_0$, then

$$\liminf_{n \to \infty} \|x_n - x_0\| < \liminf_{n \to \infty} \|x_n - x\|, \quad (x \neq x_0).$$
It is well known [16] that \( L^p \) spaces, \( p \neq 2 \), do not satisfy Opial's condition while all the \( \ell^p \) spaces do \((1 < p < \infty)\). Thus Opial's condition is independent of uniform convexity.

Spaces which satisfy Opial's condition have another nice property related to fixed point theory. Also, a function \( f : D \subset X \to X \) is demiclosed at \( w \) if
\[
x_n \to x, \quad \|f(x_n) - w\| \to 0 \implies x \in D, \quad f(x) = w.
\]

The following theorem was well known; for an example, see Theorem 10.3 in [16].

**Theorem 3.1.** ([16]) Let \( C \) be a nonempty closed convex subset of a reflexive Banach space \( X \) satisfying Opial's condition and let \( T : C \to X \) be nonexpansive. Then \( f = I - T \) is demiclosed on \( C \).

For the demiclosedness principle on uniformly convex spaces, We need the following useful lemmas; see Proposition 10.2 in [16].

**Lemma 3.2.** ([16]) Let \( C \) be a bounded closed convex subset of a uniformly convex space \( X \), and let \( T : K \to X \) be nonexpansive such that \( \inf \{\|x - Tx\| : x \in K\} = 0 \). Then \( F(T) \neq \emptyset \).

Lemma 3.2 is a crucial tool to prove the following well known demiclosedness principle for nonexpansive mappings on uniformly convex Banach spaces; see Theorem 10.4 in [16] or [6].

**Theorem 3.3.** (Demiclosedness Principle; see [16] or [6]) Let \( C \) be a nonempty closed convex subset of a uniformly convex space \( X \) and let \( T : K \to X \) be nonexpansive. Then \( f = I - T \) is demiclosed on \( C \).

We need the following notations.
\[
\Delta^{n-1} = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \geq 0, \sum \lambda_i = 1 \}.
\]

and \( \phi \in \Gamma_c \) if and only if \( \phi \in \Gamma(\mathbb{R}^+) \) and \( \phi \) is convex.

Recall that \( T : C \to X \) is said to be of type \( \gamma \) [3] if \( \gamma \in \Gamma_c \) and
\[
\gamma(\|cTx + (1 - c)Ty - T(cx + (1 - c)y)\|) \leq \|x - y\| - \|Tx - Ty\| \tag{3.1}
\]
for all \( x, y \in C \) and \( c \in [0, 1] \).

**Remark 3.4.** (a) Every type \( \gamma \) mapping is nonexpansive, and every affine nonexpansive mapping is of type \( \gamma \); but not every nonexpansive mapping is of type \( \gamma \) because \( F(T) \) is obviously convex by (3.1) if \( T : C \to X \) is of type \( \gamma \).

(b) Note that if \( T \) is nonexpansive, \( F(T) \) is generally not convex; let us recall an example due to DeMarr [11]. Let \( X \) be the space of all ordered pairs \((a, b)\) of real numbers. Define \( \|x\| = \max\{|a|, |b|\} \) for \( x = (a, b) \in X \). For \( C := \{ x : \|x\| \leq 1 \} \), define \( T : C \to C \) by
\[
Tx = (|b|, b) \quad \forall x = (a, b) \in C.
\]
Then $T$ is nonexpansive because
\[ \|Tx - Ty\| = \|(b, b) - (d, d)\| = |b - d| = \max\{|a - c|, |b - d|\} = \|x - y\| \]
for all $x = (a, b)$ and $y = (c, d)$ in $C$. However, note that $x = (1, 1) < y = (1, -1) \in F(T)$ but $\frac{1}{2}(x + y) = (1, 0) \notin F(T)$.

(c) If $X$ is uniformly convex and $C$ is a bounded closed convex subset of $X$, there exists $\gamma \in \Gamma_c$ such that every nonexpansive mapping is of type $(\gamma)$; moreover, $\gamma$ can be chosen to depend only on $\text{diam}(C)$ and not on $T$; see Lemma 1.1 in [3].

(d) If $T : C \to X$ is of type $(\gamma)$, then $f = I - T$ is demiclosed on $C$; see Lemma 1.3 in [3].

Now recall the following subsequent results due to Bruck [4]; see Lemma 2.1 of [4] for the second lemma.

Lemma 3.5. ([4]) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex $X$. Then there exists $\gamma \in \Gamma_c$ such that
\[ \|T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i Tx_i\| \leq L\gamma^{-1}\left(\max_{1 \leq i,j \leq n} (\|x_i - x_j\| - L^{-1}\|Tx_i - Tx_j\|)\right) \quad (3.2) \]
for any Lipschitzian mapping $T : C \to X$ with its Lipschitz constant $L \geq 1$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$ and $x_1, \ldots, x_n \in C$.

Lemma 3.6. ([4]) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex $X$, $\gamma \in \Gamma_c$, and let $T : C \to X$ be of type $(\gamma)$. Then there exists $\gamma_p \in \Gamma_c$ such that
\[ \gamma_p(\|T(\sum_{i=1}^{p} \lambda_i x_i) - \sum_{i=1}^{p} \lambda_i Tx_i\|) \leq \max_{1 \leq i,j \leq p} (\|x_i - x_j\| - \|Tx_i - Tx_j\|) \]
for $\lambda = (\lambda_1, \ldots, \lambda_p) \in \Delta^{p-1}$ and $x_1, \ldots, x_p \in C$.

Using Lemma 3.5 (Bruck), Xu [36] also established the following subsequent results for asymptotically nonexpansive mappings; see Theorem 2 of [36], Lemma 2.3 of [32], respectively.

Theorem 3.7. ([36]) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be a asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero.

Theorem 3.8. ([32]) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be a asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero in the sense that whenever $x_n \rightharpoonup x$ and
\[ \limsup_{k \to \infty} \limsup_{n \to \infty} \|x_n - T^k x_n\| = 0 \]
it follows that $x = Tx$.

In 2001, Chang et all [8] removed the assumption of boundedness of $C$ in Theorem 3.7; see Theorem 1 of [8].
**Theorem 3.9.** ([8]) Let $C$ be a nonempty closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero.

More generally, we easily observe the following demiclosedness principle for asymptotically nonexpansive mappings.

**Theorem 3.10.** Let $C$ be a nonempty closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $f = I - T$ is demiclosed at zero in the sense that whenever $x_n \to x$ and $\limsup_{k \to \infty} \limsup_{n \to \infty} \|x_n - T^k x_n\| = 0$ (3.3) it follows that $x = Tx$.

**Proof.** Let $x_n \to x$ and $\limsup_{k \to \infty} \limsup_{n \to \infty} \|x_n - T^k x_n\| = 0$. Then, since $\{x_n\}$ is bounded, $\exists r > 0$ such that $\{x_n\} \subset K := C \cap B_r$, where $B_r$ denotes the closed ball of $X$ with center 0 and radius $r$. Then $K$ is a nonempty bounded closed convex subset in $C$. For arbitrary $\epsilon > 0$, choose $k_0$ such that

$$\limsup_{n \to \infty} \|x_n - T^k x_n\| < \epsilon, \quad k \geq k_0$$ (3.4)

by (3.15). Since $x \in \overline{co}(\{x_n\})$, for each $n \geq 1$, we can also choose a convex combination

$$y_n := \sum_{i=1}^{m(n)} \lambda_i^{(n)} x_{i+n}, \quad \lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_{m(n)}^{(n)}) \in \Delta^{m(n)-1}$$

such that

$$\|y_n - x\| < \frac{1}{n}.$$(3.5)

Now for any (fixed) $k \geq k_0$, using (3.4), we can choose $n_0$ such that

$$\|x_n - T^k x_n\| < \epsilon, \quad n \geq n_0.$$ (3.6)

Since $T^k : K \to X$ is AN (hence Lipschitzian with its Lipschitz constants $L_k := 1 + c_k$), use to Bruck's Lemma 3.5 (with $d := \text{diam } K$) to derive

$$\|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\|$$

$$\leq L_k \gamma^{-1} \left( \max_{1 \leq i,j \leq m(n)} (\|x_{i+n} - x_{j+n}\| - L_k^{-1} \|T^k x_{i+n} - T^k x_{j+n}\|) \right)$$

$$\leq L_k \gamma^{-1} \left( \max_{1 \leq i,j \leq m(n)} (\|x_{i+n} - T^k x_{i+n}\| + \|x_{j+n} - T^k x_{j+n}\|$$

$$+ (1 - L_k^{-1}) \|T^k x_{i+n} - T^k x_{j+n}\|) \right)$$

$$\leq L_k \gamma^{-1} (2\epsilon + (1 - L_k^{-1})d) \quad \text{by (3.6)}.$$
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Also, for $k \geq k_0$ and $n \geq n_0$, it follows that
\[
\|T^k y_n - y_n\| 
\leq \|T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n}\| \leq \gamma^{-1}(2\epsilon + (1-L_k^{-1})d) + \epsilon \quad \text{by (3.6) again.}
\]

Taking $\limsup_{n \to \infty}$ firstly on both sides, we have
\[
\limsup_{n \to \infty} \|T^k y_n - y_n\| 
\leq L_k \gamma^{-1}(2\epsilon + (1-L_k^{-1})d) + \epsilon 
\quad \text{for all } k \geq k_0.
\]

Therefore, for $k \geq k_0$,
\[
\|T^k x - x\| 
\leq \|T^k x - T^k y_n\| + \|T^k y_n - y_n\| + \|y_n - x\|
\leq (1 + L_k)\|y_n - x\| + \|T^k y_n - y_n\|
\]

By virtue of (3.7) and $\|y_n - x\| \to 0$, we see
\[
\|T^k x - x\| 
\leq L_k \gamma^{-1}(2\epsilon + (1-L_k^{-1})d) + \epsilon \to 0
\]
as $k \to \infty$ and $\epsilon \to 0$. This shows $x = \lim_{k \to \infty} T^k x$. Hence $Tx = x$ by continuity of $T$. The proof is complete. \[\square\]

Recall that $X$ is said to satisfy the uniform Opial property [28] if for each $c > 0$, \exists $r > 0$ such that
\[
1 + r \leq \liminf_{n \to \infty} \|x + x_n\| 
\quad \text{for each } x \in X \text{ with } \|x\| \geq c \text{ and each weakly null sequence } \{x_n\} \text{ in } X \text{ with } \liminf_{n \to \infty} \|x_n\| \geq 1.
\]

Remark 3.11. (a) It suffices to take the weakly null sequence $\{x_n\}$ with $\|x_n\| = 1$ for all $n \geq 1$ instead of asking that $\liminf_{n \to \infty} \|x_n\| \geq 1$ in (3.8).

(b) We can substitute both $\liminf$ by $\limsup$ in (3.8).

Proof. (a) Let $x \in X$ with $\|x\| \geq c$ and $\liminf_{n \to \infty} \|x_n\| \geq 1$. Assume $\liminf_{n \to \infty} \|x + x_n\| = \lim_{m} \|x + x_m\|$ for some subsequence $\{m\}$ of $\{n\}$. Also, assume without loss of generality that $\liminf_{m} \|x_m\| = \liminf_{m} \|x_{m_k}\| = 1$; put $y_k := x_{m_k}/\|x_{m_k}\|$ for all sufficient large $k \geq k_0$ (otherwise, i.e., if $d := \liminf_{m} \|x_m\| > 1$, consider $z_m := x_m/d$; put $y_k := z_{m_k}/\|z_{m_k}\| = x_{m_k}/\|x_{m_k}\|$). Since $\{y_k\}$ is a weakly null sequence with $\|y_k\| = 1$ for all $k \geq k_0$, it follows from assumption that
\[
1 + r \leq \liminf_{k \to \infty} \|x + y_k\|
= \liminf_{k \to \infty} \|x + x_{m_k} - (1 - 1/\|x_{m_k}\|) x_{m_k}\|
\leq \liminf_{k \to \infty} \|x + x_{m_k}\| + \limsup_{k \to \infty} 1 - 1/\|x_{m_k}\| \cdot \|x_{m_k}\|
= \lim_{k \to \infty} \|x + x_{m_k}\| = \lim_{m \to \infty} \|x + x_m\| = \liminf_{n \to \infty} \|x + x_n\|.
\]
Hence (3.8) is required.

(b) Given $c > 0$, $\exists r > 0$ such that the inequality (3.8) replaced with $\limsup_n$ is satisfied. Let $x \in X$ with $\|x\| \geq c$ and $\|x_n\| = 1$ for all $n \geq 1$. Assume $\liminf_n \|x + x_n\| = \lim_k \|x + x_{n_k}\|$ for some subsequence $\{n_k\}$ of $\{n\}$. Then it follows from (a) (with $y_k := x_{n_k}$) and hypothesis that

$$1 + r \leq \limsup_{k \to \infty} \|x + y_k\| = \lim_{k \to \infty} \|x + x_{n_k}\| = \liminf_{n \to \infty} \|x + x_n\|.$$

Hence (3.8) is satisfied with $\liminf_n$. □

Recall also that $X$ satisfies the $\liminf$-locally uniform Opial condition (in brief, $\liminf\text{-LUO}$) [19] if for any weakly null sequence $\{x_n\}$ in $X$ with $\liminf_{n \to \infty} \|x_n\| \geq 1$ and any $c > 0$, $\exists r > 0$ such that

$$1 + r \leq \liminf_{n \to \infty} \|x + x_n\| \quad (3.9)$$

for all $x \in X$ with $\|x\| \geq c$.

**Definition 3.12.** ([13]) A Banach space $X$ has the $\limsup\text{-LUO}$ if for any weakly null sequence $\{x_n\}$ in $X$ with $\limsup_{n \to \infty} \|x_n\| \geq 1$ and any $c > 0$, $\exists r > 0$ such that (3.8) replaced with $\limsup_n$ holds for all $x \in X$ with $\|x\| \geq c$.

**Remark 3.13.** Note that (UO) $\Rightarrow$ (liminf-LUO) $\Rightarrow$ (limsup-LUO). But the converse implications don’t remain true in general. Consider $X = (\sum_{i=2}^{\infty} \ell_i)_{\ell_1}$. Then $X = (\limsup\text{-LUO})$ but it lacks (UO); see [37]. Also, by [13], $X \not\in$ (liminf-LUO). If we take $X = (\sum_{i=2}^{\infty} \ell_i)_{\ell_1}$, it has liminf-LUO but not (UO); see [13].

For the following lemma, see Lemma 2.3 of Oka [23] or Lemma 1.5 of [38].

**Lemma 3.14.** Let $C$ be a nonempty bounded closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. For each $\epsilon > 0$, $\exists K_\epsilon > 0$ and $\delta_\epsilon > 0$ such that if $k \geq K_\epsilon$, $z_1, \ldots, z_n \in C$ ($n \geq 2$) and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\epsilon$ for $1 \leq i, j \leq n$, then

$$\left\| T^k \left( \sum_{i=1}^{n} t_i z_i \right) - \sum_{i=1}^{n} t_i T^k z_i \right\| \leq \epsilon$$

for all $t = (t_1, \ldots, t_n) \in \Delta^{n-1}$.

Using Lemma 3.14, Yang, Xie, Peng and Hu [38] recently proved the following demiclosedness principle of $I - T$ for a mapping $T$ which is asymptotically nonexpansive in the intermediate sense.

**Theorem 3.15.** ([38]) Let $C$ be a nonempty closed convex subset of a uniformly convex space $X$ and let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that whenever $x_n \rightharpoonup x$ and

$$\limsup_{m \to \infty} \limsup_{n \to \infty} \|x_n - T^m x_n\| = 0$$

it follows that $x = Tx$. 
Remark 3.16. Note that Theorem 3.15 is the slight modification of Lemma 2.5 in [23].

Here we give an easy example of an asymptotically nonexpansive mapping which is not nonexpansive.

Example 3.17. Let \( X = \mathbb{R}, C = [0, 1], \) and \( 1/2 < k < 1. \) For each \( x \in C, \) define
\[
Tx = \begin{cases} 
  kx, & \text{if } 0 \leq x \leq 1/2; \\
  \frac{k}{2k-1}(x-k), & \text{if } 1/2 \leq x \leq k; \\
  0, & \text{if } k \leq x \leq 1.
\end{cases}
\]
Then \( T : C \rightarrow C \) is asymptotically nonexpansive but not nonexpansive.

Now we shall give the demiclosedness principle of \( I - T \) for a TAN mapping \( T. \) We first begin with following slight modification of Lemma 2.1 in [38].

Lemma 3.18. Let \( C \) be a nonempty closed convex subset of a uniformly convex \( X \) and let \( T : C \rightarrow C \) be a TAN mapping and let \( K \) a nonempty bounded closed convex subset of \( C. \) Then, for each \( \epsilon > 0, \) \( \exists N_\epsilon \geq 1 \) and \( \delta_{2,\epsilon} \) with \( 0 < \delta_{2,\epsilon} \leq \epsilon \) such that if \( k \geq N_\epsilon, x_1, x_2 \in K \) and if \( \|x_1 - x_2\| - \|T^kx_1 - T^kx_2\| \leq \delta_{2,\epsilon}, \) then
\[
\|T^k(\lambda_1x_1 + \lambda_2x_2) - \lambda_1T^kx_1 - \lambda_2T^kx_2\| \leq \epsilon
\]
for all \( \lambda = (\lambda_1, \lambda_2) \in \Delta^1. \)

Proof. We employ the method of the proof in [23]. Since \( X \) is uniformly convex, the modulus of convexity \( \delta \) is a continuous and strictly increasing function on \( [0, 2] \) (see [16] for more details). Then the function \( F : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) defined by
\[
F(x) = \begin{cases} 
  \frac{1}{2} \int_0^x \delta(t)dt, & \text{if } 0 \leq x \leq 2; \\
  \frac{1}{2}(x - 2) + F(2), & \text{if } x > 2.
\end{cases}
\]
is clearly strictly increasing, continuous and convex on \( \mathbb{R}^+. \) Obviously, since \( F(x) \leq \delta(x) (0 \leq x \leq 2), \) the uniform convexity of \( X \) implies that
\[
2\lambda_1\lambda_2F(\|x - y\|) \leq 1 - \|\lambda_1x + \lambda_2y\| \quad (3.10)
\]
for \( \lambda = (\lambda_1, \lambda_2) \in \Delta^1, \|x\| \leq 1 \) and \( \|y\| \leq 1. \) If either \( \lambda_1 \) or \( \lambda_2 \) is 1 or 0, our conclusion is clearly satisfied. So assume that \( 0 < \lambda_1, \lambda_2 < 1 \) and let \( \epsilon > 0 \) be arbitrary given. Set
\[
M := \text{diam } K \vee \sup_{x,y \in K} \phi(\|x - y\|) < \infty.
\]
(Note that \( \sup_{x,y \in K} \phi(\|x - y\|) \leq \phi(\text{diam } K) \) because \( \phi \) is strictly increasing.)

Choose \( d_\epsilon > 0 \) such that \( \frac{M}{2}F^{-1}(\frac{2d_\epsilon}{M}) < \epsilon \) and put \( \delta_{2,\epsilon} = \min \{\epsilon, d_\epsilon, \frac{M}{4}\}. \) For \( \tilde{\delta}_{2,\epsilon} = \min\{\lambda_i\delta_{2,\epsilon} : i = 1, 2\} > 0, \) since \( c_n, d_n \rightarrow 0, \) there exists an integer \( N_\epsilon \geq 1 \) (depending on the set \( K \)) such that if \( k \geq N_\epsilon, \)
\[
c_k < \tilde{\delta}_{2,\epsilon}/2M \quad \text{and} \quad d_k < \tilde{\delta}_{2,\epsilon}/2.
\]
Then, by (1.6), we have
\[
\|T^k x - T^k y\| \leq \|x - y\| + c_k \phi(\|x - y\|) + d_k \\
\leq \|x - y\| + c_k M + d_k \\
< \|x - y\| + \delta_{2, \epsilon}
\] (3.11)
for all \(k \geq N_\epsilon, x, y \in K\). Now let \(k \geq N_\epsilon\) and let \(x_1, x_2 \in K\) with \(\|x_1 - x_2\| - \|T^k x_1 - T^k x_2\| \leq \delta_{2, \epsilon}\). On letting
\[
x := \frac{T^k x_2 - T^k (\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1 (\|x_1 - x_2\| + \delta_{2, \epsilon})}
\quad\text{and} \quad
y := \frac{T^k (\lambda_1 x_1 + \lambda_2 x_2) - T^k x_1}{\lambda_2 (\|x_1 - x_2\| + \delta_{2, \epsilon})},
\]
we have \(\|x\| \leq 1, \|y\| \leq 1\) by with help of (3.11) and
\[
\lambda_1 x + \lambda_2 y = \frac{T^k x_2 - T^k x_1}{\|x_1 - x_2\| + \delta_{2, \epsilon}}.
\]
From these facts, on letting
\[
0 < t := \frac{2}{M} \lambda_1 \lambda_2 (\|x_1 - x_2\| + \delta_{2, \epsilon}) \leq \frac{2}{M} \frac{1}{4} \left( M + \frac{M}{4} \right) < 1,
\]
we have
\[
\frac{2}{M} \|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k (\lambda_1 x_1 + \lambda_2 x_2)\| = t \|x - y\|
\] (3.12)
and
\[
\frac{1}{2 \lambda_1 \lambda_2} \left( 1 - \|\lambda_1 x + \lambda_2 y\| \right) = \frac{\|x_1 - x_2\| - \|T^k x_1 - T^k x_2\| + \delta_{2, \epsilon}}{2 \lambda_1 \lambda_2 (\|x_1 - x_2\| + \delta_{2, \epsilon})}
\leq \frac{2 \delta_{2, \epsilon}}{t M}.
\] (3.13)
Using (3.10), (3.12), (3.13) and the convexity of \(F\) with \(F(0) = 0\), we have
\[
F \left( \frac{2}{M} \|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k (\lambda_1 x_1 + \lambda_2 x_2)\| \right)
= F(t \|x - y\|) = F(t \|x - y\| + (1 - t)0)
\leq t F(\|x - y\|) + (1 - t) F(0)
\leq \frac{t}{2 \lambda_1 \lambda_2} \left( 1 - \|\lambda_1 x + \lambda_2 y\| \right) \leq \frac{2 \delta_{2, \epsilon}}{M} \leq \frac{2 \delta_{\epsilon}}{M}
\]
and so we have
\[
\|\lambda_1 T^k x_1 + \lambda_2 T^k x_2 - T^k (\lambda_1 x_1 + \lambda_2 x_2)\| \leq \frac{M}{2} F^{-1} \left( \frac{2 \delta_{\epsilon}}{M} \right) < \epsilon
\]
from the choice of \(d_\epsilon\) and the proof is complete. \(\square\)

Now on mimicking Lemma 2.2 and 2.3 in Oka [23] we have the following result.
Lemma 3.19. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : C \to C$ be a TAN mapping and let $K$ a bounded closed convex subset of $C$. Then, for $\epsilon > 0$ there exists an integers $N_\epsilon \geq 1$ and $\delta_\epsilon$ with $0 < \delta_\epsilon \leq \epsilon$ such that $k \geq N_\epsilon$, $x_1, x_2, \ldots, x_n \in K$ and if $\|x_i - x_j\| - \|T^k x_i - T^k x_j\| \leq \delta_\epsilon$ for $1 \leq i, j \leq n$, then

$$\|T^k \left( \sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i T^k x_i \| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta^{n-1}$.

As a direct application of Lemma 3.19, we have the following demiclosedness principle for continuous TAN mapping.

Theorem 3.20. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : C \to C$ be a continuous TAN mapping. Then $I - T$ is demiclosed at zero in the sense that whenever $\{x_n\}$ is a sequence in $C$ such that $x_n \rightharpoonup x (\in C)$ and it satisfies (3.15), namely,

$$\lim_{k \to \infty} \limsup_{n \to \infty} \|x_n - T^k x_n\| = 0.$$

Then $x \in F(T)$.

Proof. First, we claim that $\lim_{k \to \infty} T^k x = x$. For this end, since $\{x_n\}$ is bounded in $C$, take the bounded set $K$ in Lemma 3.19 by the closed convex hull of $\{x_n : n \geq 1\}$. For $\epsilon > 0$, take $N_\epsilon \geq 1$ and $\delta_\epsilon$ with $0 < \delta_\epsilon \leq \epsilon$ as in Lemma 3.19. From (3.15), there exists an integer $k_0 (\geq N_\epsilon)$ such that

$$\lim_{n \to \infty} \sup_{\infty} \|x_n - T^k x_n\| < \delta_\epsilon/2 \quad (k \geq k_0).$$

Also, we can choose an integer $n_0 (\geq k_0)$ such that

$$\|x_n - T^k x_n\| \leq \delta_\epsilon/2 \quad (k, n \geq n_0).$$

(3.14)

Since $x_n \rightharpoonup x$ and $x \in \overline{co}\{x_i : i \geq n\}$ for each $n \geq 1$, we can choose for each $n \geq 1$ a convex combination

$$y_n = \sum_{i=1}^{m(n)} \lambda_i^{(n)} x_{i+n}, \quad \text{where } \lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{m(n)}^{(n)}) \in \Delta^{m(n)-1}$$

such that $\|y_n - x\| \to 0$. Let $k, n \geq n_0$. Then it follows from (3.14) that, for $1 \leq i, j \leq m(n)$,

$$\|x_{i+n} - x_{j+n}\| - \|T^k x_{i+n} - T^k x_{j+n}\| \leq \|x_{i+n} - T^k x_{i+n}\| + \|x_{j+n} - T^k x_{j+n}\| \leq \delta_\epsilon/2 + \delta_\epsilon/2 = \delta_\epsilon.$$
and so applying Lemma 3.19 yields
\[ \left\| T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n} \right\| < \epsilon \]
and hence
\[ \left\| T^k y_n - y_n \right\| \leq \left\| T^k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k x_{i+n} \right\| + \left| \sum_{i=1}^{m(n)} \lambda_i^{(n)} (T^k x_{i+n} - x_{i+n}) \right| \]
\[ < \epsilon + \delta \epsilon / 2 \leq (3/2) \epsilon \]
for \( k, n \geq n_0 \). Since \( \mathcal{S} = \{ T_n : C \rightarrow C \} \) is TAN on \( C \), this implies that, for \( k, n \geq n_0 \),
\[ \| T^k x - x \| \leq \| T^k x - T^k y_n \| + \| T^k y_n - y_n \| + \| y_n - x \| \]
\[ \leq \| x - y_n \| + c_k \phi(\| x - y_n \|) + d_k + (3/2) \epsilon + \| y_n - x \| \]
\[ = 2 \| y_n - x \| + c_k \phi(\| x - y_n \|) + d_k + (3/2) \epsilon. \]
(3.15)
Taking the lim sup as \( n \rightarrow \infty \) at first and next the lim sup as \( k \rightarrow \infty \) in both sides of (3.15), we have \( \limsup_{k \rightarrow \infty} \| T^k x - x \| \leq (3/2) \epsilon \) and since \( \epsilon \) is arbitrary given, \( T^k x \rightarrow x \). Therefore \( x \in F(T) \) by continuity of \( T \). The proof is complete.  \[ \square \]

References


