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Existence of positive solution for the Cauchy problem for an ordinary differential equation

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Abstract

In this paper we consider the existence of positive solution for the Cauchy problem of the second order differential equation $u''(t) = f(t, u(t))$.

1 Introduction

The following ordinary differential equations arise in many different areas of applied mathematics and physics; see [2, 4]. In [3] Knežević-Miljanović considered the Cauchy problem

$$\begin{align*}
\begin{cases}
    u''(t) = P(t)u(t)^\sigma, & t \in (0, 1], \\
u(0) = 0, & u'(0) = \lambda,
\end{cases}
\end{align*}$$

(1)

where $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$, and $P$ is a continuous mapping of $[0, 1]$ such that $\int_0^1 |P(t)| t^{a+\sigma} dt < \infty$. On the other hand in [1] Erbe and Wang considered the equation

$$u''(t) = f(t, u(t)), \quad t \in (0, 1].$$

(2)

In this paper we consider the second order Cauchy problem

$$\begin{align*}
\begin{cases}
    u''(t) = f(t, u(t)), & \text{for almost every } t \in [0, 1], \\
u(0) = 0, & u'(0) = \lambda,
\end{cases}
\end{align*}$$

(3)

where $f$ is a mapping from $[0, 1] \times (0, \infty)$ into $\mathbb{R}$ satisfying the Carathéodory condition and $\lambda \in \mathbb{R}$ with $\lambda > 0$. 
2 Main results

Theorem 2.1. Suppose that a mapping $f$ from $[0,1] \times (0, \infty)$ into $\mathbb{R}$ satisfies the following.

(a) The mapping $f$ satisfies the Carathéodory condition, that is, the mapping $t \mapsto f(t,u)$ is measurable for any $u \in (0, \infty)$ and the mapping $u \mapsto f(t,u)$ is continuous for almost every $t \in [0,1]$.

(b) $|f(t,u_1)| \geq |f(t,u_2)|$ for almost every $t \in [0,1]$ and for any $u_1, u_2 \in (0, \infty)$ with $u_1 \leq u_2$.

(c) There exists $\alpha \in \mathbb{R}$ with $0 < \alpha < \lambda$ such that
\[ \int_0^1 |f(t,\alpha t)|dt < \infty. \]

(d) There exists $\beta \in \mathbb{R}$ with $\beta > 0$ such that
\[ \left| \frac{\partial f}{\partial u}(t,u) \right| \leq \frac{\beta |f(t,u)|}{u} \]
for almost every $t \in [0,1]$ and for any $u \in (0, \infty)$.

Then there exist $h \in \mathbb{R}$ with $0 < h \leq 1$ such that the Cauchy problem (3) has a unique solution in $X$, where $X$ is a subset
\[ X = \left\{ u \left| \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \text{ for any } t \in [0,h] \end{array} \right. \right\} \]
of $C[0,h]$, which is the class of continuous mappings from $[0,h]$ into $\mathbb{R}$.

Proof. It is noted that $C[0,h]$ is a Banach space by the maximum norm
\[ \|u\| = \max\{|u(t)| \mid t \in [0,h]\}. \]
Instead of the Cauchy problem (3) we consider the integral equation
\[ u(t) = \lambda t + \int_0^t (t-s)f(s,u(s))ds. \]
By the condition (c) there exists $h \in \mathbb{R}$ with $0 < h \leq 1$ such that
\[ \int_0^h |f(t,\alpha t)|dt < \min \left\{ \lambda - \alpha, \frac{\alpha}{\beta} \right\}. \]
Let $A$ be an operator from $X$ into $C[0,h]$ defined by
\[ Au(t) = \lambda t + \int_0^t (t-s)f(s,u(s))ds. \]
Since a mapping $t \mapsto \lambda t$ belongs to $X$, $X \neq \emptyset$. Moreover $A(X) \subset X$. Indeed by the condition (a) $Au \in C[0, h]$, $Au(0) = 0$, 

$$(Au)'(0) = \left[ \lambda + \int_0^t f(s, u(s))ds \right]_{t=0} = \lambda$$

and by the condition (b) 

$$Au(t) = \lambda t + \int_0^t (t-s)f(s, u(s))ds$$

$$\geq \lambda t - t \int_0^h |f(s, u(s))|ds$$

$$\geq \lambda t - t \int_0^h |f(s, \alpha s)|ds$$

$$\geq \alpha t$$

for any $t \in [0, h]$. We will find a fixed point of $A$. Let $\varphi$ be an operator from $X$ into $C[0, h]$ defined by 

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{\lambda}, & \text{if } t \in (0, h], \\ \lambda, & \text{if } t = 0, \end{cases}$$

and 

$$\varphi[X] = \{ \varphi[u] \mid u \in X \}$$

$$= \{ v \mid v \in C[0, h], v(0) = \lambda \text{ and } \alpha \leq v(t) \text{ for any } t \in [0, h] \}.$$ 

Then $\varphi[X]$ is a closed subset of $C[0, h]$ and hence it is a complete metric space. Let $\Phi$ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by 

$$\Phi \varphi[u] = \varphi[Au].$$

By the mean value theorem for any $u_1, u_2 \in X$ there exists a mapping $\xi$ such that 

$$\frac{f(t, u_1(t)) - f(t, u_2(t))}{u_1(t) - u_2(t)} = \frac{\partial f}{\partial u}(t, \xi(t))$$

and 

$$\min\{u_1(t), u_2(t)\} \leq \xi(t) \leq \max\{u_1(t), u_2(t)\}$$

for any $t \in [0, h]$. By the conditions (b) and (d) 

$$|f(t, u_1(t)) - f(t, u_2(t))| = \left| \frac{\partial f}{\partial u}(t, \xi(t))(u_1(t) - u_2(t)) \right|$$

$$\leq \left| \frac{\beta f(t, \xi(t))}{\xi(t)} \right| |u_1(t) - u_2(t)|$$

$$\leq \frac{\beta f(t, \alpha t)}{\alpha t} |u_1(t) - u_2(t)|$$
for almost every $t \in [0, h]$. Therefore
\[
|\Phi \varphi[u_1](t) - \Phi \varphi[u_2](t)| = \left| \frac{1}{t} \int_0^t (t-s)(f(s, u_1(s)) - f(s, u_2(s)))ds \right|
\leq \int_0^h \left| \frac{\beta f(s, \alpha s)}{\alpha s} \right| |u_1(s) - u_2(s)|ds
\leq \frac{\beta}{\alpha} \int_0^h |f(s, \alpha s)|ds \|\varphi[u_1] - \varphi[u_2]\|
\]
for any $t \in [0, h]$. Therefore
\[
\|\Phi \varphi[u_1] - \Phi \varphi[u_2]\| \leq \frac{\beta}{\alpha} \int_0^h |f(s, \alpha s)|ds \|\varphi[u_1] - \varphi[u_2]\|.
\]
By the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi \varphi[u] = \varphi[u]$. Then $Au = u$. \hfill \square

**Theorem 2.2.** Suppose that a mapping $f$ from $[0, 1] \times (0, \infty)$ into $\mathbb{R}$ satisfies the following.

(a) The mapping $f$ satisfies the Carathéodory condition, that is, the mapping $t \mapsto f(t, u)$ is measurable for any $u \in (0, \infty)$ and the mapping $u \mapsto f(t, u)$ is continuous for almost every $t \in [0, 1]$.

(e) $|f(t, u_1)| \leq |f(t, u_2)|$ for almost every $t \in [0, 1]$ and for any $u_1, u_2 \in (0, \infty)$ with $u_1 \leq u_2$.

(f) There exists $\alpha \in \mathbb{R}$ with $0 < \alpha < \lambda$ such that
\[
\int_0^1 |f(t, (2\lambda - \alpha)t)|dt < \infty.
\]

(d) There exists $\beta \in \mathbb{R}$ with $\beta > 0$ such that
\[
\left| \frac{\partial f}{\partial u}(t, u) \right| \leq \frac{\beta |f(t, u)|}{u}
\]
for almost every $t \in [0, 1]$ and for any $u \in (0, \infty)$.

Then there exist $h \in \mathbb{R}$ with $0 < h \leq 1$ such that the Cauchy problem (3) has a unique solution in $X$, where $X$ is a subset
\[
X = \left\{ u \in C[0, h], u(0) = 0, u'(0) = \lambda \right. \quad \text{and} \left. \alpha t \leq u(t) \leq (2\lambda - \alpha)t \text{ for any } t \in [0, h] \right\}
\]
of $C[0, h]$. 

Proof. By the condition (f) there exists $h \in \mathbb{R}$ with $0 < h \leq 1$ such that
\[
\int_0^h |f(t, (2\lambda - \alpha)t)| dt < \min \left\{ \lambda - \alpha, \frac{\alpha}{\beta} \right\}
\]
and let $A$ be an operator from $X$ into $C[0, h]$ defined by
\[
Au(t) = \lambda t + \int_0^t (t - s)f(s, u(s))ds.
\]
Since a mapping $t \mapsto \lambda t$ belongs to $X$, $X \neq \emptyset$. Moreover $A(X) \subset X$. Indeed by the condition (a) $Au \in C[0, h]$, $Au(0) = 0$,
\[
(Au)'(0) = \left[ \lambda + \int_0^t f(s, u(s))ds \right]_{t=0} = \lambda
\]
and by the condition (e)
\[
Au(t) = \lambda t + \int_0^t (t - s)f(s, u(s))ds \\
\geq \lambda t - t \int_0^h |f(s, u(s))| ds \\
\geq \lambda t - t \int_0^h |f(s, (2\lambda - \alpha)s)| ds \\
\geq \alpha t
\]
and
\[
Au(t) = \lambda t + \int_0^t (t - s)f(s, u(s))ds \\
\leq \lambda t + t \int_0^h |f(s, u(s))| ds \\
\leq \lambda t + t \int_0^h |f(s, (2\lambda - \alpha)s)| ds \\
\leq (2\lambda - \alpha)t
\]
for any $t \in [0, h]$. We will find a fixed point of $A$. Let $\varphi$ be an operator from $X$ into $C[0, h]$ defined by
\[
\varphi[u](t) = \begin{cases} \frac{u(t)}{t}, & t \in (0, h], \\ \lambda, & t = 0, \end{cases}
\]
and
\[
\varphi[X] = \{ \varphi[u] \mid u \in X \} = \{ v \mid v \in C[0, h], v(0) = \lambda \text{ and } \alpha \leq v(t) \leq (2\lambda - \alpha) \text{ for any } t \in [0, h] \}.\]
Then $\varphi[X]$ is a closed subset of $C[0,h]$ and hence it is a complete metric space. Let $\Phi$ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi \varphi[u] = \varphi[Au].$$

Then we can show just like Theorem 2.1 that by the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi \varphi[u] = \varphi[u]$ and hence $Au = u$. \qed

## 3 Examples

In this section we give some examples to illustrate the results above.

**Example 3.1.** In [3] the Cauchy problem (1) is considered. Since $f(t,u) = P(t)t^{a}u^{\sigma}$, $a, \sigma, \lambda \in R$ with $\sigma < 0$ and $\lambda > 0$ and $P$ is a continuous mapping such that $\int_{0}^{1}|P(t)|t^{a+\sigma}dt < \infty$, the conditions (a), (b), (c) and (d) are satisfied. Indeed (a), (b) and (c) are clear and since

$$\left| \frac{\partial f}{\partial u}(t,u) \right| = |P(t)t^{a}\sigma u^{\sigma-1}| = \frac{\sigma|f(t,u)|}{u},$$

(d) holds. By Theorem 2.1 the Cauchy problem (1) has a unique solution in

$$X = \left\{ u \left| \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \text{ for any } t \in [0,h] \end{array} \right. \right\}. $$

**Example 3.2.** We consider the Cauchy problem

$$\begin{cases} u''(t) = a(t) + u(t)^{\sigma}, & t \in [0,1], \\ u(0) = 0, & u'(0) = \lambda, \end{cases}$$

(4)

where $a$ is positive and integrable, $\sigma \in R$ with $\sigma > 0$ and $\lambda \in R$ with $\lambda > 0$. Since $f(t,u) = a(t) + u^{\sigma}$, the conditions (a), (e), (f) and (d) are satisfied. Indeed (a), (e) and (f) are clear and since

$$\left| \frac{\partial f}{\partial u}(t,u) \right| = \sigma u^{\sigma-1} \leq \max\{\sigma,1\}(a(t) + u^{\sigma}) \leq \frac{\max\{\sigma,1\}|f(t,u)|}{u},$$

(d) holds. By Theorem 2.2 the Cauchy problem (4) has a unique solution in

$$X = \left\{ u \left| \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \leq (2\lambda - \alpha)t \text{ for any } t \in [0,h] \end{array} \right. \right\}. $$

**Example 3.3.** We consider the Cauchy problem

$$\begin{cases} u''(t) = a(t)u(t)^{\sigma}, & t \in [0,1], \\ u(0) = 0, & u'(0) = \lambda, \end{cases} $$

(5)
where \( \int_0^1 |a(t)| t^\sigma dt < \infty \) and \( \sigma, \lambda \in \mathbb{R} \) with \( \lambda > 0 \). Since \( f(t, u) = a(t)u^\sigma \), the conditions (a), (b), (c) and (d) are satisfied if \( \sigma < 0 \) and the conditions (a), (e), (f) and (d) are satisfied if \( \sigma \geq 0 \). Indeed (a) is clear, (b) and (c) are clear if \( \sigma < 0 \), (e) and (f) are clear if \( \sigma \geq 0 \), and since

\[
\left| \frac{\partial f}{\partial u}(t, u) \right| = \begin{cases} |a(t)\sigma u^{\sigma-1}|, & \text{if } \sigma \neq 0, \\ 0, & \text{if } \sigma = 0, \end{cases} = \frac{|\sigma||f(t, u)|}{u},
\]

(d) holds. By Theorem 2.1 if \( \sigma < 0 \) and by Theorem 2.2 if \( \sigma > 0 \) the Cauchy problem (5) has a unique solution in

\[
X = \left\{ u \left| u \in C[0, h], u(0) = 0, u'(0) = \lambda \right. \right. \\
\left. \left. \text{and } \alpha t \leq u(t) \text{ for any } t \in [0, h] \right\}
\]

and

\[
X = \left\{ u \left| u \in C[0, h], u(0) = 0, u'(0) = \lambda \right. \right. \\
\left. \left. \text{and } \alpha t \leq u(t) \leq (2\lambda - \alpha)t \text{ for any } t \in [0, h] \right\}
\],

respectively.

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**References**


