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L^p norms of nonnegative Schrödinger heat semigroup and the large time behavior of hot spots

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1 Introduction

The large time behavior of the solutions of parabolic equations is a classical subject and has fascinated many mathematicians. In this paper we investigate the large time behavior of the solution of the Cauchy problem for the heat equation with a potential,

$$\begin{cases} \partial_t u = \Delta u - V(|x|)u & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $N \geq 3$, and $\phi \in L^2(\mathbf{R}^N)$. Here $V = V(|x|)$ is a smooth, nonpositive, and radially symmetric function satisfying

$$V(x) = \omega|x|^{-2}(1 + o(1)) \quad \text{as} \quad |x| \to \infty \tag{1.2}$$

with $\omega \in (-\omega_*, 0]$ and $\omega_* = (N-2)^2/4$. More precisely, we assume the following condition:

$$(V) \quad \begin{cases} \text{(i)} & V = V(r) \in C^1([0,\infty)) \text{ and } V \leq 0 \ (\not\equiv 0) \text{ on } [0,\infty); \\ \text{(ii)} & \text{there exist constants } \omega \in (-\omega_*, 0] \text{ and } \theta > 0 \text{ such that} \\ & \qquad V(r) = \omega r^{-2} + O(r^{-2-\theta}) \quad \text{as } r \to \infty; \\ \text{(iii)} & \sup_{r\geq 1} \left| r^3 V'(r) \right| < \infty. \end{cases}$$

We say that $H := -\Delta + V$ is nonnegative (which is abbreviated as $H \geq 0$) if

$$\int_{\mathbf{R}^N} \left\{ |\nabla\varphi|^2 + V(|x|)\varphi^2 \right\} dx \geq 0, \qquad \varphi \in C_0^\infty(\mathbf{R}^N).$$

Furthermore we say that H is subcritical if for any $W \in C_0^\infty(\mathbf{R}^N)$, one has $H - \epsilon W \geq 0$ for small enough $\epsilon > 0$. In addition, a subcritical operator H is said to be strongly subcritical if $H - \epsilon V_- \geq 0$ for small enough $\epsilon > 0$, where $V_- = \max\{-V, 0\}$. In [10] the authors of this paper studied the following two subjects:

- the decay rate of $L^q(\mathbf{R}^N)$-norm ($q \geq 2$) of the solution u as $t \to \infty$;
- the large time behavior of the solution u and its hot spots

$$H(t) = \left\{ x \in \mathbf{R}^N \, : \, u(x,t) = \max_{y \in \mathbf{R}^N} u(y,t) \right\},$$

and in this paper we introduce some results of [10]. Since the results of [10] on the decay rate of $L^q(\mathbf{R}^N)$-norm ($q \geq 2$) of the solution u were given in [9], we focus on the large time behavior of the hot spots $H(t)$.

The movement of hot spots for the heat equation in unbounded domains was first studied by Chavel and Karp [1]. They proved that, for any nonzero, nonnegative initial data $\phi \in L^\infty_c(\mathbf{R}^N)$, the hot spots $H(t)$ of the solution of the heat equation are contained in the closed convex hull of the support of ϕ for any $t > 0$, and the hot spots $H(t)$ tend to the center of mass of ϕ

$$\int_{\mathbf{R}^N} x\phi(x)dx \Big/ \int_{\mathbf{R}^N} \phi(x)dx$$

as $t \to \infty$. Subsequently the movement of hot spots has been studied in several papers, see [3], [4], and [6]–[11]. Among others, in [6]–[8] the authors of this paper studied the movement of hots spots of the solution of the heat equation (1.1) with a potential V for the case where V is a nonnegative function satisfying (1.2) with $\omega \geq 0$. In this case the hot spots move to the space infinity as $t \to \infty$, and they gave the rate and the direction for hot spots to tend to the space infinity. The behavior of hot spots is determined by the initial function ϕ and the harmonic functions for the operator $H = -\Delta + V$, and depends on the constant ω and the dimension N.

In this paper, under condition (V), we study the movement of hot spots of the solution of (1.1). This is a continuation of our previous papers [6]–[8]. We emphasize that, in our case, the hot spots stay in a bounded set for all sufficiently large t, and its behavior is completely different from in the cases treated in [6]–[8]. We prove that:

- if $\omega < 0$, then the hot spots converge to the origin as $t \to \infty$;
- if $\omega = 0$, then the hot spots converge to the one point x^* as $t \to \infty$. In particular, if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, then the point x^* does not necessarily coincide with the origin and depends on the initial function ϕ.

These assertions include an interesting fact in the study of the behavior of the hot spots. Consider the case where $V \equiv 0$ in $[0, R]$ for some $R > 0$ and assume that the hot spots stay in the ball $B(0, R) := \{x \in \mathbf{R}^N : |x| < R\}$ for all sufficiently large t. Then, since the harmonic functions for $H = -\Delta + V$ are independent of ω in the ball $B(0, R)$, the results in [6]–[8] suggest that the behavior of hot pots in the case $\omega < 0$ is similar to that in the case $\omega = 0$. However the behavior of hot spots for the case $\omega < 0$ is not necessarily similar to that in the case $\omega = 0$. (See Theorem 1.2.) This means that the analysis of the behavior of hots spots for the case we treat in this paper is more delicate and requires more careful calculations than in [6]–[8].

We introduce some notation. For $1 \leq p \leq \infty$, we denote by $\|\cdot\|_p$ the norm of the $L^p(\mathbf{R}^N)$ space. We also denote by $\|\cdot\|$ the norm of the $L^2(\mathbf{R}^N)$ space with weight $e^{|x|^2/4}$, that is,

$L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$. Let $|\mathbf{S}^{N-1}|$ be the volume of the $(N-1)$-dimensional unit sphere $\mathbf{S}^{N-1}$. Let $\Delta_{\mathbf{S}^{N-1}}$ be the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$ and $\{\omega_k\}_{k=0}^{\infty}$ the eigenvalues of

$$-\Delta_{\mathbf{S}^{N-1}}Q = \omega Q \quad \text{on} \quad \mathbf{S}^{N-1}, \qquad Q \in L^2(\mathbf{S}^{N-1}), \tag{1.3}$$

that is,

$$\omega_k := k(N+k-2), \qquad k = 0,1,2,\dots. \tag{1.4}$$

Furthermore let $\{Q_{k,i}\}_{i=1}^{l_k}$ and l_k be the orthonormal system and the dimension of the eigenspace corresponding to ω_k, respectively. In particular, $l_0 = 1$, $l_1 = N$, and we may write

$$Q_{0,1}\left(\frac{x}{|x|}\right) = \kappa_0, \qquad Q_{1,i}\left(\frac{x}{|x|}\right) = \kappa_1 \frac{x_i}{|x|}, \quad i = 1,\dots,N, \tag{1.5}$$

where κ_0 and κ_1 are positive constants.

For any sets Λ and Σ, let $f = f(\lambda,\sigma)$ and $h = h(\lambda,\sigma)$ be maps from $\Lambda \times \Sigma$ to $(0,\infty)$. Then we say

$$f(\lambda,\sigma) \preceq h(\lambda,\sigma)$$

for all $\lambda \in \Lambda$ if, for any $\sigma \in \Sigma$, there exists a positive constant C such that $f(\lambda,\sigma) \le Ch(\lambda,\sigma)$ for all $\lambda \in \Lambda$. In addition, we say $f(\lambda,\sigma) \asymp h(\lambda,\sigma)$ for all $\lambda \in \Lambda$ if $f(\lambda,\sigma) \preceq h(\lambda,\sigma)$ and $f(\lambda,\sigma) \succeq h(\lambda,\sigma)$ for all $\lambda \in \Lambda$.

Assume condition (A). Let $k = 0,1,2,\dots.$ Put

$$\alpha_N(\omega) := \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2}. \tag{1.6}$$

Then we have

$$\alpha_{N+2k}(\omega) + k = \alpha_N(\omega + \omega_k) \tag{1.7}$$

for $k = 0,1,2,\dots.$ Furthermore there exists a unique positive solution $U_{N,k} = U_{N,k}(r)$ of

$$U'' + \frac{N-1}{r}U' - \left(V(r) + \frac{\omega_k}{r^2}\right)U = 0 \quad \text{in} \quad (0,\infty) \tag{1.8}$$

such that

$$d_{N,k} := \lim_{r\to 0} r^{-k}U_{N,k}(r) > 0, \tag{1.9}$$

$$U_{N,k}(r) = r^{\alpha_N(\omega+\omega_k)}(1+o(1)) \qquad \text{as} \quad r \to \infty. \tag{1.10}$$

In addition, $r^{-k}U_k(r)$ is monotone decreasing in $[0,\infty)$ and

$$U'_{N,k}(r) = \begin{cases} O(r) & \text{as} \quad r \to 0 \quad \text{if} \quad k = 0, \\ O(r^{k-1}) & \text{as} \quad r \to 0 \quad \text{if} \quad k \ge 1, \end{cases} \tag{1.11}$$

$$U'_{N,k}(r) = (\alpha_N(\omega+\omega_k) + o(1))r^{\alpha_N(\omega+\omega_k)-1} \quad \text{as} \quad r \to \infty, \tag{1.12}$$

$$U_{N,k}(r) = r^k U_{N+2k,0}(r), \quad r \ge 0, \quad k = 0,1,2,\dots. \tag{1.13}$$

See [18] and [10]. In what follows, for notational simplicity, if there occurs no confusion, then we use

$$\alpha(\omega), \qquad \beta(\omega), \qquad U_k(r),$$

instead of $\alpha_N(\omega)$, $\beta_N(\omega)$, and $U_{N,k}(r)$, respectively. Put

$$(1.14)\qquad \begin{aligned} &M_0 := \int_{\mathbf{R}^N} \phi(x)U_0(|x|)dx, \qquad M_i := \int_{\mathbf{R}^N} \phi(x)U_1(|x|)\frac{x_i}{|x|}dx \quad (i = 1,\dots,N),\\ &\mathcal{M} := \gamma_N\left(\frac{M_1}{M_0},\cdots,\frac{M_N}{M_0}\right), \qquad \gamma_N := \frac{U_1'(0)}{U_0(0)}. \end{aligned}$$

Furthermore, for any $k = 0, 1, 2, \dots$, since $\alpha(\omega + \omega_k) > -N/2$, we can define $\varphi_{N,k}$ by

$$\varphi_{N,k}(y) := c_{N,k}|y|^{\alpha_N(\omega+\omega_k)}e^{-|y|^2/4},$$

where $c_{N,k}$ is a positive constant such that $\|\varphi_{N,k}\| = 1$. Here, by (1.7) we have

$$(1.15)\qquad |\mathbf{S}^{N-1}|^{1/2}c_{N,k} = |\mathbf{S}^{N+2k-1}|^{1/2}c_{N+2k,0}, \qquad \varphi_{N,k}(y) = \frac{|\mathbf{S}^{N+2k-1}|^{1/2}}{|\mathbf{S}^{N-1}|^{1/2}}|y|^k\varphi_{N+2k,0}(y).$$

We write $\varphi_k = \varphi_{N,k}$ and $c_k = c_{N,k}$ for simplicity.

We are ready to state the main results of this paper. In the first theorem we give a result on the large time behavior of solution of (1.1).

Theorem 1.1 *Let $N \geq 3$. Assume condition (V) and that $H := -\Delta + V$ is subcritical. Let u be a solution of* (1.1) *with the initial function $\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$. Then there exists a constant C such that*

$$(1.16)\qquad \|u(t)\|_2 \leq Ct^{-\frac{N}{4}-\frac{\alpha(\omega)}{2}}\|\phi\|, \qquad t \geq 1.$$

Furthermore there hold

$$(1.17)\qquad \lim_{t\to\infty}\sup_{x\in B(0,L)}\left|t^{\frac{N}{2}+\alpha(\omega)}u(x,t) - c_0^2M_0U_0(x,t)\right| = 0, \quad L > 0$$

and

$$(1.18)\quad \lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}}u\left((1+t)^{\frac{1}{2}}y, t\right) = c_0M_0\varphi_0(y) \ \ in \ \ C_{loc}(\mathbf{R}^N\setminus\{0\}) \cap L^2(\mathbf{R}^N, e^{|y|^2/4}dy).$$

Next we give a result on the large time behavior of hot spots $H(t)$ of the solution u. Let

$$R_* = \inf\{r > 0 : V(r) < 0\}.$$

In the second theorem we prove that the hot spots converges to one point x_*, which is given exactly by the initial function and the functions $U_0(|x|)$ and $U_1(|x|)$. The point x_* can be characterized as the nearest point to the limit of $\gamma_N A(t)$ as $t \to \infty$ over the ball $\overline{B(0, R_*)}$, where $A(t)$ is the center of the mass of the solution u at the time t, that is,

$$A(t) := \int_{\mathbf{R}^N} xu(x,t)dx \Big/ \int_{\mathbf{R}^N} u(x,t)dx.$$

Theorem 1.2 *Assume the same conditions as in Theorem* 1.1 *and* $M_0 > 0$. *Then* $H(t) \neq \emptyset$ *for any* $t > 0$. *Furthermore there hold the following:*

(i) *For any sufficiently large* t,

$$\int_{\mathbf{R}^N} u(x,t)dx > 0$$

holds, and $A(t)$ *can be defined for all sufficiently large* t. *Furthermore there holds*

$$\lim_{t\to\infty} \gamma_N A(t) = \begin{cases} 0 & \textit{if} \quad \omega < 0, \\ \mathcal{M} & \textit{if} \quad \omega = 0; \end{cases} \tag{1.19}$$

(ii) *There holds*

$$\lim_{t\to\infty} \sup\{|x - x^*| : x \in H(t)\} = 0, \tag{1.20}$$

where

$$x^* := \begin{cases} 0 & \textit{if} \quad \omega < 0, \\ \mathcal{M} & \textit{if} \quad \omega = 0 \textit{ and } |\mathcal{M}| < R_*, \\ R_* \dfrac{\mathcal{M}}{|\mathcal{M}|} & \textit{if} \quad \omega = 0 \textit{ and } |\mathcal{M}| \geq R_*. \end{cases}$$

Next we give a sufficient condition for the set of the hot spots to consist of only one point and to move along a smooth curve on $\mathbf{R}^N$ for all sufficiently large t.

Theorem 1.3 *Assume the same conditions as in Theorem* 1.1 *and* $M_0 > 0$. *If* $V(0) = 0$ *and* $|x^*| = R_*$, *further assume that* $-V(r)$ *is monotone increasing on* $[R_*, R_* + \delta]$ *for some* $\delta > 0$. *Then there exist a constant* $T > 0$ *and a curve* $x(t) \in C^1([T,\infty) : \mathbf{R}^N)$ *such that*

$$H(t) = \{x(t)\}, \qquad t \geq T. \tag{1.21}$$

The rest of this paper is organized as follows. In Section 2 we give preliminary results in order to prove our theorems. In Section 3 we study the large time behavior of the solution u and prove Theorem 1.1. Sections 4 and 5 are devoted to the proofs of Theorems 1.2 and 1.3, respectively.

2 Preliminaries

In this section we give preliminary results in order to prove our theorems. Assume condition (V). Then, by the standard arguments for ordinary differential equations, we see that there exists a unique solution U of

$$U'' + \frac{N-1}{r}U' - V(r)U = 0 \quad \text{in} \quad (0,\infty) \tag{O}$$

with

$$\lim_{r\to 0} U(r) = 1. \tag{2.1}$$

Furthermore, by the same argument as in [6] we have:

(P) for any solution $\tilde{U}$ of (O) satisfying $\limsup_{r\to 0}|\tilde{U}(r)| < \infty$, there exists a constant c' such that $\tilde{U}(r) = c'U(r)$ on $[0,\infty)$.

Let $k = 0$ and $d_0 := d_{N,0}$ be the constant given in (1.9). Since the function

$$U_0(0) + \int_0^r s^{1-N}\left(\int_0^s \tau^{N-1}V(\tau)U_0(\tau)d\tau\right)ds$$

is also a solution of (O), the property (P) implies

$$U_0(r) = U_0(0) + \int_0^r s^{1-N}\left(\int_0^s \tau^{N-1}V(\tau)U_0(\tau)d\tau\right)ds \quad \text{on} \quad [0,\infty). \tag{2.2}$$

Then we have

$$U_0'(r) = r^{1-N}\int_0^r \tau^{N-1}V(\tau)U_0(\tau)d\tau \le (\not\equiv)\, 0 \quad \text{on} \quad [0,\infty), \tag{2.3}$$

$$U_0'(r) = \frac{V(0)U_0(0)}{N}r(1+o(1)) \quad \text{as} \quad r\to 0. \tag{2.4}$$

In particular, (2.4) yields (1.11) with $k = 0$. Furthermore we have:

Lemma 2.1 *Assume condition (V), and let $H := -\Delta + V$ be a nonnegative operator on $L^2(\mathbf{R}^N)$. Let $f \in C([0,\infty))$ and v be a solution of*

$$U'' + \frac{N-1}{r}U' - V(r)U = f \quad \textit{in} \quad (0,\infty)$$

such that $\limsup_{r\to 0}|v(r)| < \infty$. Then there exists a constant c such that

$$v(r) = cU_0(r) + F[f](r), \qquad r \ge 0, \tag{2.5}$$

where

$$F[f](r) := U_0(r)\int_0^r s^{1-N}[U_0(s)]^{-2}\left(\int_0^s \tau^{N-1}U_0(\tau)f(\tau)d\tau\right)ds.$$

Proof. The function

$$\tilde{v}(r) := v(r) - F[f](r)$$

is a solution of (O) such that $\limsup_{r\to 0}|\tilde{v}(r)| < \infty$. Then the property (P) implies (2.5), and Lemma 2.1 follows. □

On the other hand, by similar arguments as in [5]–[8] we have the following lemma.

Lemma 2.2 *Assume condition (V). Let $T > 0$ and ϵ be a sufficiently small positive constant. Let $u = e^{-tH}\phi$ be a solution of (1.1) such that*

$$\|u(t)\|_2 \le C_1(1+t)^{-d}\|\phi\|_2, \qquad t > 0, \tag{2.6}$$

for some constants $C_1 > 0$ *and* $d \geq 0$. *Then there exists a constant* C_2 *such that*

$$|u(x,t)| \leq C_2\|\phi\|_2 \times \begin{cases} (1+t)^{-d-\frac{N}{4}} & \text{if } A > -N/2, \\ (1+t)^{-d-\frac{N}{4}}[\log(2+t)]^{\frac{N}{4}} & \text{if } A = -N/2, \\ (1+t)^{-d-\frac{N}{2(2-N-2A)}} & \text{if } A < -N/2, \end{cases} \tag{2.7}$$

for all $x \in \mathbf{R}^N$ *and* $t > T$ *with* $|x| \geq h_\epsilon(t)$. *Furthermore there exists a constant* C_3 *such that*

$$|u(x,t)| \leq C_3\|\phi\|_2 U_0(|x|) \times \begin{cases} (1+t)^{-d-\frac{N}{4}-\frac{A}{2}} & \text{if } A > -N/2, \\ (1+t)^{-d-\frac{N+2A}{2(2-N-2A)}} & \text{if } A \leq -N/2, \end{cases} \tag{2.8}$$

for all $(x,t) \in D_\epsilon(T)$.

Next we consider the radial solutions of problem (1.1), and give the following proposition.

Proposition 2.1 *Assume condition* (V), *and let* $H := -\Delta + V(|x|)$ *be a subcritical operator on* $L^2(\mathbf{R}^N)$. *Let* ϕ *be a radial function such that* $\phi \in L^2(\mathbf{R}^N, e^{|x|^2/4}dx)$, *and put* $v(t) = e^{-tH}\phi$. *Then there holds the following:*
(i) *There exists a constant* C *such that*

$$\begin{aligned} &\|w(s)\| \leq Ce^{-\frac{\alpha(\omega)}{2}s}\|\phi\|, && s > 0, \\ &\|v(t)\|_{L^2(\mathbf{R}^N,\rho_{N,t}dx)} \leq C(1+t)^{-\frac{\alpha(\omega)}{2}}\|\phi\|, && t > 0, \end{aligned} \tag{2.9}$$

where $\rho_{N,t}(x) = (1+t)^{N/2}\exp(|x|^2/4(1+t))$;
(ii) *There hold*

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} v\left((1+t)^{\frac{1}{2}}y, t\right) = a(\phi)\varphi_0(y) \quad \text{in} \quad L^2(\mathbf{R}^N, e^{|y|^2/4}dy) \tag{2.10}$$

and

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)+l}{2}} (\nabla_x^l v)\left((1+t)^{\frac{1}{2}}y, t\right) = a(\phi)(\nabla_y^l \varphi_0)(y) \quad \text{in} \quad C(\{L^{-1} \leq |y| \leq L\}) \tag{2.11}$$

for any $L > 0$ *and* $l \in \{0,1,2\}$, *where*

$$a(\phi) = c_0 \int_{\mathbf{R}^N} \phi(x)U_0(|x|)dx. \tag{2.12}$$

In particular, if $a(\phi) = 0$, *for any* $L > 0$, *there exists a constant* C_2 *such that*

$$(1+t)^{\frac{N+\alpha(\omega)}{2}} \left|v\left((1+t)^{\frac{1}{2}}y, t\right)\right| \leq C_2(1+t)^{-1} \tag{2.13}$$

for all $L^{-1} \leq |y| \leq L$ *and* $t \geq 1$;
(iii) *There exists a function* $c(t)$ *in* $(0,\infty)$ *satisfying*

$$v(x,t) = c(t)U_0(|x|) + F[(\partial_t v)(\cdot,t)](|x|) \quad \text{in} \quad \mathbf{R}^N \times (0,\infty) \tag{2.14}$$

such that

$$t^{\frac{N}{2}+\alpha(\omega)}c(t) = c_0 a(\phi)(1+o(1)) + O(t^{-1}) \quad as \quad t \to \infty. \tag{2.15}$$

Furthermore there exists a function $d(t)$ in $(0,\infty)$ satisfying

$$t^{\frac{N}{2}+\alpha(\omega)+1}d(t) = -c_0(a(\phi)+o(1))\left(\frac{N}{2}+\alpha(\omega)\right) \quad as \quad t \to \infty \tag{2.16}$$

such that, for any sufficiently small $\epsilon > 0$ and $l \in \{0,1,2\}$,

$$\begin{aligned} &t^{\frac{N}{2}+\alpha(\omega)}\partial_r^l F[(\partial_t v)(\cdot,t)](|x|) \\ &= t^{\frac{N}{2}+\alpha(\omega)}d(t)(\partial_r^l F[U_0])(|x|) + O(t^{-2}|x|^{4-l}U_0(|x|)) = O(t^{-1}|x|^{2-l}U_0(|x|)) \end{aligned} \tag{2.17}$$

for all $(x,t) \in D_\epsilon(1)$.

Proof. Since

$$\alpha(\omega) + \frac{N-2}{2} > 0,$$

we can apply the same argument as in the proof of [6, Proposition 3.1] (see also [6, Theorem 1.1]), and obtain assertion (i). Furthermore, by the same argument as in the proof of [6, Proposition 3.2, Proposition 3.3] we have assertions (ii) and (iii), respectively. We leave the details of the proof to the reader. □.

3 Large time behavior of solutions

In this section we study the large time behavior of solution of (1.1), and prove Theorem 1.1. Put

$$H_N := -\Delta_N + V(|x|), \quad H_{N,k} := -\Delta_N + V(|x|) + \frac{\omega_k}{|x|^2}, \quad \rho_{N,t}(x) := (1+t)^{\frac{N}{2}} e^{\frac{|x|^2}{4(1+t)}},$$

where $k = 1,2,\dots$. Let $u = e^{-tH_N}\phi$ be the solution of (1.1). Then there exists a family of radially symmetric functions $\{\phi_{k,i}\} \subset L^2(\mathbf{R}^N, \rho dx)$ such that

$$\phi = \sum_{k=0}^{\infty}\sum_{i=1}^{l_k} \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\mathbf{R}^N, \rho dx). \tag{3.1}$$

(See [3, Section 6].) For any $k = 0,1,2,\dots$ and $i = 1,\dots,l_k$, let

$$\Phi_{k,i}(x) := \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right), \quad u_{k,i}(x,t) := (e^{-tH_N}\Phi_{k,i})(x), \quad v_{k,i}(x,t) := (e^{-tH_{N,k}}\phi_{k,i})(x).$$

Then we have

$$u_{k,i}(x,t) = v_{k,i}(x,t)Q_{k,i}\left(\frac{x}{|x|}\right). \tag{3.2}$$

Furthermore, putting

$$\tilde{\phi}_{k,i}(x) := |x|^{-k}\phi_{k,i}(x) \in L^2(\mathbf{R}^{N+2k}, \rho dx), \tag{3.3}$$

we have

$$v_{k,i}(x,t) = (e^{-tH_{N,k}}\phi_{k,i})(x) = |x|^k(e^{-tH_{N+2k}}\tilde{\phi}_{k,i})(x). \tag{3.4}$$

For any $m = 0, 1, 2, \dots$, let

$$u_0(x,t) := u(x,t), \quad u_m(x,t) := \sum_{k=m}^{\infty}\sum_{i=1}^{l_k} u_{k,i}(x,t) = u(x,t) - \sum_{k=0}^{m-1}\sum_{i=1}^{l_k} u_{k,i}(x,t).$$

Then we prove the following lemma.

Lemma 3.1 *Assume the same conditions as in Theorem* 1.1. *Let u be the solution of* (1.1). *Then, for any $m = 0, 1, 2, \dots$, there exists a constant C_1 such that*

$$\|u_m(t)\|_{L^2(\mathbf{R}^N, \rho_{N,t}dx)} \le C_1 t^{-\frac{\alpha(\omega+\omega_m)}{2}}\|u_m(0)\| \le C_1 t^{-\frac{\alpha(\omega+\omega_m)}{2}}\|\phi\| \tag{3.5}$$

for all $t > 0$. Furthermore there holds the following:
(i) *For any $\epsilon > 0$, there exists a positive constant L_1 such that*

$$|u_m(x,t)| \le \epsilon t^{-\frac{N+\alpha(\omega+\omega_m)}{2}}\|\phi\| \tag{3.6}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$ with $|x| \ge L_1(1+t)^{1/2}$. Furthermore, for any $L_2 > 0$,

$$\left|u_m\left((1+t)^{\frac{1}{2}}y, t\right)\right| = O\left(t^{-\frac{N+\alpha(\omega+\omega_m)}{2}}\right) \tag{3.7}$$

for all $L_2^{-1} \le |y| \le L_2$ and all sufficiently large t;
(ii) *For any $T > 0$ and any sufficiently small $\epsilon > 0$, there exist constants C_3 and C_4 such that*

$$|u_m(x,t)| \le C_3 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}(1+U_m(|x|))\|\phi\| \le C_4(t^{-\frac{N}{2}-\alpha(\omega+\omega_m)} + t^{-\frac{N}{2}-\frac{\alpha(\omega+\omega_m)}{2}})\|\phi\| \tag{3.8}$$

for all $(x,t) \in D_\epsilon(T)$. Furthermore, for any $L_3 > 0$ and $l \in \{0,1,2\}$, there exists a constant C_5 such that

$$|(\nabla_x^l u_m)(x,t)| \le C_5 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}\|\phi\| \tag{3.9}$$

for all $x \in B(0, L_3)$ and all sufficiently large t.

Here we remark that $\alpha(\omega+\omega_m)$ is not necessarily of definite sign.
Proof. Let $m = 0, 1, 2 \dots$ For any $k \ge m$ and $i = 0, \dots, l_k$, put

$$\tilde{\phi}_{k,i}^m(x) := |x|^{-m}\phi_{k,i}(|x|) \in L^2(\mathbf{R}^{N+2m}, \rho dx)$$

and

$$\tilde{v}_{k,i}(x,t) = (e^{-tH_{N,m}}|\phi_{k,i}|)(x) = |x|^m(e^{-tH_{N+2m}}|\tilde{\phi}^m_{k,i}|)(x) \tag{3.10}$$

(see also (3.4)). Then, since $\omega_k \geq \omega_m$, the comparison principle together with (3.4) and (3.10) yields

$$|v_{k,i}(x,t)| \leq \tilde{v}_{k,i}(x,t) \quad \text{in} \quad \mathbf{R}^N \times (0,\infty). \tag{3.11}$$

Furthermore the operator H_{N+2m} is a subcritical operator on $L^2(\mathbf{R}^{N+2m})$, and we can apply Proposition 2.1 (i) with the dimension N replaced by $N+2m$. Then, by (1.7) and (3.10) we obtain

$$\begin{aligned} \|\tilde{v}_{k,i}(t)\|_{L^2(\mathbf{R}^N,\rho_{N,t}dx)} &= \frac{|\mathbf{S}^{N-1}|^{1/2}}{|\mathbf{S}^{N+2m-1}|^{1/2}}(1+t)^{-\frac{m}{2}}\|e^{-tH_{N+2m}}|\tilde{\phi}^m_{k,i}|\|_{L^2(\mathbf{R}^{N+2m},\rho_{N+2m,t}dx)} \\ &\leq C_1\frac{|\mathbf{S}^{N-1}|^{1/2}}{|\mathbf{S}^{N+2m-1}|^{1/2}}t^{-\frac{m}{2}-\frac{\alpha_{N+2m}(\omega)}{2}}\|\tilde{\phi}^m_{k,i}\|_{L^2(\mathbf{R}^{N+2m},\rho dx)} = C_1 t^{-\frac{\alpha(\omega+\omega_m)}{2}}\|\phi_{k,i}\| \end{aligned} \tag{3.12}$$

for all $t \geq 1$, where C_1 is a constant independent of k and i. Furthermore we have

$$\|e^{-tH_{N+2m}}|\tilde{\phi}^m_{k,i}|\|_{L^2(\mathbf{R}^{N+2m})} \preceq t^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_m)}{2}}\|\phi_{k,i}\| \tag{3.13}$$

for all sufficiently large t. By (3.13), applying (2.8) with the dimension N replaced by $N+2m$, for any $T > 0$ and any sufficiently small $\epsilon > 0$, we obtain

$$|e^{-tH_{N+2m}}|\tilde{\phi}^m_{k,i}|(x)| \leq C_2 t^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_m)}{2}}t^{-\frac{N+2m}{4}-\frac{\alpha_{N+2m}(\omega)}{2}}U_{N+2m,0}(|x|)\|\phi_{k,i}\|$$

for all $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x| \leq C_3\epsilon^{1/2}(1+t)^{1/2}$, where C_2 and C_3 are constants independent of k and i. This together with (1.7), (1.13), (3.10), and (3.11) implies

$$|v_{k,i}(x,t)| \leq \tilde{v}_{k,i}(x,t) \leq C_2 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}U_{N,m}(|x|)\|\phi_{k,i}\| \tag{3.14}$$

for all $(x,t) \in \mathbf{R}^N \times (T,\infty)$ with $|x| \leq C_3\epsilon^{1/2}(1+t)^{1/2}$. In addition, for any $L > 0$, by (1.7), (2.11), (3.10), and (3.11) we obtain

$$\begin{aligned} \left|v_{k,i}\left((1+t)^{\frac{1}{2}}y,t\right)\right| &\leq \tilde{v}_{k,i}\left((1+t)^{\frac{1}{2}}y,t\right) \\ &= (1+t)^{\frac{m}{2}}|y|^m(e^{-tH_{N+2m}}|\tilde{\phi}^m_{k,i}|)\left((1+t)^{\frac{1}{2}}y,t\right) \preceq t^{\frac{m}{2}}t^{-\frac{N+2m+\alpha_{N+2m}(\omega)}{2}} = t^{-\frac{N+\alpha(\omega+\omega_m)}{2}} \end{aligned} \tag{3.15}$$

for all $L^{-1} \leq |y| \leq L$ and all sufficiently large t.

We prove (3.5). By the orthonormality of $\{Q_{k,i}\}$, (3.2), (3.11), and (3.12) we have

$$\begin{aligned} \|u_m(t)\|^2_{L^2(\mathbf{R}^N,\rho_t dx)} &= \sum_{k=m}^{\infty}\sum_{i=1}^{l_k}\|u_{k,i}(t)\|^2_{L^2(\mathbf{R}^N,\rho_t dx)} \\ &\leq C_4\sum_{k=m}^{\infty}\sum_{i=1}^{l_k}\|v_{k,i}(t)\|^2_{L^2(\mathbf{R}^N,\rho_t dx)} \leq C_4\sum_{k=m}^{\infty}\sum_{i=1}^{l_k}\|\tilde{v}_{k,i}(t)\|^2_{L^2(\mathbf{R}^N,\rho_t dx)} \\ &\leq C_5 t^{-\alpha(\omega+\omega_m)}\sum_{k=m}^{\infty}\sum_{i=1}^{l_k}\|\phi_{k,i}\|^2 \leq C_6 t^{-\alpha(\omega+\omega_m)}\sum_{k=m}^{\infty}\sum_{i=1}^{l_k}\|\Phi_{k,i}\|^2 = C_6 t^{-\alpha(\omega+\omega_m)}\|u_m(0)\|^2 \end{aligned}$$

for all $t \geq 1$, where C_4, C_5, and C_6 are constants. Therefore, since $\|u_m(0)\| \leq \|\phi\|$, we have (3.5). Furthermore, by (3.5) we apply the similar argument as in the proof of (2.7) to obtain (3.6) (see also the proof of Lemma 4.1 in [6]).

Next we prove (3.7) and (3.8). Let M be a sufficiently large integer such that

$$\alpha(\omega+\omega_M)+\alpha(\omega)\geq 2\alpha(\omega+\omega_m). \tag{3.16}$$

Inequality (3.5) implies that

$$\|u_M(t)\|_2 \preceq t^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_M)}{2}}\|u_M(0)\|$$

for all sufficiently large t. This together with (3.16) implies

$$\begin{aligned}\|u_M(t)\|_\infty &\leq \|e^{-tH/2}\|_{q,2}\|u_M(t/2)\|_2 \\ &\preceq t^{-\frac{N}{2}-\frac{\alpha(\omega)}{2}-\frac{\alpha(\omega+\omega_M)}{2}}\|u_M(0)\|_2 \preceq t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}\|\phi\|_2\end{aligned} \tag{3.17}$$

for all $t > T$. Then, since it follows from the definition of u_m and (3.17) that

$$\begin{aligned}|u_m(x,t)| &\leq \sum_{k=m}^{M-1}\sum_{i=1}^{l_k}|v_{k,i}(x,t)|\left|Q_{k,i}\left(\frac{x}{|x|}\right)\right| + |u_M(x,t)| \\ &\preceq \sum_{k=m}^{M-1}\sum_{i=1}^{l_k}|v_{k,i}(x,t)| + t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}\|\phi\|\end{aligned}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large t, by (3.14) and (3.15) we have (3.7) and (3.8). Furthermore (3.8) implies (3.9) with $l = 0$. Moreover, by (3.8) we apply the regularity theorems for the parabolic equations, and obtain (3.9) with $l = 1, 2$. Thus Lemma 3.1 follows. □

Next we give a lemma on the asymptotics of $u_{0,1}$ and $u_{1,i}$ $(i = 1, \dots, N)$. Lemma 3.2 is proved by Proposition 2.1.

Lemma 3.2 *Assume the same conditions as in Theorem* 1.1. *Let* $i = 1, \dots, N$. *Then there hold*

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u_{0,1}\left((1+t)^{\frac{1}{2}}y,t\right) = c_0 M_0 \varphi_0(y), \tag{3.18}$$

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_1)}{2}} u_{1,i}\left((1+t)^{\frac{1}{2}}y,t\right) = c_1 N M_i \varphi_1(y)\frac{y_i}{|y|}, \tag{3.19}$$

in $C_{loc}(\mathbf{R}^N\setminus\{0\})$ *and* $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$. *Furthermore, for any* $l = 0, 1, 2$ *and any sufficiently small* $\epsilon > 0$, *there hold*

$$\begin{aligned} t^{\frac{N}{2}+\alpha(\omega)}(\nabla_x^l u_{0,1})(x,t) &= c_0^2(M_0+o(1))(\nabla_x^l U_0)(x) \\ &\quad -c_0^2\left(\frac{N}{2}+\alpha(\omega)\right)t^{-1}(M_0+o(1))(\nabla_x^l F[U_0])(x) + O(t^{-2}|x|^{4-l}U_0(|x|)),\end{aligned} \tag{3.20}$$

$$t^{\frac{N}{2}+\alpha(\omega+\omega_1)}(\nabla_x^l u_{1,i})(x,t) = c_1^2 N(M_i+o(1))(\nabla_x^l Z_i)(x) + O(t^{-1}|x|^{2-l}U_1(|x|)), \tag{3.21}$$

as $t \to \infty$, *uniformly for all* $x \in \mathbf{R}^N$ *with* $|x| \leq \epsilon t^{1/2}$. *Here* $Z_i(x) := U_1(|x|)x_i/|x|$.

Proof. By (1.5), (1.14), (2.12), (3.1), and the orthonormality of $\{Q_{k,i}\}$ we have

$$a(\phi_{0,1}) = \frac{c_0}{\kappa_0}\int_{\mathbf{R}^N}\kappa_0\phi_{0,1}(x)U_0(|x|)dx = \frac{c_0}{\kappa_0}\int_{\mathbf{R}^N}\phi(x)U_0(|x|)dx = \frac{c_0}{\kappa_0}M_0.$$

Then, since $u_{0,1}(x,t) = \kappa_0 v_{0,1}(x,t)$, we apply Proposition 2.1 to the function $v_{0,1}(x,t)$, and we obtain (3.18) and (3.20).

We prove (3.19) and (3.21). Let $i = 1,\dots,N$. By (1.13), (1.15), and (3.3) we have

$$\begin{aligned}\tilde{a}(\tilde{\phi}_{1,i}) &:= c_{N+2,0}\int_{\mathbf{R}^{N+2}}\tilde{\phi}_{1,i}(x)U_{N+2,0}(|x|)dx = c_{N+2,0}\frac{|\mathbf{S}^{N+1}|}{|\mathbf{S}^{N-1}|}\int_{\mathbf{R}^N}\phi_{1,i}(x)U_1(|x|)dx\\ &= c_1\frac{|\mathbf{S}^{N+1}|^{1/2}}{|\mathbf{S}^{N-1}|^{1/2}}\int_{\mathbf{R}^N}\phi_{1,i}(x)U_1(|x|)dx\\ &= c_1\frac{|\mathbf{S}^{N+1}|^{1/2}}{|\mathbf{S}^{N-1}|^{1/2}}N\kappa_1^{-1}\int_{\mathbf{R}^N}\kappa_1\phi_{1,i}(x)U_1(|x|)\frac{x_i^2}{|x|^2}dx.\end{aligned}$$

Then, by (1.5), (1.14), (3.1), and the orthonormality of $\{Q_{k,i}\}$ we have

$$\tilde{a}(\tilde{\phi}_{1,i}) = c_1\frac{|\mathbf{S}^{N+1}|^{1/2}}{|\mathbf{S}^{N-1}|^{1/2}}N\kappa_1^{-1}M_i. \tag{3.22}$$

On the other hand, applying Proposition 2.1 (ii) with the dimension N replaced by $N+2$ to the function $\hat{v}_{1,i}(x,t) := (e^{-tH_{N+2}}\tilde{\phi}_{1,i})(x)$, by (1.15) and (3.22) we obtain

$$\lim_{t\to\infty} t^{\frac{N+2+\alpha_{N+2}(\omega)}{2}}\hat{v}_{1,i}\left((1+t)^{1/2}y,t\right) = \tilde{a}(\tilde{\phi}_{1,i})\varphi_{N+2,0}(y) = c_1N\kappa_1^{-1}M_i|y|^{-1}\varphi_1(y) \tag{3.23}$$

in $C_{loc}(\mathbf{R}^{N+2}\setminus\{0\})$ and $L^2(\mathbf{R}^{N+2}, e^{|y|^2/4}dy)$. Similarly, applying Proposition 2.1 (iii), by (1.7), (1.13), (1.15), and (3.22) we obtain

$$\begin{aligned}(\nabla_x^l\hat{v}_{1,i})(x,t) &= c_i(t)(\nabla_x^lU_{N+2,0})(x) + O(t^{-\frac{N+2}{2}-\alpha_{N+2}(\omega)-1}|x|^{2-l}U_{N+2,0}(|x|))\\ &= c_i(t)\nabla_x^l\left[\frac{U_1(|x|)}{|x|}\right] + O(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)-1}|x|^{2-l}|x|^{-1}U_1(|x|))\end{aligned} \tag{3.24}$$

as $t\to\infty$, uniformly for all $x\in\mathbf{R}^N$ with $|x|\le\epsilon t^{1/2}$, where

$$\begin{aligned}c_i(t) &= c_{N+2,0}t^{-\frac{N+2}{2}-\alpha_{N+2}(\omega)}(\tilde{a}(\tilde{\phi}_{1,i})+o(1))\\ &= c_1^2N\kappa_1^{-1}t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}(M_i+o(1)) \quad\text{as}\quad t\to\infty.\end{aligned} \tag{3.25}$$

Furthermore, since it follows from (1.5), (3.2), and (3.4) that

$$u_{1,i}(x,t) = |x|\hat{v}_{1,i}(x,t)\cdot\kappa_1\frac{x_i}{|x|} = \kappa_1x_i\hat{v}_{1,i}(x,t),$$

by (1.7), (3.23), (3.24), and (3.25) we have

$$\begin{aligned}&\lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_1)}{2}}u_{1,i}\left((1+t)^{1/2}y,t\right)\\ &= \lim_{t\to\infty} t^{\frac{N+1+\alpha_{N+2}(\omega)}{2}}u_{1,i}\left((1+t)^{1/2}y,t\right) = c_1NM_i\varphi_1(y)\frac{y_i}{|y|}\end{aligned}$$

in $C_{loc}(\mathbf{R}^N \setminus \{0\})$ and $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$ and

$$(\nabla_x^l u_{1,i})(x,t) = c_1^2 N t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}(M_i + o(1))(\nabla_x^l Z_i)(x) + O(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)-1}|x|^{2-l}U_1(|x|))$$

as $t \to \infty$, uniformly for all $x \in \mathbf{R}^N$ with $|x| \le \epsilon t^{1/2}$. Thus we have (3.19) and (3.21), and the proof of Lemma 3.2 is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By (3.5) with $m = 0$ we have (1.16). Since $u(x,t) = u_{0,1}(x,t) + u_1(x,t)$, by (3.9) with $l = 0$ and (3.20), for any $L > 0$, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega)} u(x,t) = \lim_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega)} u_{0,1}(x,t) = c_0^2 M U_0(|x|)$$

in $C(B(0,L))$, and obtain (1.17). Furthermore, applying (3.5) and (3.7) to the function u_1, by (3.18) we have

$$\lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u\left((1+t)^{1/2}y, t\right) = \lim_{t\to\infty} t^{\frac{N+\alpha(\omega)}{2}} u_{0,1}\left((1+t)^{1/2}y, t\right) = c_0 M_0 \varphi_0(y)$$

in $C_{loc}(\mathbf{R}^N \setminus \{0\})$ and in $L^2(\mathbf{R}^N, e^{|y|^2/4}dy)$. This implies (1.18), and Theorem 1.1 follows. □

4 Movement of hot spots

In this section we study the behavior of hot spots of the solution u of (1.1), and prove Theorem 1.2. In what follows we write $\alpha_k = \alpha_N(\omega + \omega_k)$ for simplicity.

Assume the same conditions as in Theorem 1.2. We first prove that $H(t) \ne \emptyset$ for all $t > 0$. Since

$$\int_{\mathbf{R}^N} u(x,t_0)U_0(|x|)dx = \int_{\mathbf{R}^N} \phi(x)U_0(|x|)dx = M_0 > 0, \qquad t_0 > 0,$$

for any $t_0 > 0$, there exists a point x_0 such that $u(x_0,t_0) > 0$. On the other hand, by (3.6) we can find a constant L such that

$$|u(x,t_0)| < u(x_0,t_0) \quad \text{for all} \quad |x| \ge L.$$

This implies that $\emptyset \ne H(t_0) \subset B(0,L)$.

Next we study the behavior of $A(t)$ and the hot spots $H(t)$, and prove Theorem 1.2 (i) and (ii).

Proof of Theorem 1.2 (i). By (1.18) we have

$$\lim_{t\to\infty} (1+t)^{\frac{\alpha_0}{2}} \int_{\mathbf{R}^N} u(x,t)dx = c_0 M_0 \int_{\mathbf{R}^N} \varphi_0(y)dy > 0, \tag{4.1}$$

and see that $\int_{\mathbf{R}^N} u(x,t)dx > 0$ for all sufficiently large t. Then $A(t)$ can be defined for all sufficiently large t. Furthermore, since it follows from (3.5) that

$$\int_{\mathbf{R}^N} |x||u_2(x,t)|dx \le \left(\int_{\mathbf{R}^N} |x|^2 \rho_t(x)^{-1}dx\right)^{1/2} \left(\int_{\mathbf{R}^N} |u_2(x,t)|^2 \rho_t(x)dx\right)^{1/2} \preceq t^{-\frac{\alpha_2}{2}+\frac{1}{2}}$$

for all sufficiently large t, by the radial symmetry of $u_{0,1}$ and (3.19) we obtain

$$(4.2)\quad \begin{aligned}&(1+t)^{\frac{\alpha_1-1}{2}}\int_{\mathbf{R}^N} x_i u(x,t)dx\\ &= (1+t)^{\frac{\alpha_1-1}{2}}\int_{\mathbf{R}^N} x_i u_{1,i}(x,t)dx + (1+t)^{\frac{\alpha_1-1}{2}}\int_{\mathbf{R}^N} x_i u_2(x,t)dx\\ &= (1+t)^{\frac{N+\alpha_1}{2}}\int_{\mathbf{R}^N} y_i u_{1,i}\left((1+t)^{\frac{1}{2}}y,t\right)dy + o(1) = c_1 N M_i \int_{\mathbf{R}^N}\varphi_1(y)\frac{y_i^2}{|y|}dy + o(1)\end{aligned}$$

as $t \to \infty$, where $i = 1, \ldots, N$. Since

$$(4.3)\qquad \alpha(\omega+\omega_k) > \alpha(\omega) + k, \qquad k = 1,2,3,\ldots,$$

we have $\alpha_1 > \alpha_0 + 1$ for the case $\omega < 0$, and by (4.1) and (4.2) we have

$$(4.4)\qquad \lim_{t\to\infty} A(t) = 0 \quad \text{if} \quad \omega < 0.$$

On the other hand, if $\omega = 0$, then $\alpha_0 = 0$, $\alpha_1 = 1$, $c_0 \int_{\mathbf{R}^N} \varphi_0(y)dy = \|\varphi_0\|^2 = 1$, and

$$c_1\int_{\mathbf{R}^N}\varphi_1(y)\frac{y_i^2}{|y|}dy = c_1^2\int_{\mathbf{R}^N} e^{-\frac{|y|^2}{4}} y_i^2 dy = \frac{c_1^2}{N}\int_{\mathbf{R}^N} e^{-\frac{|y|^2}{4}}|y|^2 dy = \frac{1}{N}\|\varphi_1\|^2 = \frac{1}{N},$$

and by (4.1) and (4.2) we obtain

$$(4.5)\qquad \lim_{t\to\infty} A(t) = \left(\frac{M_1}{M_0}, \ldots, \frac{M_N}{M_0}\right).$$

Therefore, by (4.4) and (4.5) we obtain (1.19), and Theorem 1.2 (i) follows. □

Proof of Theorem 1.2 (ii). We first prove

$$(4.6)\qquad \lim_{t\to\infty} \sup\{|x| : x \in H(t)\} \le R_*.$$

Since $M_0 > 0$ and $\alpha_0 \le 0$, by (1.17) and (3.6) we can take a sufficiently large L so that

$$(4.7)\qquad t^{\frac{N}{2}+\alpha_0}u(0,t) \ge \frac{1}{2}c_0^2 M_0 U_0(0) > t^{\frac{N}{2}+\alpha_0} \sup_{|x|\ge L(1+t)^{1/2}} u(x,t)$$

for all sufficiently large t. Furthermore, for any sufficiently small $\epsilon > 0$, it follows from (1.18), $M_0 > 0$, and the monotonicity of the function φ_0 that

$$(4.8)\qquad \sup_{\epsilon^{1/2}(1+t)^{1/2}\le|x|\le L(1+t)^{1/2}} u(x,t) < \inf_{|x|=2^{-1}\epsilon^{1/2}(1+t)^{1/2}} u(x,t)$$

for all sufficiently large t. By (4.7) and (4.8) we have

$$(4.9)\qquad H(t) \subset B(0, \epsilon^{1/2}(1+t)^{1/2})$$

for all sufficiently large t. On the other hand, by (2.3) and the definition of R_* we have

(4.10) $$U_0(r) = U_0(0) \quad \text{in} \quad r \in [0, R_*], \qquad U_0'(r) < 0 \quad \text{in} \quad r \in (R_*, \infty).$$

Then, by (3.8) with $m = 1$, (3.20), and (4.10), for any $\delta > 0$, we have

$$t^{\frac{N}{2}+\alpha_0} \sup_{R_*+\delta<|x|\le\epsilon^{1/2}(1+t)^{1/2}} u(x,t) = c_0^2(M_0 + o(1))U_0(R_* + \delta) + o(1) < t^{\frac{N}{2}+\alpha_0}u(0,t)$$

for all sufficiently large t. This together with (4.9) and the arbitrariness of δ implies (4.6). In particular, by (4.6) we have (1.20) for the case $R_* = 0$.

Next we prove (1.20) for the case $R_* > 0$. We divide the proof into the following three cases:

$$\text{(a)} \quad \omega < 0; \qquad \text{(b)} \quad \omega = 0 \text{ and } |\mathcal{M}| < R_*; \qquad \text{(c)} \quad \omega = 0 \text{ and } |\mathcal{M}| \ge R_*.$$

We consider case (a). Let $0 < \delta < R_* < R$. Then, by (1.11) and the definition of F we can take a constant C_1 satisfying

(4.11) $$F[U_0](r) \ge C_1, \qquad r \in [\delta, R].$$

Since $F[U_0](0) = 0$, $U_0'(r) \le 0$, and $\alpha_0 > -N/2$, by (3.9) with $m = 1$, (3.20), (4.3), and (4.11) we have

$$\begin{aligned} & t^{\frac{N}{2}+\alpha_0}[u(x,t) - u(0,t)] \\ & \le -c_0^2\left(\frac{N}{2} + \alpha_0\right)t^{-1}(M_0 + o(1))F[U_0](|x|) + O(t^{-2}) + O(t^{\alpha_0-\alpha_1}) \\ & \le -C_2t^{-1} + C_3t^{\alpha_0-\alpha_1} < 0 \end{aligned}$$

for all $x \in B(0,R)\setminus B(0,\delta)$ and all sufficiently large t, where C_2 and C_3 are positive constants. This together with (4.6) implies that $H(t) \subset B(0,\delta)$ for all sufficiently large t. Therefore, since δ is arbitrary, we have (1.20) for case (a).

Next we consider case (b). By $\omega = 0$ we have $c_0^2 = (4\pi)^{-\frac{N}{2}}$, $c_1^2 = c_0^2/2N$, and

(4.12) $$U_0(r) = U_0(0), \quad U_1(r) = U_1'(0)r, \quad F[U_0](r) = \frac{U_0(0)}{2N}r^2$$

for all $r \in [0, R_*]$. Furthermore, by (3.9) we have

(4.13) $$\sup_{x\in\overline{B(0,R)}} |u_2(x,t)| = O(t^{-\frac{N}{2}-\alpha(\omega_2)}) = O(t^{-\frac{N}{2}-2})$$

for any $R > 0$. Since

$$x_i^* = \frac{U_1'(0)}{U_0(0)}\frac{M_i}{M_0}, \qquad i = 1,\dots,N,$$

by (3.20), (3.21), (4.12), and (4.13) we have

(4.14) $$\begin{aligned} & (4\pi t)^{\frac{N}{2}}t\,[u(x^*,t) - u(x,t)] \\ & = \frac{U_0(0)}{4}(M_0 + o(1))(|x|^2 - |x^*|^2) + \sum_{i=1}^{N}\frac{U_1'(0)}{2}(M_i + o(1))(x_i^* - x_i) + O(t^{-1}) \\ & = \frac{U_0(0)}{4}M_0\sum_{i=1}^{N}\left[x_i^2 - (x_i^*)^2 - 2x_i^*(x_i - x_i^*)\right] + o(1) = \frac{U_0(0)}{4}M_0|x - x^*|^2 + o(1) \end{aligned}$$

for all $x \in \overline{B(0, R_*)}$ and all sufficiently large t.

Let $\delta_1 > 0$ and $x \in B(0, R_* + \delta_1)$ with $|x| > R_*$. Put $\tilde{x} = R_* x/|x|$ for $x \in \mathbf{R}^N \setminus \{0\}$. Since $|\tilde{x}| = R_*$ and $|x_*| = |\mathcal{M}| < R_*$, by (4.14) we can find a positive constant C_4 satisfying

$$(4.15) \qquad (4\pi t)^{\frac{N}{2}} t\,[u(x^*, t) - u(\tilde{x}, t)] \geq C_4$$

for all sufficiently large t. Furthermore, by (3.20), (3.21), (4.10), (4.13), and the continuity of the functions $F[U_0](r)$ and $U_1(r)$ at $r = R_*$, taking a sufficiently small δ_1 if necessary, we have

$$(4.16) \qquad \begin{aligned} &(4\pi t)^{\frac{N}{2}} t\,[u(\tilde{x}, t) - u(x, t)] \\ &\geq -\frac{N}{2}(M_0 + o(1))\,\{F[U_0](\tilde{x}) - F[U_0](x)\} \\ &\quad + \sum_{i=1}^{N} \frac{x_i}{2}(M_i + o(1)) \left\{ \frac{U_1(|\tilde{x}|)}{R_*} - \frac{U_1(|x|)}{|x|} \right\} + O(t^{-1}) \geq -\frac{C_4}{2} \end{aligned}$$

for all sufficiently large t. This together with (4.15) yields

$$(4.17) \qquad \begin{aligned} &(4\pi t)^{\frac{N}{2}} t\,[u(x^*, t) - u(x, t)] \\ &= (4\pi t)^{\frac{N}{2}} t\,[u(x^*, t) - u(\tilde{x}, t)] + (4\pi t)^{\frac{N}{2}} t\,[u(\tilde{x}, t) - u(x, t)] \geq \frac{C_4}{2} > 0 \end{aligned}$$

for all $x \in B(0, R_* + \delta_1)$ with $|x| > R_*$ and all sufficiently large t. Therefore, since

$$(4\pi t)^{\frac{N}{2}} t\,[u(x^*, t) - u(x, t)] \leq 0 \qquad \text{if} \qquad x \in H(t),$$

by (4.6), (4.14), and (4.17) we obtain (1.20) for case (b).

Next we consider case (c). Then we can assume, without loss of generality, that $\mathcal{M} = (|\mathcal{M}|, 0, \ldots, 0)$. Then, since

$$x^* = (R_*, 0, \ldots, 0), \qquad \gamma_N \frac{M_1}{M_0} = \frac{U_1'(0) M_1}{U_0(0) M_0} \geq R_*,$$

by the same argument as in (4.14) we have

$$\begin{aligned} &(4\pi t)^{\frac{N}{2}} t\,[u(x^*, t) - u(x, t)] \\ &= \frac{U_0(0)}{4}(M_0 + o(1))(|x|^2 - |x^*|^2) + \frac{U_1'(0)}{2}(M_1 + o(1))(R_* - x_1) + O(t^{-1}) \\ &= \frac{U_0(0)}{4} M_0 |x - x^*|^2 + o(1) \end{aligned}$$

for all $x \in B(0, R_*)$ and all sufficiently large t. This implies that, for any $\delta_2 > 0$,

$$(4.18) \qquad \{x \in B(0, R_*) \,:\, |x - x^*| > \delta_2\} \cap H(t) = \emptyset$$

for all sufficiently large t.

Let $\theta > 0$ and put

$$C(\theta) := \left\{ x \in \mathbf{R}^N \setminus \{0\} : \frac{x_1}{|x|} < 1 - \theta \right\}.$$

Then, similarly to (4.16), by (3.20), (3.21), (4.10), (4.13), and the continuity of the functions $F[U_0](r)$ and $U_1(r)$ at $r = R_*$, taking a sufficiently small $\delta_3 > 0$, we see that there exist positive constant C such that

$$(4\pi t)^{\frac{N}{2}} t\,[u(x^*,t) - u(x,t)]$$
$$\ge -\frac{N}{2} M_0[F[U_0](R_*) - F[U_0](|x|)] + \frac{M_1}{2}\left[U_1(R_*) - U_1(|x|)\frac{x_1}{|x|}\right] + o(1) \ge \frac{M_1\theta}{4} U_1(R_*)$$

for all $x \in C(\theta) \cap [B(0, R_* + \delta_3) \setminus B(0, R_*)]$ and all sufficiently large t. This implies that

$$\{x \in C(\theta) : R_* \le |x| < R_* + \delta_3\} \cap H(t) = \emptyset \tag{4.19}$$

for all sufficiently large t. Therefore, since θ and δ_3 are arbitrary, by (4.6), (4.18), and (4.19) we have

$$\lim_{t\to\infty} \sup\{|x - R_* e_1| : x \in H(t)\} = 0,$$

and obtain (1.20) for case (c). Therefore the proof of Theorem 1.2 (iii) is complete, and Theorem 1.2 follows. □

5 Number of hot spots

In this section we study the number of hot spots by obtaining the large time behavior of the Hesse matrix of the solution u near its hot spots, and prove Theorem 1.3. The proof of Theorem 1.3 is divided into the following cases:

(a) $R_* = 0$ and $V(0) \ne 0$; (b) $R_* = 0$ and $V(0) = 0$;
(c) $R_* > 0$ and $x^* \in B(0, R_*)$; (d) $R_* > 0$ and $x^* \notin B(0, R_*)$.

Proof of Theorem 1.3 for case (a). By (2.4) we have

$$U_0''(0) = \lim_{r\to 0} \frac{U_0'(r)}{r} = \frac{1}{N} V(0) U_0(0) < 0. \tag{5.1}$$

Then, for any sufficiently small $\delta > 0$, there exists a positive constant C_1 such that

$$\xi \cdot (\nabla_x^2 U_0)(x)\xi \le -C_1 < 0, \qquad \xi \in \mathbf{S}^{N-1}, \tag{5.2}$$

for all $x \in B(0,\delta)$. Therefore, by (3.9) with $m = 1$, (3.20), and (5.2) we have

$$\begin{aligned} &\xi \cdot t^{\frac{N}{2}+\alpha_0} (\nabla_x^2 u)(x,t)\xi \\ &= c_0(M_0 + o(1))\xi \cdot (\nabla_x^2 U_0)(x)\xi + o(1) \le -\frac{1}{2} c_0^2 M_0 C_1 < 0, \qquad \xi \in \mathbf{S}^{N-1}, \end{aligned} \tag{5.3}$$

for all $x \in B(0,\delta)$ and all sufficiently large t. On the other hand, Theorem 1.2 implies that $H(t) \subset B(0,\delta)$ for all sufficiently large t. Therefore, due to (5.3), any maximum point is non-degenerate and we see that $H(t)$ consists of only one point for all sufficiently large t. Furthermore, by the implicit function theorem we see that there exist a constant $T > 0$ and a curve $x(t) \in C^1([T,\infty) : \mathbf{R}^N)$ such that $H(t) = \{x(t)\}$ for $t \ge T$. Therefore the proof of

Theorem 1.3 for case (a) is complete. □

Proof of Theorem 1.3 for case (b). By Theorem 1.2 we have $|x^*| = 0 = R_*$. Due to the assumption of Theorem 1.3, $-V(r)$ is monotone increasing in $[0,\delta]$ for some $\delta > 0$. Then, by (2.3) we have

$$0 \le -U_0'(r) \le -\frac{V(r)U_0(0)}{N}r, \qquad r \in [0,\delta]. \tag{5.4}$$

This together with (O) and the continuity of U_0 implies

$$\begin{aligned} U_0''(r) &= -\frac{N-1}{r}U_0'(r) + V(r)U_0(r) \\ &\le -V(r)\left[\frac{N-1}{N}U_0(0) - U_0(r)\right] \le \frac{1}{N}V(r)(U_0(0)+o(1)) \le \frac{1}{2N}V(r)U_0(0) \le 0 \end{aligned} \tag{5.5}$$

for all sufficiently small $r \ge 0$. On the other hand, by (O), (5.4), and (5.5) we can take a sufficiently small $\delta > 0$ so that

$$\begin{aligned} \xi\cdot(\nabla_x^2 U_0)(x)\xi &= \frac{U_0'(|x|)}{|x|}|\xi|^2 + \left[U_0''(|x|) - \frac{U_0'(|x|)}{|x|}\right]\xi\cdot\left[\frac{x_ix_j}{|x|^2}\right]_{i,j=1}^N \xi \\ &= \frac{U_0'(|x|)}{|x|}\left[1-\left(\sum_{i=1}^N \frac{x_i}{|x|}\xi_i\right)^2\right] + U_0''(r)\left(\sum_{i=1}^N\frac{x_i}{|x|}\xi_i\right)^2 \le 0 \end{aligned} \tag{5.6}$$

for all $x \in B(0,\delta)$ and $\xi \in \mathbf{S}^{N-1}$. Furthermore, since

$$\begin{aligned} &F[U_0](0) = 0, \quad F[U_0]'(0) = 0, \\ &F[U_0]''(0) = \lim_{r\to 0} r^{-1}F[U_0]'(r) = \frac{1}{N}U_0(0) > 0, \end{aligned}$$

by the similar argument as in (5.6), taking a sufficiently small δ if necessary, we have

$$\begin{aligned} &\xi\cdot(\nabla_x^2 F[U_0])(|x|)\xi \\ &= \frac{F[U_0]'(|x|)}{|x|} + \left[F[U_0]''(|x|) - \frac{F[U_0]'(|x|)}{|x|}\right]\left(\sum_{i=1}^N\frac{x_i}{|x|}\xi_i\right)^2 \ge \frac{1}{2N}U_0(0) \end{aligned} \tag{5.7}$$

for all $x \in B(0,\delta)$ and $\xi \in \mathbf{S}^{N-1}$. On the other hand, by (2.4), (5.1), and $V(0)=0$ we have $U'_{N+2,0}(0) = U''_{N+2,0}(0) = 0$. Then, since

$$Z_i(x) = \frac{x_i}{|x|}U_1(|x|) = \frac{x_i}{|x|}\cdot|x|U_{N+2,0}(|x|) = x_iU_{N+2,0}(|x|),$$

we have

$$(\nabla_x^2 Z_i)(0) = 0. \tag{5.8}$$

Then, for any $\epsilon > 0$, since $\alpha_2 > \alpha_1 \ge \alpha_0 + 1$ and $\alpha_0 > -N/2$, by (3.9) with $m = 2$, (3.20), (3.21), (5.6), and (5.8), taking a sufficiently small δ if necessary, we have

$$\begin{aligned} &t^{\frac{N}{2}+\alpha_0+1}\xi\cdot(\nabla_x^2 u)(x,t)\xi \\ &\le -c_0^2\left(\frac{N}{2}+\alpha_0\right)M_0\xi\cdot(\nabla_x^2F[U_0])(x,t)\xi + o(1) + \epsilon t^{\alpha_0+1-\alpha_1}, \quad \xi\in\mathbf{S}^{N-1}, \end{aligned} \tag{5.9}$$

for all $x \in B(0,\delta)$ and all sufficiently large t. Therefore, taking a sufficiently small δ if necessary, by (5.7) and (5.9) we have

$$t^{\frac{N}{2}+\alpha_0+1}\xi\cdot(\nabla_x^2 u)(x,t)\xi \le -c_0^2 M_0\left(\frac{N}{2}+\alpha_0\right)\frac{U_0(0)}{4N}<0, \quad \xi\in\mathbf{S}^{N-1},$$

for all $x \in B(0,\delta)$ and all sufficiently large t. Since δ is arbitrary and $x^*=0$, by the same argument as in the proof for case (a) we obtain the desired conclusion, and the proof of Theorem 1.3 for case (b) is complete. □

Proof of Theorem 1.3 for case (c). Since (4.12) remains true in case (c), we have

$$(5.10)\qquad (\nabla_x^2 U_0)(x)=0,\qquad (\nabla_x^2 F[U_0])(x)=\frac{U_0(0)}{N}I_N,\qquad (\nabla_x^2 Z_i)(x)=0$$

in $B(0,R_*)$, where I_N is the identity matrix on $\mathbf{R}^N$. Therefore, since $\alpha_2>\alpha_1\ge\alpha_0+1$, by (3.9) with $m=2$, (3.20), (3.21), and (5.10) we have

$$(5.11)\qquad (4\pi t)^{\frac{N}{2}+\alpha_0+1}\xi\cdot(\nabla_x^2 u)(x,t)\xi=-\frac{M_0U_0(0)}{2}|\xi|^2+o(1)\le-\frac{M_0U_0(0)}{4},\quad \xi\in\mathbf{S}^{N-1},$$

for all $x\in B(0,R_*)$ and all sufficiently large t. Then, since $H(t)\subset B(0,R_*)$ for all sufficiently large t, by the same argument as in the proof of case (a) we obtain the desired conclusion, and the proof of Theorem 1.3 for case (c) is complete. □

Proof of Theorem 1.3 for case (d). By Theorem 1.2 we see $\omega=0$. Due to the assumption of Theorem 1.3, $-V$ is a monotone increasing positive function in $(R_*,R_*+\delta)$ for some $\delta>0$. Then, by (2.3) we have

$$(5.12)\qquad 0\le -U_0'(r)\le-\frac{1}{N}V(r)U_0(R_*)\left(r-\left(\frac{R_*}{r}\right)^{N-1}R_*\right),\qquad r\in(R_*,R_*+\delta).$$

By the similar argument as in (5.5), taking a sufficiently small $\delta>0$ if necessary, we have $U_0''(r)\le 0$ for $r\in[R_*,R_*+\delta)$. Then, by (5.12) we apply the same argument as in (5.6) to obtain

$$(5.13)\qquad \xi\cdot(\nabla_x^2 U_0)(|x|)\xi\le 0,\qquad \xi\in\mathbf{S}^{N-1},$$

for all $x\in B(0,R_*+\delta)\setminus B(0,R_*)$. On the other hand, by (5.10) and the continuity of $\nabla_x^2 F[U_0]$ and $\nabla_x^2 Z_i$, for any sufficiently small $\epsilon>0$, taking a sufficiently small δ if necessary, we have

$$(5.14)\qquad \xi\cdot(\nabla_x^2 F[U_0])(x)\xi\ge\frac{U_0(0)}{2N},\qquad |\xi\cdot(\nabla_x^2 Z_i)(x)\xi|\le\epsilon,\qquad \xi\in\mathbf{S}^{N-1},$$

for all $x\in B(0,R_*+\delta)$. Therefore, by (3.9) with $m=2$, (3.20), (3.21), (5.13), and (5.14) we can take a sufficiently small δ so that

$$(5.15)\qquad (4\pi t)^{\frac{N}{2}+1}\xi\cdot(\nabla_x^2 u)(x,t)\xi\le-\frac{N}{2}(M_0+o(1))\xi\cdot\nabla_x^2 F[U_0](x)\xi$$
$$+C\sum_{i=1}^{N}\xi\cdot\nabla_x^2 Z_i(x)\xi+o(1)\le-\frac{M_0U_0(0)}{8},\qquad \xi\in\mathbf{S}^{N-1},$$

for all $x\in B(0,R_*+\delta)\setminus B(0,R_*)$ and all sufficiently large t, where C is a constant. Then, by (4.6), (5.11), and (5.15), taking a sufficiently small δ again if necessary, we apply the same argument as in the proof for case (a) to obtain the desired conclusion. Therefore the proof of Theorem 1.3 for case (d) is complete, and Theorem 1.3 follows. □

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